Preserving and constructing combinatorial families of reals with forcing

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20.11.2023

Origins of combinatorial set theory

$$|\mathbb{N}| := \aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \cdots < \aleph_{\aleph_0} < \ldots$$

Theorem (Cantor, 1874)

The set of real numbers is uncountable, i.e. $\mathfrak{c} := |\mathbb{R}| > \aleph_0$.

Question (Cantor, Hilbert's problem No.1)

Is $\mathfrak{c} = \aleph_1$ *provable in* ZFC*?*

Independence of the continuum hypothesis


Theorem (Gödel, 1938, [12])

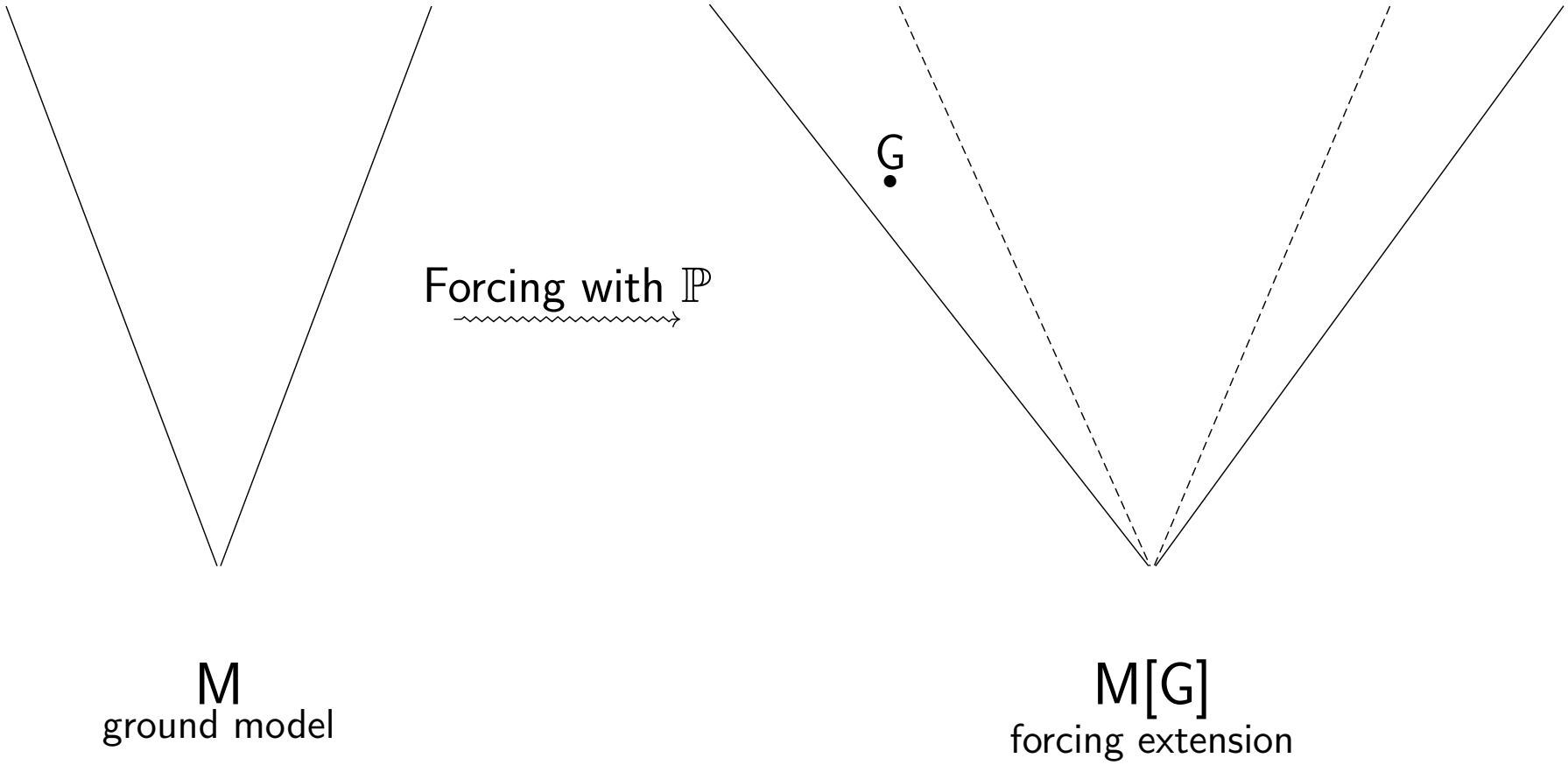
$\mathfrak{c} = \aleph_1$ *is consistent with* ZFC. *(→ constructible universe)*

