Suslin's hypothesis and Aronszajn trees

John Krueger

Department of Mathematics
University of North Texas

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Outline

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1. Suslin's Hypothesis and Aronszajn Trees
   - A Brief History of the Suslin Problem
   - A Survey of Aronszajn Trees

Mikhail Suslin, Originator of the Suslin Problem

Михаи́л Я́ковлевич Су́слин

Transliteration: Mikhail Yakovlevich Suslin (Souslin)

Mikhail Suslin: Brief Biography

Mikhail Suslin (1894-1919) was a Russian mathematician. He was a student of Nikolai Luzin starting in the 1914-15 academic year and studied descriptive set theory and topology.

In his short mathematical career of around five years, his main mathematical contributions are:

1. introducing the idea of analytic sets in descriptive set theory;
2. asking a question now known as the famous *Suslin problem*, which remained open for around 50 years and eventually led to major advances in set theory.

Mikhail Suslin: Brief Biography

Suslin published a total of three short articles, only one of which appeared during his lifetime.

The Suslin problem was published as *Problem 3* in a list of ten open problems by various mathematicians published in 1920 in the very first issue of Fundamenta Mathematica ([S1920]).

Suslin passed away in 1919 as a result of typhus at the age of 24.

The Suslin Problem

3) Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel que tout ensemble de ses intervalles (contenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?

Problème de M. M. Souslin.

Translation:

Problem 3). Let a linearly ordered set without gaps or jumps have the property that every set of non-overlapping intervals (each containing at least one element) is at most countable. Will this set necessarily be the (usual) linear continuum?

The Suslin Problem

Theorem (Cantor)

Let L be a dense linear order without endpoints which is complete and separable (that is, has a countable dense subset). Then L is isomorphic to the real line $\mathbb{R}$.

Question (Suslin's Problem)

Let L be a dense linear order without endpoints which is complete and has the countable chain condition (that is, every pairwise disjoint family of non-empty open intervals is countable). Is L isomorphic to the real line $\mathbb{R}$ (or equivalently, is L separable)?

Question (Equivalent to Suslin's Problem)

Is every linear order L with the countable chain condition necessarily separable?

The Suslin Hypothesis

A *Suslin continuum* is a complete dense linear order without endpoints which is c.c.c. but not separable.

A *Suslin line* is a linear order which is c.c.c. but not separable.

A Suslin continuum exists iff a Suslin line exists.

Definition

The *Suslin Hypothesis* (SH) is the statement that there does not exist a Suslin line.

Đuro Kurepa

One of the earliest and most notable scholars to work on the Suslin problem was the Serbian mathematician Đuro Kurepa (1907-1993).

In his doctoral dissertation of 1935, written under the supervision of M. Fréchet, Kurepa gave the first ever systematic study of trees.

In his dissertation, Kurepa introduced and analyzed many fundamental ideas which are now considered the foundation of the subject.

Kurepa's Contributions

An overview of some of Kurepa's many important contributions to the theory of trees:

(1) Introduced Aronszajn, Suslin, and Kurepa trees ([K1935], [K1937], [K1943]);
(2) In unpublished work Nachman Aronszajn proved the existence of an Aronszajn tree in 1934. Kurepa produced other Aronszajn trees including the first example of a special Aronszajn tree ([K1937]);
(3) Introduced normal trees and lexicographical orderings of trees ([K1935]);
(4) Proved that there is a Suslin line iff there is a Suslin tree ([K1935]);
(5) Proved that any two infinitely branching trees of countable height are isomorphic, and posed the question of whether this is also true for Aronszajn trees ([K1935]);

Kurepa's Contributions

(6) Proved that for any tree *T* of height a regular uncountable cardinal κ such that for some $\lambda < \kappa$, the levels of *T* have size less than λ, *T* has a cofinal branch ([K1935]);

(7) Introduced special Aronszajn trees ([K1937]), and proved the existence of an Aronszajn tree which embeds into the rationals ([K1937]);

(8) Proved that any partial order is the union of countably many antichains iff it embeds into the rationals ([K1940]);

(9) Supervised the PhD dissertation of Stevo Todorčević, who went on to become one of the world's leading set theorists (1979).

Suslin Lines and Suslin Trees

The earliest major result concerning the Suslin problem is the equivalence between the existence of a Suslin line and a Suslin tree.

Theorem

There exists a Suslin line iff there exists a Suslin tree.

This theorem was proven independently by three authors:

1. Kurepa in his 1935 dissertation ([K1935]);
2. Edwin Miller in a 1943 article ([M1943] published posthumously);
3. Wacław Sierpiński in a 1948 article ([S1948]).

