Inquisitive Logic and Homotopy Type Theory

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The outline of the talk

- inquisitive semantics
- homotopy type theory
- some connections between the two

Inquisitive Semantics

- a framework for a logical analysis of questions

Predecessors of inquisitive semantics

Alternative semantics:

- Hamblin, C. L. (1973). Questions in Montague English.
- Karttunen, L. (1977). Syntax and Semantics of Questions.

Partition semantics:

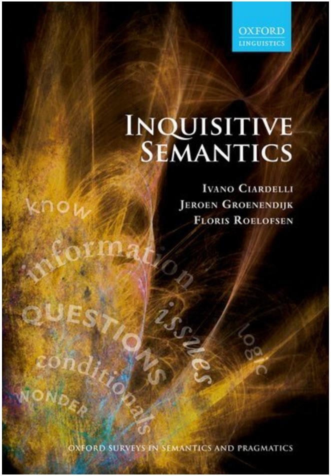
- Groenendijk, J., Stokhof, M. (1984). Studies in the Semantics of Questions and the Pragmatics of Answers.
- Groenendijk, J. (1999). The Logic of Interrogation.

Inquisitive indifference semantics:

- Groenendijk, J. (2009). Inquisitive Semantics: Two Possibilities for Disjunction.
- Mascarenhas, S. (2009). Inquisitive Semantics and Logic. (Master thesis)

The current framework of inquisitive semantics

- Ciardelli, I. (2009). Inquisitive Semantics and Intermediate Logics. (Master thesis)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INQUISITIVE
SEMANTICS
IVANO CIARDELLI
JEROEN GROENENDIJK
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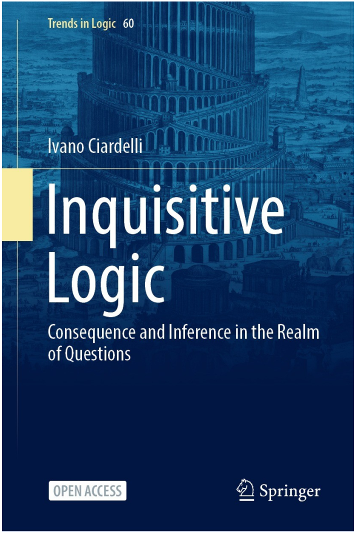
Trends in Logic 60
Ivano Ciardelli
Inquisitive Logic
Consequence and Inference in the Realm of Questions
OPEN ACCESS
Springer

Arguments with questions

$S1, S2$ are statements and $Q1, Q2$ questions

an argument	its intended interpretation
$S1/S2$	$S1$ implies $S2$
$Q1/S2$	$Q1$ presupposes $S2$
$S1/Q2$	$S1$ resolves $Q2$
$Q1/Q2$	any information that resolves $Q1$ resolves also $Q2$

Examples

(a) The question *who is Peter's father: John or George?* pressuposes that *John or Georg is Peter's father*.

(b) The conjunction of the statements *if Mary is Peter's mother, then John is not Peter's father* and *John is Peter's father* resolves the question *whether Mary is Peter's mother*.

(c) Assuming that Peter will go to the pub if and only if Ann will go, any information that resolves the question *whether Ann will go to the pub* resolves also the question *whether Peter will go*.

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Questions are types of information

- Ciardelli, I. (2018). Questions as information types. *Synthese*, 195, 321–365.

Ciardelli's example

- a certain disease may give rise to two symptoms: S_1, S_2
- hospital's protocol:

  > if a patient presents symptom S_2, the treatment is always prescribed; if the patient only presents symptom S_1, the treatment is prescribed just in case the patient is in good physical condition; if not, the risk associated with the treatment outweigh the benefits, and the treatment is not prescribed

A formalization of the protocol

The protocol:

- $t \leftrightarrow s_2 \vee (s_1 \wedge g)$

where

- s_1: the patient has symptom S_1
- s_2: the patient has symptom S_2
- g: the patient is in good physical condtion
- t: the treatment is prescribed

Types of information

Examples of types of information:

- patient's symptoms ($S_1, S_2, \ldots$)
- patient's conditions (good, bad)
- treatment (prescribed, not prescribed)

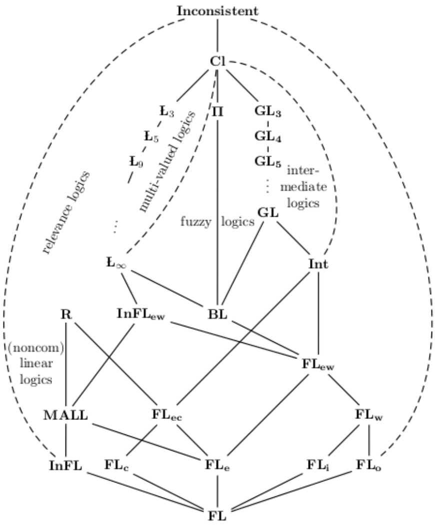
Types of information

Types of information correspond to questions:

- what are the patient's symptoms: $?s_1 \wedge ?s_2$
- whether the patient is in good physical conditions: $?g$
- whether the treatment is prescribed: $?t$

Dependencies among information types correspond to logical relations among questions

$$t \leftrightarrow s_2 \vee (s_1 \wedge g), ?s_1 \wedge ?s_2, ?g \vDash ?t$$



Picture taken from Galatos, N. Jipsen, P. Kowalski, T., Ono, H. (2007) *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier Science.