Theorem (Cohen, 1963, [7])

$\mathfrak{c} > \aleph_1$ *is consistent with* ZFC. *(→ forcing method)*

Idea of forcing

Fundamental idea of forcing:



One may force ZFC $+\neg$ CH when G is a set of $\aleph_2$-many new reals.

Combinatorial families of reals

Expanding on Cohen's result one may ask what possible sizes subset of reals with additional combinatorial properties may have in models with large continuum.

Example (Mad families)

A family $\mathcal{A}$ of infinite subsets of natural numbers is called almost disjoint iff $A \cap B$ is finite for all $A \neq B \in \mathcal{A}$. It is called maximal iff it is maximal with respect to inclusion.

For any such type of combinatorial family of reals we may define its associated cardinal characteristic and spectrum:

$$\mathrm{spec}(\mathfrak{a}) := \{|\mathcal{A}| \mid \mathcal{A} \text{ is an infinite maximal almost disjoint family}\}$$
$$\mathfrak{a} := \min(\mathrm{spec}(\mathfrak{a})).$$

Dominating and unbounded families

For $f, g \in {}^\omega\omega$ we say g dominates f and write $f <^* g$ iff

$$\exists N \, \forall n > N : f(n) < g(n).$$

We say $\mathcal{B} \subseteq {}^\omega\omega$ is unbounded iff

$$\forall g \in {}^\omega 2 \;\; \exists f \in \mathcal{B} : f \not<^* g.$$

Similarly, we say $\mathcal{D} \subseteq {}^\omega\omega$ is dominating iff

$$\forall g \in {}^\omega 2 \;\; \exists f \in \mathcal{D} : g <^* f.$$

Finally, we define the cardinal characteristics

$$\mathfrak{b} := \min \{|\mathcal{B}| \mid \mathcal{B} \subseteq {}^\omega\omega \text{ is unbounded}\},$$
$$\mathfrak{d} := \min \{|\mathcal{D}| \mid \mathcal{D} \subseteq {}^\omega\omega \text{ is dominating}\}.$$

Classical combinatorial set theory

A Cantor-style diagonalization argument shows that

$$\aleph_0 < \mathfrak{a}, \mathfrak{b}, \mathfrak{d} \leq \mathfrak{c},$$

so we call $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{d}$ cardinal characteristics of the continuum.

Guiding Question

What values can some cardinal characteristic take?

For example, consistently $\mathrm{cof}(\mathfrak{a}) = \aleph_0$ [3], but provably $\mathrm{cof}(\mathfrak{d}) > \aleph_0$.