Miller's Theorem, Excerpt From His 1943 Article

THEOREM. *In order that there exist a linear order which possesses properties* (1), (2) *and* (4) *without possessing property* (3) *it is necessary and sufficient that there exist a partial order P of power $\aleph_1$ such that*

(a) *if $Q \subset P$ and $\bar{\bar{Q}} = \aleph_1$, then Q contains two comparable elements and two non-comparable elements;*

(b) *if x and y are non-comparable elements of P, then there exists no z in P such that $x < z$ and $y < z$.*

Miller's proof used some ideas from his earlier paper with B. Dushnik, "Partially Ordered Sets", which contains the famous theorem that for every infinite cardinal κ, $\kappa \to (\kappa, \omega)^2$ ([DM1941]).

According to a note by the publisher, Miller passed away two weeks after submitting the article in July 1942.

The Suslin Number

Definition

The *Suslin number* of a topological space is the supremum of the set of cardinalities of any family of pairwise disjoint open sets.

Note that by definition, the Suslin number of a Suslin line (in the order topology) is ω.

Here is another early theorem related to the Suslin problem due to Kurepa:

Theorem (Kurepa [K1950])

Suppose that L is a Suslin line. Then the Suslin number of $L \times L$ (in the product topology) is equal to ω_1.

The Gaifman and Specker Theorem

Kurepa asked whether any two infinitely splitting Aronszajn trees are isomorphic ([K1935]), a problem which he referred to as "premier problème miraculeux." It took almost 30 years to solve.

Theorem (Gaifman and Specker [GS1964])

There exists a family of 2^{ω_1}-many pairwise non-isomorphic infinitely splitting normal Aronszajn trees.

Cohen Invents Forcing

In 1963, Paul Cohen invented the method of forcing, which provided a powerful technique for proving independence results in set theory ([C1966]).

Previously, Kurt Gödel had shown that ZF is consistent with the axiom of choice (AC) and the continuum hypothesis (CH) by developing the idea of the constructible universe L ([G1940]).

In the other direction, Cohen used this new technique of forcing to construct models of $\mathrm{ZF} + \neg\mathrm{AC}$ and $\mathrm{ZFC} + \neg\mathrm{CH}$. In combination with Gödel's work, these models demonstrated that the axiom of choice does not follow from ZF and the continuum hypothesis does not follow from ZFC.

With the method of forcing now available, a few years later the Suslin problem was finally solved.

A Solution to the Suslin Problem

Suslin's problem was solved by showing that the existence of a Suslin tree can neither be proved nor disproved in the theory ZFC.

The consistency of the negation of Suslin's hypothesis was established independently by Thomas Jech and Stanley Tennenbaum.

Theorem (Jech [J1967], Tennenbaum [T1968])

There exists a forcing poset which forces the existence of a Suslin tree. Therefore, $\neg$SH is consistent with ZFC.

Jech's forcing adds a Suslin tree with countable conditions, and Tennenbaum's forcing adds a Suslin tree with finite conditions.

A Solution to the Suslin Problem

A non-forcing proof of the consistency of ZFC $+ \neg$SH was given by Jensen using Gödel's constructible universe L.

Theorem (Jensen [J1968])

If $\diamondsuit$ holds, then there exists a Suslin tree. In particular, if $V = L$ then $\neg$SH holds.

The more difficult direction in the independence of the Suslin hypothesis was proven later by Solovay and Tennenbaum, who built a model of ZFC $+$ SH.

Theorem (Solovay and Tennenbaum [ST1971])

There exists a forcing poset which forces Martin's axiom plus $\neg$CH, and in particular, forces that there does not exist a Suslin tree.

Invention of Iterated Forcing and Forcing Axioms

The Solovay and Tennenbaum proof of the consistency of SH involved:

1. developing the new technique of *iterated forcing* (specifically, finite support forcing iterations of c.c.c. forcings), and
2. establishing the consistency of the first *forcing axiom*, Martin's axiom (named after its originator Donald Martin).

These two developments had a transformative effect on the field of set theory. For these and other reasons, such as its impact on the theory of trees, the Suslin problem ranks among the most significant problems in the history of set theory, comparable to Cantor's continuum problem.

Suslin's Hypothesis and the Continuum Hypothesis

A natural question is whether there is any relationship between SH and CH.

The Jech and Tennenbaum models of $\neg$SH satisfy CH. Adding any number of Cohen reals preserves a Suslin tree, so $\neg\text{SH} + \neg\text{CH}$ is consistent as well.

The Solovay and Tennenbaum model of SH satisfies $\neg$CH. It took an ingenious argument of Jensen to prove the consistency of $\text{SH} + \text{CH}$.

Theorem (Jensen; Devlin and Johnsbraten [DJ1974])

Assume GCH, $\diamondsuit^*$, and $\square_{\omega_1}$. Then there exists a forcing poset which forces that CH holds and there does not exist a Suslin tree.