Language of propositional InqIL

Formulas:

$$\varphi, \psi ::= p \mid \bot \mid \varphi \to \psi \mid \varphi \wedge \psi \mid \varphi \vee\!\!\!\vee \psi$$

Defined symbols:

- $\neg\varphi =_{def} \varphi \to \bot$,
- $\varphi \leftrightarrow \psi =_{def} (\varphi \to \psi) \wedge (\psi \to \varphi)$,

Declarative formulas:

$$\alpha, \beta ::= p \mid \bot \mid \alpha \to \beta \mid \alpha \wedge \beta$$

Kripke frames and models

- Kripke frame $\mathcal{F} = \langle W, \leq \rangle$ is a partial order,
- proposition in $\mathcal{F}$ is an upward closed set in $\mathcal{F}$,
- Kripke model is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a Kripke frame, and V assigns propositions in $\mathcal{F}$ to atomic formulas.

Support relation

In any Kripke model the support relation $\Vdash$ between states and formulas is defined as follows:

- $w \Vdash p$ iff $w \in V(p)$;
- $w \nVdash \bot$;
- $w \Vdash \varphi \to \psi$ iff for any $v \geq w$, if $v \Vdash \varphi$, $v \Vdash \psi$;
- $w \Vdash \varphi \wedge \psi$ iff $w \Vdash \varphi$ and $w \Vdash \psi$;
- $w \Vdash \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $w \Vdash \varphi$ or $w \Vdash \psi$.

Inquisitive variant of a Kripke model

Let $\mathcal{M} = \langle S, \leq, V \rangle$ be a Kripke model. The inquisitive variant of $\mathcal{M}$ is defined as the Kripke model

$$inq(\mathcal{M}) = \langle Up_{\emptyset}S, \supseteq, V^* \rangle,$$

where

- $Up_{\emptyset}S$ is the set of all *nonempty* propositions in $\mathcal{M}$,
- $\supseteq$ is the superset relation,
- $V^*(p) = \{s \in Up_{\emptyset}S \mid s \subseteq V(p)\}$.

Inquisitive variant of a Kripke model

- a Kripke model $\mathcal{N}$ is called inquisitive if it is the inquisitive variant of a Kripke model, i.e. $\mathcal{N} = \mathit{inq}(\mathcal{M})$, for some Kripke model $\mathcal{M}$.
- an inquisitive Kripke model is called classical if it is the inquisitive variant of a Kripke model where the ordering is identity.

Claim

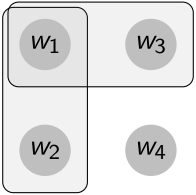
Let $\mathcal{M}$ be a Kripke model, s a state in $inq(\mathcal{M})$ and α an declarative formula. Then

$$s \Vdash \alpha \text{ in } inq(\mathcal{M}) \text{ iff for all } w \in s,\ w \Vdash \alpha \text{ in } \mathcal{M}.$$

As a consequence, α is valid in $inq(\mathcal{M})$ iff α is valid in $\mathcal{M}$.

Let $\mathcal{M} = \langle W, =, V \rangle$, where $W = \{w_1, w_2, w_3, w_4\}$, $V(p) = \{w_1, w_2\}$, and $V(q) = \{w_1, w_3\}$. Then:

- $||p|| = \{w_1, w_2\}$ in $\mathcal{M}$,
- $||p|| = \{\{w_1, w_2\}, \{w_1\}, \{w_2\}\}$ in $inq(\mathcal{M})$,
- $||p \vee\!\!\!\vee q|| = \{w_1, w_2, w_3\}$ in $\mathcal{M}$,
- $||p \vee\!\!\!\vee q|| = \{\{w_1, w_2\}, \{w_1, w_3\}, \{w_1\}, \{w_2\}, \{w_3\}\}$ in $inq(\mathcal{M})$.



Intuitionistic and classical inquisitive logic

The logic of all inquisitive models InqIL is obtained as intuitionistic logic plus one extra principle called split:

$$\frac{\alpha \to (\varphi \mathbin{\vee\!\!\!\vee} \psi)}{(\alpha \to \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \to \psi)}$$

The logic of all classical inquisitive models InqB is obtained as InqIL plus double negation for declarative formulas:

$$\frac{\neg\neg\alpha}{\alpha}$$

(α is declarative)

Adding presupposition

Language of propositional InqIL$^\circ$

$$\varphi, \psi ::= p \mid \bot \mid \varphi \to \psi \mid \varphi \wedge \psi \mid \varphi \mathbin{\vee\!\!\!\vee} \psi \mid \circ\varphi$$

$\circ\varphi$ expresses the information presupposed by φ

Defined symbols:

- $\neg\varphi =_{def} \varphi \to \bot$,
- $\varphi \leftrightarrow \psi =_{def} (\varphi \to \psi) \wedge (\psi \to \varphi)$,
- $\varphi \vee \psi =_{def} \circ(\varphi \mathbin{\vee\!\!\!\vee} \psi)$.