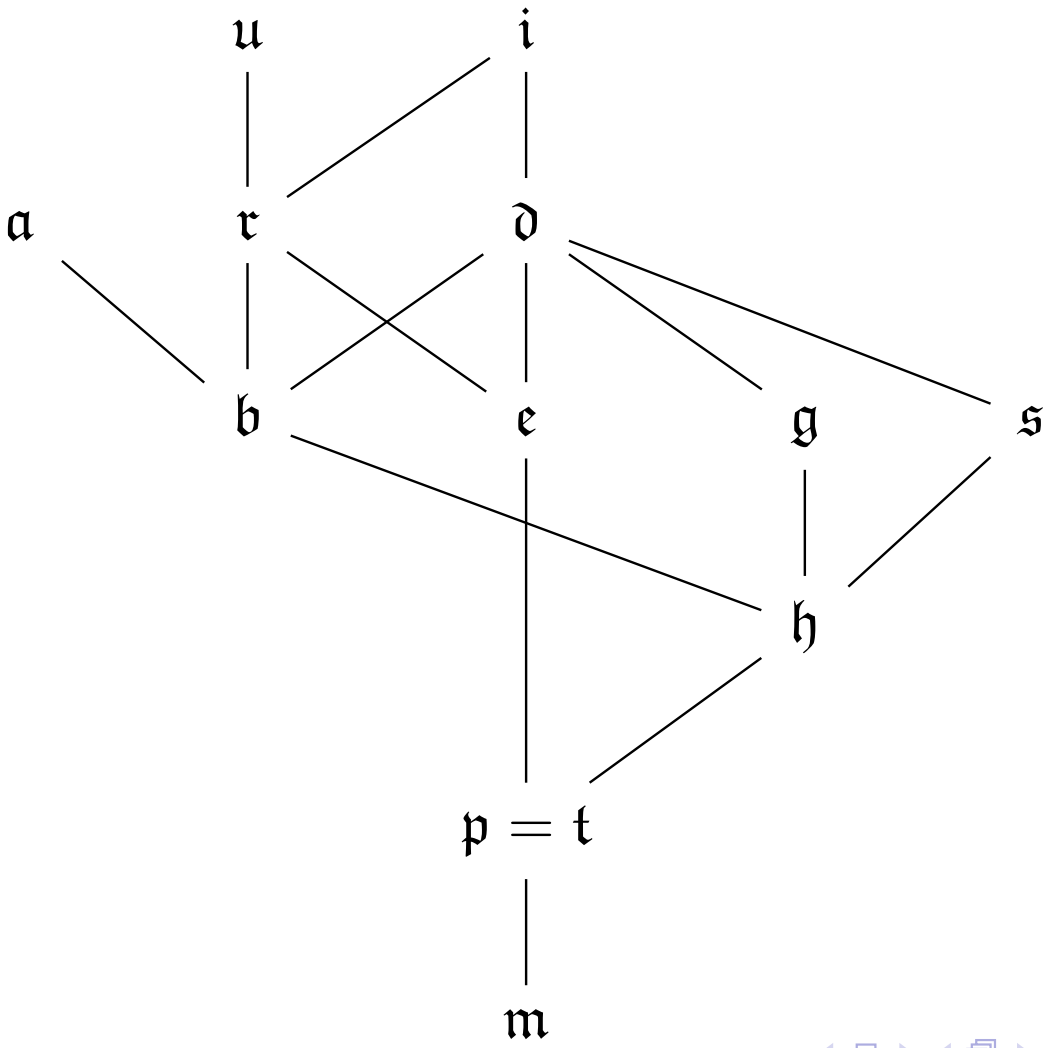
Guiding Question

What are the provable relations between different cardinal characteristics?

For example $\mathfrak{b} \leq \mathfrak{d}, \mathfrak{a}$.

Provable relations between cardinal characteristics

The provable relations between classical cardinal characteristics yield a very rich picture:



Changing spectra with forcing

The point of forcing in the context of combinatorial set theory is that the forcing extension may have a different spectrum than the ground model:

Theorem (Hechler, 1972, [14])

For any $\kappa > \aleph_0$ there is a c.c.c. forcing which adds a new maximal almost disjoint family of size κ.

Generally, we may obtain different spectra with forcing by ...

- ... constructing new maximal families in the forcing extension.
- ... preserving/destroying maximal families from the ground model.

Forcing in combinatorial set theory

We may also use forcing to construct models in which different cardinal characteristics are separated. Tackling these types of questions has proven very fruitful for the development of new forcing techniques, e.g.:

- $\mathfrak{u} < \mathfrak{d}$ led to the development of matrix iterations [2],
- $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$ led to the development of creature forcings [17],
- $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ led to the development of template iterations [18].