Jensen's Proof and Shelah's Proper Forcing

Jensen's model of SH $+$ CH did not use iterated forcing in the way we think of it nowadays, but rather involved defining a sequence of Suslin trees

$$\langle T^{\nu} : \nu < \omega_2 \rangle,$$

together with projection mappings, and forcing with the direct limit. Given T^{ν}, a Suslin tree $T^{\nu+1}$ is defined which adds a cofinal branch to T^{ν} and specializes an Aronszajn tree in $V^{T^{\nu}}$. No reals are added because forcing with Suslin trees does not add countable sets.

In the 1980's Shelah developed a more general and flexible method for iterating forcing while not adding reals, as part of his theory of proper forcing, and used his method to produce an alternative model of SH $+$ CH ([S1982]).

Outline

Review of Basic Definitions and Notation

Definition

A strict partial order $(T, <_T)$ is *tree-like* if for all $x \in T$, the set $\{y \in T : y <_T x\}$ is linearly ordered by $<_T$.

Definition

A *tree* is a strict partial order $(T, <_T)$ such that for every x, the set $\{y \in T : y <_T x\}$ is well-ordered by $<_T$.

Review of Basic Definitions and Notation

Let T be a tree. For $x \in T$, the order type of $\{y \in T : y <_T x\}$ is the *height of x in T*, denoted $\mathrm{ht}_T(x)$.

For all $\alpha < \mathrm{ht}_T(x)$, we write $x \restriction \alpha$ for the unique $y <_T x$ with height α.

For any ordinal α, $T_\alpha := \{x \in T : \mathrm{ht}_T(x) = \alpha\}$ is the *α-th level of T*. The *height of T* is the least δ such that $T_\delta = \emptyset$.

For any ordinal α, $T \restriction \alpha := \{x \in T : \mathrm{ht}_T(x) < \alpha\}$. More generally, if A is a set of ordinals, then $T \restriction A := \{x \in T : \mathrm{ht}_T(x) \in A\}$.

In these talks we will mostly be interested in trees of height ω_1.

Review of Basic Definitions and Notation

Elements x and y of T are *comparable* if $x \leq_T y$ or $y <_T x$, and otherwise are *incomparable*.

A *chain* is a subset of T consisting of comparable elements, and an *antichain* is a subset of T consisting of incomparable elements.

A *branch* of T is a maximal chain. A branch is *cofinal* if it meets every level of T.

If b is a branch of T and α is less than the order type of b, we will write $b(\alpha)$ for the unique element of b with height α.

Review of Basic Definitions and Notation

Definition

An *ω_1-tree* is a tree of height ω_1 whose levels are countable.

Definition

An *Aronszajn tree* is an ω_1-tree with no cofinal branch (equivalently, no uncountable chain).

Definition

A *Suslin tree* is a tree of height ω_1 with no uncountable chains and no uncountable antichains.

Since the levels of a tree are antichains, every Suslin tree has countable levels and hence is an Aronszajn tree.

Review of Basic Definitions and Notation

Definition

A tree T is *normal* if it satisfies the following properties:

1. T has a *root*, which is the unique element with height 0;
2. every element of T not at the maximal level of the tree (if it exists), has at least two immediate successors;
3. (*unique limits*) if x and y have height δ, where δ is a limit ordinal, then there exists some $\alpha < \delta$ such that $x \restriction \alpha \neq y \restriction \alpha$;
4. if x is in T then there exists an element above x at any higher level of T.

Different authors use somewhat different definitions of normal, but in all variations (4) is always required.

Review of Basic Definitions and Notation

If T is a normal ω_1-tree, then T is Suslin iff T has no uncountable antichains.

For any cardinal λ, a tree T is *λ-ary* if every element of T has exactly λ-many immediate successors.

We are mostly interested in normal ω_1-trees which are either *binary*, which means 2-ary, or *infinitely splitting*, which means ω-ary.

Review of Basic Definitions and Notation

If T is a tree, a *subtree* of T is any subset of T considered as a tree equipped with the ordering inherited from T.

A subset $X \subseteq T$ is *downwards closed* if whenever $x \in X$ and $y <_T x$, then $y \in X$.

If X is a downwards closed subtree of T then the height functions ht_T and ht_X agree on X.

The *downward closure* of a set $X \subseteq T$ is the set

$$\{y \in T : \exists x \in X \ y \leq_T x\}.$$

If $X \subseteq T$ and X is a Suslin tree, then the downward closure of X is also a Suslin tree.

Nowhere Suslin Trees

Definition

A tree T of height ω_1 is *nowhere Suslin* if every uncountable subset of T contains an uncountable antichain.

Lemma

If T is an Aronszajn tree, then T is nowhere Suslin iff T has no Suslin subtree.

So for an Aronszajn tree T, T being *not nowhere Suslin* means that T contains a Suslin subtree, not that T itself is Suslin.