Declarative formulas:

$$\alpha, \beta ::= p \mid \bot \mid \alpha \to \beta \mid \alpha \wedge \beta \mid \circ\varphi$$

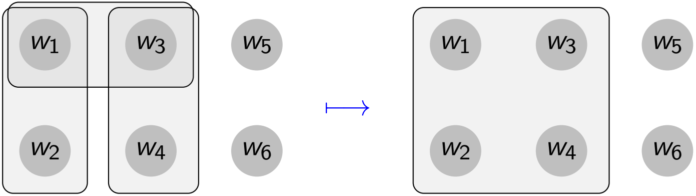
Support condition for the presupposition

s is a state of an inquisitive model:

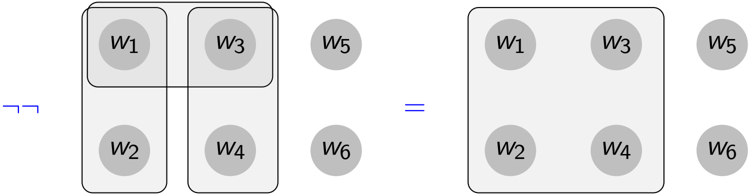
$$s \Vdash \circ\varphi \text{ iff for all } w \in s, \uparrow w \Vdash \varphi.$$

$\uparrow w = \{v \mid w \leq v\}$

Presupposition

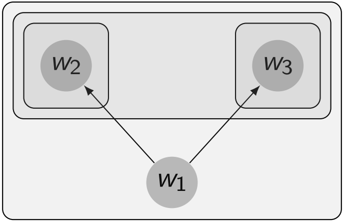


Presupposition as double negation



In intuitionistic inquisitive logic the presupposition modality is not double negation

$$||\varphi|| = \{\{w_2\}, \{w_3\}\},$$
$$||\circ\varphi|| = \uparrow\{w_2, w_3\},$$
$$||\neg\neg\varphi|| = \uparrow\{w_1, w_2, w_3\},$$



The inquisitive nucleus

Let P be a proposition in inquisitive semantics (non-empty downward closed set of states). We define:

$$jP = \{s \in States \mid s \subseteq \bigcup P\}$$

jP is the strongest declarative proposition implied by P. Then it holds:

(a) $P \subseteq jP$,

(b) $jjP \subseteq jP$,

(c) $j(P \cap Q) = jP \cap jQ$,

(d) $j(\emptyset) = \emptyset$.

So, j is a dense nucleus on the algebra of propositions.

The logic InqIL$^{\circ}$

- Intuitionistic logic plus split plus $\circ$-intro and $\circ$-elim

Split (alternative schematic formulation)

$$\frac{\circ\chi \to (\varphi \Vvee \psi)}{(\circ\chi \to \varphi) \Vvee (\circ\chi \to \psi)}$$

$\circ$-intro and $\circ$-elim

$$\frac{\varphi}{\circ\varphi}\ \circ\text{-intro}$$

$$\frac{\circ\varphi \qquad \begin{array}{c}[\varphi]\\ \alpha\end{array}}{\alpha}\ \circ\text{-elim}$$

where α is declarative

Compare $\circ$ and $\vee$

$$\frac{\varphi}{\circ\varphi}\ \circ\text{-intro} \qquad\qquad \frac{\circ\varphi \qquad \begin{array}{c}[\varphi]\\ \alpha\end{array}}{\alpha}\ \circ\text{-elim}$$

$$\frac{\varphi(\psi)}{\varphi \vee \psi}\ \vee\text{-intro} \qquad\qquad \frac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi]\\ \alpha\end{array} \qquad \begin{array}{c}[\psi]\\ \alpha\end{array}}{\alpha}\ \vee\text{-elim}$$

$$\varphi \vee \psi =_{def} \circ(\varphi \Vee \psi) \qquad\qquad \circ\varphi =_{def} \varphi \vee \varphi$$

Compare $\circ$ and $\vee$

$$\frac{\varphi}{\circ\varphi}\ \circ\text{-intro} \qquad\qquad \frac{\circ\varphi \qquad \begin{matrix}[\varphi]\\ \alpha\end{matrix}}{\alpha}\ \circ\text{-elim}$$

$$\frac{\varphi(\psi)}{\varphi \vee \psi}\ \vee\text{-intro} \qquad\qquad \frac{\varphi \vee \psi \qquad \begin{matrix}[\varphi]\\ \alpha\end{matrix} \qquad \begin{matrix}[\psi]\\ \alpha\end{matrix}}{\alpha}\ \vee\text{-elim}$$

$$\varphi \vee \psi =_{def} \circ(\varphi \Vee \psi) \qquad\qquad \circ\varphi =_{def} \varphi \vee \varphi$$

Axiomatic formulation

$\circ$-intro and $\circ$-elim are equivalent to the following set of axioms:

(a) $\circ\alpha \to \alpha$, for declarative α,

(b) $\varphi \to \circ\varphi$,

(c) $(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$.