Partitions of Baire space into compact sets

Definition

We define the spectrum

$$\mathrm{spec}(\mathfrak{a}_T) := \{|\mathcal{P}| \mid \mathcal{P} \text{ is a partition of } {}^{\omega}\omega \text{ into compact sets}\}$$

and define the cardinal characteristic $\mathfrak{a}_T := \min(\mathrm{spec}(\mathfrak{a}_T))$.

Theorem (Spinas, 1997, [20])

$\mathfrak{a}_T$ is independent of the underlying Polish space.

Theorem (Brian, 2021, [6])

$\mathrm{spec}(\mathfrak{a}_T)$ is independent of the choice of the underlying Polish space.

The cardinal characteristic $\mathfrak{a}_T$

Theorem (Miller, 1980, [15])

Consistently, $\aleph_1 = \mathrm{cov}(\mathcal{M}) < \mathfrak{a}_T = \aleph_2$.

Theorem (Spinas, 1997, [20])

$\mathfrak{d} \leq \mathfrak{a}_T$ *and consistently* $\aleph_1 = \mathfrak{d} < \mathfrak{a}_T = \aleph_2$.

Theorem (Fischer, S., 2022, [8])

Let κ *be measurable and* μ, λ *be regular with* $\kappa < \mu < \lambda$. *Then consistently* $\mathfrak{b} = \mathfrak{d} = \mu$ *and* $\mathfrak{a} = \mathfrak{a}_T = \lambda = \mathfrak{c}$.

Questions regarding $\mathfrak{a}_T$

Theorem (Brendle, 2003, [3])

Consistently, $\mathfrak{a} = \aleph_{\aleph_0}$, in particular $\mathfrak{a}$ may have countable cofinality.

Question

Is $\mathrm{cof}(\mathfrak{a}_T) = \aleph_0$ consistent? In particular is $\mathfrak{a}_T = \aleph_{\aleph_0}$ consistent?

Corollary (essentially Hechler, 1967, [13])

Let κ be any cardinal of uncountable cofinality. Then consistently $\kappa = \mathfrak{d} = \mathfrak{a}_T = \mathfrak{c}$.

Question

Can $\mathfrak{a}_T$ be singular of uncountable cofinality and $\mathfrak{a}_T < \mathfrak{c}$?

Indestructibility of mad families

Definition

Let $\mathcal{A}$ be a mad family and $\mathbb{P}$ be any forcing. Then we say $\mathcal{A}$ is $\mathbb{P}$-indestructible iff $\mathcal{A}$ stays maximal after forcing with $\mathbb{P}$.

Theorem (Brendle, Yatabe, 2005, [4])

$$\begin{tikzcd}
 & \mathbb{B}\text{-indestructible} \arrow[r] & \mathbb{S}\text{-indestructible} \\
\mathbb{L}\text{-indestructible} \arrow[r] & \mathbb{M}\text{-indestructible} \arrow[ur] & \\
 & \mathbb{C}\text{-indestructible} \arrow[u] &
\end{tikzcd}$$

Other Sacks-indestructible families

Theorem (Newelski, 1987, [16])

Consistently, there is a partition of Baire space into compact sets which is indestructible by any countably supported product of Sacks-forcing.

Question (Newelski, 1987, [16])

What about countably supported iterations of Sacks-forcing?

Theorem (Fischer, Schritesser, 2021, [11])

Assume CH. *Then there is m.e.d. family, which is indestructible by any countably supported iteration or product of Sacks-forcing of any length.*

Universally Sacks-indestructible partition

Theorem (Fischer, S., 2022, [8])

Assume CH. *Then there is a partition of Baire space into compact sets, which is indestructible by any countably supported iteration or product of Sacks-forcing of any length.*

We call such families universally Sacks-indestructible.