Another axiomatic formulation

$\circ$-intro and $\circ$-elim are equivalent to the following axiom:

$(\varphi \to \alpha) \leftrightarrow (\circ\varphi \to \alpha)$ (where α is declarative)

Presupposition and double negation

- $\circ\varphi \to \neg\neg\varphi$ is generally valid but $\neg\neg\varphi \to \circ\varphi$ is not,
- $(\neg\neg\varphi \to \varphi) \to (\neg\neg\varphi \leftrightarrow \circ\varphi)$ is valid
- adding full double negation leads to the trivialization of $\circ$
- adding double negation for declarative formulas leads to the equivalence of $\circ$ and $\neg\neg$
- $(\neg\neg\alpha \to \alpha) \leftrightarrow (\neg\neg\alpha \leftrightarrow \circ\alpha)$ is valid for every declarative α

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Propositional lax logic

Fairtlough, M., Mendler, M. (1997). Propositional lax Logic. *Information and Computation*, 137, 1–33.

- model for formal verification of computer hardware

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Dense propositional lax logic

Intuitionistic logic plus:

(a) $\circ\circ\varphi \to \circ\varphi$,

(b) $\varphi \to \circ\varphi$,

(c) $(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$,

(d) $\circ\bot \to \bot$.

Compare to our three axioms:

(a)' $\circ\alpha \to \alpha$, for declarative α,

(b)' $\varphi \to \circ\varphi$,

(c)' $(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$.

(a)'-(c)' is equivalent to (a)-(d) plus $\circ p \to p$ for atomic formulas

Nucleus (the algebraic counterpart of the lax modality)

A nucleus on a Heyting algebra $\mathcal{H}$ is a function $j : H \to H$ such that for each $s, t \in H$:

(a) $s \leq j(s)$,
(b) $j(j(s)) \leq j(s)$,
(c) $j(s \wedge t) = j(s) \wedge j(t)$.

(every nucleus is a closure operator)

A nucleus is dense if $j(0) = 0$.

Nuclear algebraic semantics for first-order intuitionistic inquisitive logic

Formulas:

$$\varphi, \psi ::= Pt_1 \ldots t_n \mid \bot \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \forall x \varphi \mid \circ\varphi \mid \varphi \vee\!\!\!\vee \psi \mid \exists\!\!\!\exists x \varphi$$

Defined symbols:

$$\varphi \vee \psi =_{def} \circ(\varphi \vee\!\!\!\vee \psi), \exists x \varphi =_{def} \circ\exists\!\!\!\exists x \varphi$$

Declarative formulas:

$$\alpha, \beta ::= Pt_1 \ldots t_n \mid \bot \mid \alpha \wedge \beta \mid \alpha \to \beta \mid \forall x \alpha \mid \circ\varphi$$

Complete Heyting algebra

A complete Heyting algebra (cHA) is any structure

$$\mathcal{H} = \langle H, \bigvee, \bigwedge, \Rightarrow, 0 \rangle,$$

where

- $\langle H, \bigvee, \bigwedge \rangle$ is a complete lattice,
- 0 is its least element
- $\Rightarrow$ is a relative pseudocomplement, i.e. a binary operation on H satisfying the residuation condition:

$$u \leq s \Rightarrow t \text{ iff } u \wedge s \leq t.$$

Complete Heyting algebras as "frames" or "locales"

Complete Heyting algebras coincide with "frames" or "locales" complete lattices satisfying the following infinitary distributive law:

$$s \wedge \bigvee_{i \in I} t_i = \bigvee_{i \in I} (s \wedge t_i).$$

In every locale, relative pseudocomplement satisfying the residuation condition can be defined as follows:

$$s \Rightarrow t = \bigvee \{u \in H \mid s \wedge u \leq t\}.$$

Nuclear cHAs

A nuclear cHA (ncHA) is a cHA equipped with a nucleus.

If $\mathcal{H}$ is an ncHA, the set $jH = \{j(s) \mid s \in H\}$ of all its j-fixed points will be called the declarative core of $\mathcal{H}$. The j-fixed points will be called declarative propositions.

Kripkean ncHAs

Let $\mathcal{H} = \langle H, \bigvee, \bigwedge, \Rightarrow, 0 \rangle$ be a cHA. Take the structure

$$Dw(\mathcal{H}) = \langle DwH, \bigcup, \bigcap, \Rrightarrow, \{0\}, dj \rangle$$

where

- DwH is the set of all non-empty downsets of $\mathcal{H}$,
- $\bigcup$ and $\bigcap$ are (infinitary) union and intersection,
- $\Rrightarrow$ is defined as follows:

$$X \Rrightarrow Y = \bigcup \{Z \in DwH \mid Z \cap X \subseteq Y\},$$

- and dj as follows:

$$dj(X) = \downarrow \bigvee X.$$

We will call these structures Kripkean ncHAs. Moreover, if $\mathcal{H}$ is complete atomic Boolean algebra then the Kripkean ncHA $Dw(\mathcal{H})$ will be called standard.

Declarative propositions

Declarative propositions are closed under implication, conjunction and universal quantification. Moreover, the contradiction is declarative.

Proposition

The declarative core of any ncHA is closed under $\bigwedge$ and $\Rightarrow$. In dense ncHAs, 0 is declarative.

Declarative propositions form an cHA

If $\mathcal{H} = \langle H, \bigvee, \bigwedge, \Rightarrow, 0, j\rangle$ is a dense ncHA, we can define the structure

$$j\mathcal{H} = \langle jH, \bigvee^{j}, \bigwedge, \Rightarrow, 0\rangle$$

where

- $\bigwedge, \Rightarrow, 0$ are taken form $\mathcal{H}$ (but restricted to the core),
- and $\bigvee^{j} X = j(\bigvee X)$, for all $X \subseteq H$.

Proposition

$j\mathcal{H}$ is a cHA.

First-order frames

- by a first-order algebraic frame we will understand a pair $\mathcal{F} = \langle \mathcal{H}, U \rangle$, where $\mathcal{H}$ is an ncHA and U is a non-empty set (the domain of quantification).
- a valuation in $\mathcal{F}$ is defined as a function V which assigns to any n-ary predicate P a function $V(P) : U^n \to H$
- we say that a valuation V is informative, if for any n-ary predicate P we have $V(P) : U^n \to jH$

First-order models

- a first-order algebraic model is an algebraic frame equipped with a valuation
- a regular algebraic model is an algebraic model in which the valuation is informative.
- a Kripkean algebraic model is a regular algebraic model based on a Kripkean ncHA.
- a standard algebraic model is a regular algebraic model based on a standard Kripkean ncHA.

Evaluation

An evaluation in U is a function that assigns to each variable of the language an element of U. If e is an evaluation, x a variable, and $m \in U$, then $e(m/x)$ is the evaluation that assigns m to x and $e(y)$ to any other variable y. For any term t, $V^e(t)$ is identical with $V(t)$ if t is a name, and with $e(t)$ if t is a variable.

Algebraic value of a formula in a ncHA

- $|\bot|_e^{\mathcal{N}} = 0$,
- $|Pt_1 \dots t_n|_e^{\mathcal{N}} = V(P)(V^e(t_1), \dots, V^e(t_n))$,
- $|\varphi \wedge \psi|_e^{\mathcal{N}} = |\varphi|_e^{\mathcal{N}} \wedge |\psi|_e^{\mathcal{N}}$,
- $|\varphi \to \psi|_e^{\mathcal{N}} = |\varphi|_e^{\mathcal{N}} \Rightarrow |\psi|_e^{\mathcal{N}}$,
- $|\forall x\varphi|_e^{\mathcal{N}} = \bigwedge_{m\in U} |\varphi|_{e(m/x)}^{\mathcal{N}}$,
- $|\circ\varphi|_e^{\mathcal{N}} = j(|\varphi|_e^{\mathcal{N}})$
- $|\varphi \Vvee \psi|_e^{\mathcal{N}} = |\varphi|_e^{\mathcal{N}} \vee |\psi|_e^{\mathcal{N}}$,
- $|\exists x\varphi|_e^{\mathcal{N}} = \bigvee_{m\in U} |\varphi|_{e(m/x)}^{\mathcal{N}}$.

Validity

- an L-formula φ is e-valid in $\mathcal{N}$, if $|\varphi|_e^{\mathcal{N}} = 1$
- φ is valid in $\mathcal{N}$ if for every evaluation e in $\mathcal{N}$, φ is e-valid in $\mathcal{N}$
- φ is valid in an algebraic frame if it is valid in every algebraic model based on that frame.

Logics

- the logic of all algebraic models is first-order lax logic
- the logic of all regular algebraic models is first-order lax logic plus the following axiom:

$$\circ\alpha \to \alpha, \text{ for elementary formulas}$$

- the logic of all Kripkean models is first-order intuitionistic inquisitive logic
- the logic of all standard models is standard (classical) first-order inquisitive logic

Inquisitive ncHAs

Let $\mathcal{H} = \langle H, \bigvee, \bigwedge, \Rightarrow, 0, j \rangle$ be an ncHA. We say that $\mathcal{H}$ is inquisitive if j is dense and the following two conditions are satisfied for every $s \in H$ and any collection of indexed elements $t_i, u_{ik} \in H$, where $i \in I$ and $k \in K$ for some index sets I, K:

(a) $j(s) \Rightarrow \bigvee_{i \in I} t_i = \bigvee_{i \in I}(j(s) \Rightarrow t_i)$,

(b) $\bigwedge_{i \in I} \bigvee_{k \in K} j(u_{ik}) = \bigvee_{f: I \to K} \bigwedge_{i \in I} j(u_{if(i)})$.

An algebraic frame (model) is called inquisitive if it is based on an inquisitive ncHA.