Corollary (Fischer, S., 2022, [8])

Assume CH *and* $\kappa^{\aleph_0} = \kappa$. *Then* $\mathbb{S}^\kappa \Vdash \mathrm{spec}(\mathfrak{a}_T) = \{\aleph_1, \mathfrak{c}\}$.

Other Sacks-indestructible families

With a bit more work the previous theorem can be split into the following two results:

Theorem (Fischer, S., 2022, [8])

Assume CH. *Then there is a* $\mathbb{S}^{\aleph_0}$*-indestructible partition of Baire space into compact sets.*

Theorem (Fischer, S., 2023, [10])

Every $\mathbb{S}^{\aleph_0}$*-indestructible partition of Baire space into compact sets is universally Sacks-indestructible.*

Arithmetical types of combinatorial families

Our aim was to generalize these results to other types of families.

Informal Definition

We call a type of combinatorial family arithmetical iff being a family of that type and its maximality is definable by some arithmetical formulas.

Example

Mad families, m.e.d. families, ultrafilter basis, partition of Baire space into compact sets, independent families, dominating/unbounded families, ... are all arithmetical types of combinatorial families.

Sacks-indestructibility of arithmetical types

Theorem (Fischer, S., 2023, [10])

Every $\mathbb{S}^{\aleph_0}$-indestructible arithmetical combinatorial family of reals is universally Sacks-indestructible.

Corollary (Fischer, S., 2023, [10])

Every $\mathbb{S}^{\aleph_0}$-indestructible mad family/m.e.d. family/ultrafilter basis/partition of Baire space into compact sets/independent family/... is universally Sacks-indestructible.

Sacks-indestructibility of arithmetical types

The key lemma for the previous theorem is the following translation from forcing statements in $\mathbb{S}^{\aleph_0}$ to Π^1_3-formulas:

Lemma (Fischer, S., 2023, [10])

Let $\chi(v_1, \dots, v_k, w_1, \dots, w_l)$ be an arithmetical formula in $k + l$ real parameters. Further, let $p \in \mathbb{S}^{\aleph_0}$, $f_1, \dots, f_l$ be reals and $g_1, \dots, g_k$ be codes for continuous functions $g_i^* : {}^\omega({}^\omega 2) \to {}^\omega\omega$. Then the following are equivalent:

1. $p \Vdash_{\mathbb{S}_{\aleph_0}} \chi(g_1^*(s_{\dot{G}}), \dots, g_k^*(s_{\dot{G}}), f_1, \dots, f_l)$,
2. $\forall q \leq p \; \exists r \leq q \; \forall x \in [r] \; \chi(g_1^*(x), \dots, g_k^*(x), f_1, \dots, f_l)$.

Elimination of intruders

Is there a uniform way to construct $\mathbb{S}^{\aleph_0}$-indestructible families?

Lemma (Folklore)

Let $p \in \mathbb{S}^{\aleph_0}$, $\mathcal{A}$ a countable a.d. family and $\dot{B}$ be a $\mathbb{S}^{\aleph_0}$-name such that

$$p \Vdash \dot{B} \cap A \text{ is finite for every } A \in \mathcal{A}.$$

Then there is $q \leq p$ and $A \in [\omega]^\omega$ such that $\mathcal{A} \cup \{A\}$ is an a.d. family and

$$q \Vdash \dot{B} \cap A \text{ is infinite.}$$

We say that elimination of intruders holds for mad families.

Theorem (Fischer, S., 2023, [10])

Assume CH *and elimination of intruders holds for an arithmetical type of combinatorial family. Then there is a universally Sacks-indestructible family of that type.*

Elimination of intruders

Hence, for the existence of a universally Sacks-indestructible family it suffices to verify elimination of intruders:

1. Mad families - Folklore
2. M.e.d. families - Fischer, Schrittesser, 2021, [11]
3. Partitions of Baire space into compact sets - Fischer, S., 2022, [8]
4. Ultrafilter bases - Fischer, S., 2023, [10]
5. Maximal cofinitary groups - Fischer, S., 2023, [10]
6. Maximal independent families - ?

Spectra of combinatorial families

A more recent trend in combinatorial set theory is the analysis of the whole spectrum $\mathrm{spec}(\mathfrak{x})$ and not just some cardinal characteristic $\mathfrak{x}$.

Guiding Question

What properties of $\mathrm{spec}(\mathfrak{x})$ *does* ZFC *prove?*

Guiding Question

Given a set of uncountable cardinals Θ*, may we realize it as* $\mathrm{spec}(\mathfrak{x})$ *with the means of forcing?*

Spectra of almost disjoint families

Theorem (Hechler, 1972, [14])

Let Θ be any subset of regular uncountable cardinals. Then there is a c.c.c. forcing extension in which $\Theta \subseteq \mathrm{spec}(\mathfrak{a})$ holds.

Theorem (Blass, 1993, [1])

Assume GCH *and let Θ be a set of uncountable cardinals such that*

1. $\aleph_1 \in \Theta$,
2. Θ *is closed under singular limits,*
3. $\max(\Theta)$ *exists and* $\mathrm{cof}(\max(\Theta)) > \omega$,
4. *If* $\theta \in \Theta$ *with* $\mathrm{cof}(\theta) = \omega$ *then* $\theta^+ \in \Theta$.

Then there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}) = \Theta$ *holds.*

Spectra of almost disjoint families

Theorem (Shelah, Spinas, 2015, [19])

Assume GCH *and let* Θ *be a set of uncountable cardinals, which satisfies*

1. $\min(\Theta)$ *is regular,*
2. Θ *is closed under singular limits,*
3. $\max(\Theta)$ *exists and* $\mathrm{cof}(\max(\Theta)) > \omega$.

Then there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}) = \Theta$ *holds.*

Theorem (Hechler, 1972, [14])

$\mathrm{spec}(\mathfrak{a})$ *is closed under singular limits.*

Spectrum of partitions of Baire space into compact sets

Theorem (Brian, 2021, [6])

Assume GCH *and let* Θ *be a set of uncountable cardinals, which satisfies*

1. $\min(\Theta)$ *is regular,*
2. Θ *is closed under singular limits,*
3. $\max(\Theta)$ *exists and* $\mathrm{cof}(\max(\Theta)) > \omega$*,*
4. *If* $\theta \in \Theta$ *with* $\mathrm{cof}(\theta) = \omega$ *then* $\theta^+ \in \Theta$*,*
5. $|\Theta| < \min(\Theta)$*.*

Then there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}_T) = \Theta$ *holds.*

Necessary extra assumptions?

Theorem (Brian, 2021, [6])

$\mathrm{spec}(\mathfrak{a}_T)$ *is closed under singular limits.*

Theorem (Brian, 2022, [5])

Assume $0^\dagger$ *does not exist,* $\mathrm{cof}(\theta) = \omega$ *and* $\theta \in \mathrm{spec}(\mathfrak{a}_T)$*. Then* $\theta^+ \in \mathrm{spec}(\mathfrak{a}_T)$*.*

In particular $\aleph_\omega \in \mathrm{spec}(\mathfrak{a}_\mathsf{T})$ but $\aleph_{\omega+1} \notin \mathrm{spec}(\mathfrak{a}_\mathsf{T})$ would imply the existence of a measurable cardinal.