Resolutions in propositional logic

$$\mathcal{R}(p) = \{p\},\ \mathcal{R}(\bot) = \{\bot\},$$
$$\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\},$$
$$\mathcal{R}(\varphi \to \psi) = \{\textstyle\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\},$$
$$\mathcal{R}(\varphi \vee\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi).$$

Resolutions in first-order logic

$$\mathcal{R}_e^{\mathcal{N}}(Pt_1 \dots t_n) = \{|Pt_1 \dots t_n|_e^{\mathcal{N}}\},\ \mathcal{R}_e^{\mathcal{N}}(\bot) = \{0\},$$

$$\mathcal{R}_e^{\mathcal{N}}(\varphi \wedge \psi) = \{s \wedge u \mid s \in \mathcal{R}_e^{\mathcal{N}}(\varphi), u \in \mathcal{R}_e^{\mathcal{N}}(\psi)\},$$

$$\mathcal{R}_e^{\mathcal{N}}(\varphi \to \psi) = \{\textstyle\bigwedge_{s \in \mathcal{R}_e^{\mathcal{N}}(\varphi)}(s \Rightarrow f(s)) \mid f : \mathcal{R}_e^{\mathcal{N}}(\varphi) \to \mathcal{R}_e^{\mathcal{N}}(\psi)\},$$

$$\mathcal{R}_e^{\mathcal{N}}(\forall x \varphi) = \{\textstyle\bigwedge_{m \in U} f(m) \mid f : U \to \bigcup_{m \in U} \mathcal{R}_{e(m/x)}^{\mathcal{N}},$$
$$\text{s.t. } f(m) \in \mathcal{R}_{e(m/x)}^{\mathcal{N}}, \text{ for each } m \in U\},$$

$$\mathcal{R}_e^{\mathcal{N}}(\circ\varphi) = \{j(\textstyle\bigvee \mathcal{R}_e^{\mathcal{N}}(\varphi))\},$$

$$\mathcal{R}_e^{\mathcal{N}}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}_e^{\mathcal{N}}(\varphi) \cup \mathcal{R}_e^{\mathcal{N}}(\psi),$$

$$\mathcal{R}_e^{\mathcal{N}}(\exists x \varphi) = \textstyle\bigcup\{\mathcal{R}_{e(m/x)}^{\mathcal{N}}(\varphi) \mid m \in U\}.$$

One can observe that for any formula φ, $\mathcal{R}^{\mathcal{N}}_{e(m/x)}(\varphi)$ is a set of core elements (i.e. declarative propositions) in $\mathcal{N}$. Moreover:

Theorem

Let $\mathcal{N} = \langle \mathcal{H}, U, V \rangle$ be a regular algebraic model, e an evaluation in U, and φ an L-formula. If $\mathcal{N}$ is inquisitive then

$$|\varphi|^{\mathcal{N}}_{e} = \bigvee \mathcal{R}^{\mathcal{N}}_{e}(\varphi).$$

A connection between inquisitive and Kripkean ncHAs

Let $\mathcal{H}_1$, $\mathcal{H}_2$ be ncHAs. A homomorphism from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a function $h : H_1 \to H_2$ which preserves the operations $\bigvee, \bigwedge, \Rightarrow, j$ and 0. We say that $\mathcal{H}_2$ is a homomorphic j-image of $\mathcal{H}_1$ if $j_2 H_2 = h(j_1 H_1)$.

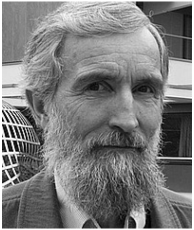
Theorem

If $\mathcal{H}_2$ is a homomorphic j-image of $\mathcal{H}_1$ then φ is valid in all regular models on $\langle \mathcal{H}_1, U \rangle$ iff φ is valid in all regular models on $\langle \mathcal{H}_2, U \rangle$.

Theorem

An ncHA is inquisitive iff it is a homomorphic j-image of a Kripkean ncHA.

Intuitionistic type theory
Per Martin-Löf (born 1942)



- Swedish logician, mathematician, philosopher
- an enthusiastic bird-watcher
- 1961: Mortality rate calculations on ringed birds with special reference to the Dunlin Calidris alpina
- 1966: The definition of random sequences. (a paper that gave the first suitable definition of a random sequence)
- 1984: Intuitionistic type theory. Napoli: Bibliopolis.

Homotopy type theory

- (relatively) new framework for the foundations of mathematics, oriented at computer proof-assistants and computer friendly formalization of mathematics
- based on Per Martin-Löf's intuitionistic type theory
- some history:
  - 1994/1998 Hofmann & Streicher: intensional groupoid model of Martin-Löf's type theory
  - 2006 Voevodsky: homotopy λ-calculus (program of studying type systems by homotopical methods
  - 2007/2009 Awodey & Warrren: homotopical model of Martin-Löf's type theory

Voevodsky's research program

The broad motivation behind univalent foundations is a desire to have a system in which mathematics can be formalized in a manner which is as neutral as possible. Whilst it is possible to encode all of mathematics into Zermelo-Fraenkel set theory, the manner in which this is done is ugly . . . This problem becomes particularly pressing in attempting a computer formalization of mathematics; in the standard foundations, to write down in full even the most basic definitions . . . requires many pages of symbols. Univalent foundations seeks to improve on this situation by providing a system, based on Martin-Löf's dependent type-theory, whose syntax is tightly wedded to the intended semantical interpretation in the world of everyday mathematics.