Spectrum of partitions of Baire space into compact sets

Theorem (Brian, 2021, [6])

Assume GCH *and let* Θ *be a set of uncountable cardinals, which satisfies*

1. $\min(\Theta)$ *is regular,*
2. Θ *is closed under singular limits,*
3. $\max(\Theta)$ *exists and* $\mathrm{cof}(\max(\Theta)) > \omega$*,*
4. *If* $\theta \in \Theta$ *with* $\mathrm{cof}(\theta) = \omega$ *then* $\theta^+ \in \Theta$*,*
5. $|\Theta| < \min(\Theta)$*.*

Then there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}_T) = \Theta$ *holds.*

Realizing arbitrarily large spectra

Theorem (Fischer, S., 2023, [9])

Assume GCH *and let* Θ *be a set of uncountable cardinals, which satisfies*

1. ~~$\min(\Theta)$ *is regular*~~, $\aleph_1 \in \Theta$,
2. Θ *is closed under singular limits,*
3. $\max(\Theta)$ *exists and* $\mathrm{cof}(\max(\Theta)) > \omega$,
4. *If* $\theta \in \Theta$ *with* $\mathrm{cof}(\theta) = \omega$ *then* $\theta^+ \in \Theta$,
5. ~~$|\Theta| < \min(\Theta)$~~.

Then there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}_T) = \Theta$ *holds.*

Bibliography I

Andreas Blass.
Simple cardinal characteristics of the continuum.
In *Set Theory of the Reals (Ramat Gan, 1991)*, pages 63–90. Israel Math. Conf. Proc. 6, Bar-Ilan Univ., Ramat Gan, 1993.

Andreas Blass and Saharon Shelah.
Ultrafilters with small generating sets.
Israel Journal of Mathematics, 65:259–271, 1989.

Jörg Brendle.
The almost-disjointness number may have countable cofinality.
Transactions of the American Mathematical Society, 355(7):2633–2649, 2003.

Jörg Brendle and Shunsuke Yatabe.
Forcing indestructibility of MAD families.
Annals of Pure and Applied Logic, 132:271–312, 2005.

Will Brian.
The Borel partition spectrum at successors of singular cardinals.
To appear in *Proceedings of the American Mathematical Society*.

Will Brian.
Partitioning the real line into Borel sets.
To appear in *The Journal of Symbolic Logic*.

Paul J. Cohen.
The independence of the continuum hypothesis.
Proceedings of the National Academy of Sciences, 50(6):1143–1148, 1963.

Bibliography II

Vera Fischer and Lukas Schembecker.
Partitions of Baire space into compact sets.
submitted, 2022.

Vera Fischer and Lukas Schembecker.
Realizing arbitrarily large spectra of $\mathfrak{a}_T$.
submitted, 2023.

Vera Fischer and Lukas Schembecker.
Universally Sacks-indestructible combinatorial sets of reals.
submitted, 2023.

Vera Fischer and David Schrittesser.
A Sacks indestructible co-analytic maximal eventually different family.
Fundamenta Mathematicae, 252:179–201, 2021.

Kurt Gödel.
The consistency of the continuum hypothesis.
Princeton University Press, 1940.

Stephen H. Hechler.
On the existence of certain cofinal subsets of ${}^{\omega}\omega$.
Proc. Sympos. Pure Math., 13(2):155–173, 1967.

Stephen H. Hechler.
Short complete nested sequences in $\beta\mathbb{N} \setminus \mathbb{N}$ and small maximal almost-disjoint families.
General Topology and its Applications, 2(3):139–149, 1972.

Bibliography III

Arnold W. Miller.
Covering 2^ω with ω_1 disjoint closed sets.
In Jon Barwise, H. Jerome Keisler, and Kenneth Kunen, editors, *The Kleene Symposium*, volume 101 of *Studies in Logic and the Foundations of Mathematics*, pages 415–421. Elsevier, 1980.

Ludomir Newelski.
On partitions of the real line into compact sets.
The Journal of Symbolic Logic, 52(2):353–359, 1987.

Saharon Shelah.
On cardinal invariants of the continuum.
American Mathematical Society, 31:183–207, 1984.

Saharon Shelah.
Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing.
Acta Mathematica, 192(2):187–223, 2004.

Saharon Shelah and Otmar Spinas.
Mad spectra.
The Journal of Symbolic Logic, 80(3):901–916, 2015.

Otmar Spinas.
Partition numbers.
Annals of Pure and Applied Logic, 90(1):243–262, 1997.

Thank you for your attention!