Curry-Howard isomorphism/correspondence

The logic of function application and function formation:

- e.g. $x : A, f : A \to B \vdash f(x) : B$

The logic of propositions:

- e.g. $A, A \to B \vdash B$

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Sequent system

- identity ($\varphi \vdash \varphi$),
- structural rules,
- introduction and elimination rules

Sequent system for intuitionistic logic

Introduction rule:

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}$$

Elimination rule:

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi \to \psi}{\Gamma \vdash \psi}$$

Adding terms

Introduction rule:

$$\frac{\Gamma, x : \varphi \vdash t : \psi}{\Gamma \vdash \lambda x.t : \varphi \to \psi}$$

Elimination rule:

$$\frac{\Gamma \vdash u : \varphi \qquad \Gamma \vdash t : \varphi \to \psi}{\Gamma \vdash t(u) : \psi}$$

Curry-Howard isomorphism/correspondence

Theorem

φ is a type of a closed λ-term iff φ is intuitionistically valid.

Theorem

A derivation is normal (contains no detours) if and only if the term assigned to it is in normal form.

$\varphi \to ((\varphi \to \psi) \to \psi)$

$$\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi}{\varphi, \varphi \to \psi \vdash \varphi} \qquad \dfrac{\varphi \to \psi \vdash \varphi \to \psi}{\varphi, \varphi \to \psi \vdash \varphi \to \psi}}{\dfrac{\varphi, \varphi \to \psi \vdash \psi}{\varphi \vdash (\varphi \to \psi) \to \psi}}}{\vdash \varphi \to ((\varphi \to \psi) \to \psi)}$$

$\lambda x.\lambda y.y(x) : \varphi \to ((\varphi \to \psi) \to \psi)$

$$
\dfrac{\dfrac{\dfrac{x : \varphi \vdash x : \varphi}{x : \varphi, y : \varphi \to \psi \vdash x : \varphi} \qquad \dfrac{y : \varphi \to \psi \vdash y : \varphi \to \psi}{x : \varphi, y : \varphi \to \psi \vdash y : \varphi \to \psi}}{\dfrac{x : \varphi, y : \varphi \to \psi \vdash y(x) : \psi}{x : \varphi \vdash \lambda y.y(x) : (\varphi \to \psi) \to \psi}}}{\vdash \lambda x.\lambda y.y(x) : \varphi \to ((\varphi \to \psi) \to \psi)}
$$

Types

Every object in mathematics is of some type:

- $3 : \mathbb{N}$
- $\pi : \mathbb{R}$
- $\lambda n.n! : \mathbb{N} \to \mathbb{N}$
- $\xi : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{N} \to \mathbb{N})$

In type theory, every object is of a unique type:

- $3_{\mathbb{N}} : \mathbb{N}$
- $3_{\mathbb{Z}} : \mathbb{Z}$
- $3_{\mathbb{Q}} : \mathbb{Q}$
- $3_{\mathbb{R}} : \mathbb{R}$

Propositions as types

The expression

$$t : A$$

has double reading:

(a) t is an object of type A

(b) t is a proof of proposition A

Dependent function types

A family of types $B(a)$

- $B(a)$ is a type, for each $a : A$

General form of function types:

- $f : \prod_{a:A} B(a)$
- that is, $f(a) : B(a)$, for each $a : A$

Judgements

- $\Gamma \vdash A \ type$
- $\Gamma \vdash A = B \ type$
- $\Gamma \vdash t : A$
- $\Gamma \vdash s = t : A$

where Γ is a context specifying the types of variables. E.g. we can have the folowing judgement:

- $x : A, f : A \to B \vdash f(x) : B$

Rules

(a) formation rules
(b) introduction rules
(c) elimination rules
(d) computation rules

Type constructions

(a) $\prod$ (universal quantification)

(b) $\sum$ (existential quantification)

(c) $\times$, $+$, $\to$ (conjunction, disjunction, implication

(d) 0 (empty type = contradiction)

(e) **Bool**, $\mathbb{N}$, $\mathbb{Z}$, ...

Coproduct: formation rule

$$\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma \vdash B \ \mathsf{type}}{\Gamma \vdash A + B \ \mathsf{type}}$$

Coproduct: introduction rules

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash \mathit{inl} : A \to (A + B)}$$

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash \mathit{inr} : B \to (A + B)}$$

Coproduct: elimination rule

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma, x : A + B \vdash P(x) \text{ type}}{\Gamma \vdash \ \mathsf{ind}_{+} : \prod_{(x:A)} P(inl(x)) \to (\prod_{(y:B)} P(inr(y)) \to \prod_{(z:A+B)} P(z))}$$

Natural numbers: formation rule

$$\frac{}{\vdash \mathbb{N}\ \mathsf{type}}$$

Natural numbers: introduction rules

$$\frac{}{\vdash 0_{\mathbb{N}} : \mathbb{N}}$$

$$\frac{}{\vdash \mathit{succ}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}}$$

Natural numbers: elimination rule

$$\frac{\Gamma, n : \mathbb{N} \vdash P(n) \text{ type}}{\Gamma \vdash \ \mathsf{ind}_{\mathbb{N}} : P(0_{\mathbb{N}}) \to (\prod_{n:\mathbb{N}} (P(n) \to P(succ_{\mathbb{N}})) \to \prod_{n:\mathbb{N}} P(n))}$$

Logical operators

- $A \to B$ corresponds to implication
- $A \to 0$ corresponds to negation
- $A \times B$ corresponds to conjunction
- $A + B$ corresponds to disjunction

Quantifiers

- $\prod_{x:A} B(x)$ corresponds to $\forall x{:}A.B(x)$
- $\sum_{x:A} B(x)$ corresponds to $\exists x{:}A.B(x)$

Identity types

- if A is a type then Id_A is a family of types
- $Id_A(x, y)$ is a type, for any $x, y : A$
- $Id_{Id_A(x,y)}(p, q)$ is a type, for any $p, q : Id_A(x, y)$
- instead of $Id_A(x, y)$, I will be using $x =_A y$

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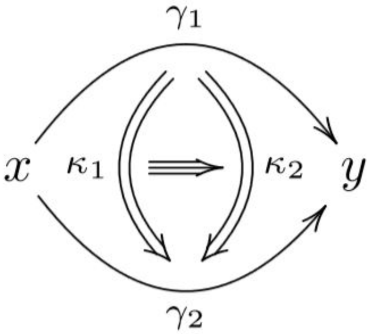

HoTT vs. set theory

- set theory: based on classical logic (not constructive)
- type theory: based on intuitionistic logic (constructive)

HoTT vs. set theory

- set theory: logic and mathematics separated
- type theory: logic and mathematics mixed

HoTT vs. set theory

- set theory: propositions and sets separated
- type theory: propositions and types identified

HoTT vs. set theory

- set theory: one category of objects (sets)
- type theory: multiplicity of types

HoTT vs. set theory

- set theory: one object is a member of many sets
- type theory: one object is a member of one single type

HoTT vs. set theory

- set theory: one closed universe
- type theory: open-ended (new inductive types can be introduced)

Constructivity of HoTT

- Homotopy type theory is essentially constructive
- it is incompatible with the law of excluded middle and with double negation law:

$$A + \neg A, \ \neg\neg A \to A$$

- can HoTT preserve classical mathematics?

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Mere propositions

- type P is a mere proposition if for all $x, y : P$ we have $x =_P y$
- more formally:

$$Mere(P) =_{def} \prod_{x,y:P} x =_P y$$

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Truncation

- for any type A there is a type $\circ A$, called the truncation of A
- $\circ A$ usually denoted as $||A||$ or $[A]$
- this modality can be characterized by type theoretic versions of $\circ$-intro and $\circ$-elim

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The logic of mere propositions

Awodey, S., Bauer, A. (2004). Propositions as [Types]. *JLC*

Definition

Let P and Q be mere propositions. Traditional logical notation is defined as follows:

$$\begin{array}{ll} \top :\equiv 1, & \bot :\equiv 0 \\ \neg P :\equiv P \to 0 & P \Rightarrow Q :\equiv P \to Q \\ P \Leftrightarrow Q :\equiv P = Q & (\forall x : A).P(x) :\equiv \prod_{x:A} P(x) \\ P \vee Q :\equiv \circ(P + Q) & (\exists x : A).P(x) \equiv \circ \sum_{x:A} P(x) \end{array}$$

Double negation law and excluded middle can be preserved on the level of mere propositions

Levels of truncation

In HoTT there are not just two levels, like in inquisitive logic, but there is whole hierarchy of the levels of truncation:

- level -2: contractible types (declarative theorems)
- level -1: mere propositions (declarative propositions)
- level 0: sets (inquisitive propositions)

  $\vdots$

- level n: n-truncated types

  $\vdots$

From the HoTT book

Using all types as propositions yields a very "constructive" conception of logic . . . Thus, from every proof we can automatically extract an algorithm; this can be very useful in applications to computer programming. On the other hand, however, this logic . . . does not faithfully represent certain important classical principles of reasoning, such as the axiom of choice and the law of excluded middle. For these we need to use the "(-1)-truncated" logic, in which only the homotopy (-1)-types represent propositions. . .

From the HoTT book

. . . while the pure propositions-as-types logic is "constructive" in the strong algorithmic sense mentioned above, the default (-1)-truncated logic is "constructive" in a different sense (namely, that of the logic formalized by Heyting under the name "intuitionistic"); and to the latter we may freely add the axioms of choice and excluded middle to obtain a logic that may be called "classical". . . The homotopical perspective reveals that classical and constructive logic can coexist. . . Indeed, one can even have useful systems in which only certain types satisfy such further "classical" principles, while types in general remain "constructive". . . Most of classical mathematics which depends on the law of excluded middle and the axiom of choice can be performed in univalent foundations, simply by assuming that these two principles hold (in their proper, (-1)-truncated, form).