Special Trees

Definition

A tree T is *special* if T is the union of countably many antichains.

Being special is equivalent to the existence of a *specializing function*, which is a function $f : T \to \omega$ such that $x <_T y$ implies $f(x) \neq f(y)$.

Note that if T is a special tree, then T does not have an uncountable chain and T has an uncountable antichain. Any subtree of a special tree is also special. Hence, any special tree is nowhere Suslin.

Theorem (Kurepa [K1937])

There exists a special Aronszajn tree.

Trees Embeddable Into Linear Orders

Definition

A map $h : P \to Q$ between partial orders is *strictly increasing* if $x <_P y$ implies $h(x) <_Q h(y)$.

Definition

For a linear order L, a tree T is *L-embeddable* if there exists a strictly increasing map $h : T \to L$.

Theorem (Kurepa [K1940])

A tree is special iff it is $\mathbb{Q}$-embeddable.

Theorem

There exists an ω_1-tree T for which there exists a continuous strictly increasing map $h : T \to \mathbb{Q}$.

Baumgartner's Dissertation of 1970

The study of embeddings of trees into linear orders was initiated by Kurepa.

This topic was explored further in the doctoral dissertation of James Baumgartner in 1970.

We survey some of the results of Baumgartner's dissertation as well as some unpublished results of Fred Galvin and Richard Laver which appear there.

Baumgartner's Dissertation

Theorem (Galvin (unpublished); Baumgartner [B1970])

Let T be a tree of height ω_1. Then T is $\mathbb{R}$-embeddable iff $T \restriction \{\alpha + 1 : \alpha < \omega_1\}$ is special.

Proposition (Baumgartner [B1970])

For a tree T of height ω_1, if $T \restriction \{\alpha + 1 : \alpha < \omega_1\}$ is nowhere Suslin, then T is nowhere Suslin.

If T is $\mathbb{R}$-embeddable, then $T \restriction \{\alpha + 1 : \alpha < \omega_1\}$ is special and hence nowhere Suslin. So by the above proposition, T is nowhere Suslin. Consequently:

Proposition

If T is a tree of height ω_1 which is $\mathbb{R}$-embeddable, then T has no uncountable chains and is nowhere Suslin.

The Tree T^*

Note that for any set X,

$$^{<\omega_1}X := \{f : \exists\alpha < \omega_1\ f : \alpha \to X\},$$

ordered by strict subset, is a tree of height ω_1.

Definition

Let T^* be the subtree of $^{<\omega_1}\omega$ consisting of injective functions.

Note that T^* has no uncountable chains.

The Tree T^*

Theorem (Laver (unpublished); Baumgartner [B1970])

The tree T^* is not special.

Theorem (Baumgartner [B1970])

A tree T is $\mathbb{R}$-embeddable iff there exists a strictly increasing map of T into T^*.

In particular, T^* itself is $\mathbb{R}$-embeddable.

The Tree T^*

So T^* is an example of a tree of height ω_1 which is $\mathbb{R}$-embeddable but not special. Consequently:

Theorem

The statement that every tree with no uncountable chains is special is disprovable in ZFC.

Of course T^* is not an ω_1-tree because it has levels of size 2^ω.

Characterizations of Being $\mathbb{R}$-Embeddable

As mentioned previously, Baumgartner proved that a tree T is $\mathbb{R}$-embeddable iff there exists a strictly increasing map of T into T^*.

A variation of this result was proved by Honzík and Stejskalová.

Theorem (Honzík and Stejskalová [HS2015])

A normal ω_1-tree T is $\mathbb{R}$-embeddable iff it is isomorphic to a subset of T^* with the induced order.

The difference between this theorem and that of Baumgartner is that it gets a stronger conclusion (namely, isomorphic instead of mapping strictly increasing into) from a stronger assumption. However, the stronger assumption is so commonly made when working with trees, that overall it is the more relevant and substantial result.

The Shift Operator $\mathcal{S}$ on Trees

Definition

Let T be a tree of height ω_1. The *shift of T*, denoted by $\mathcal{S}(T)$, is the unique smallest tree satisfying that

$$\mathcal{S}(T) \restriction \{\alpha + 1 : \alpha < \omega_1\} = T.$$

In other words, we shift every element of T up by one level, and add unique limits at limit levels to chains which already had upper bounds in T.

Note that $\mathcal{S}(T)$ also has height ω_1, and if T is an ω_1-tree then so is $\mathcal{S}(T)$.

The Tree $\mathcal{S}(T^*)$

Let us apply the shift operator to the tree T^*.

Consider $\mathcal{S}(T^*)$. Then

$$\mathcal{S}(T^*) \restriction \{\alpha + 1 : \alpha < \omega_1\} = T^*,$$

which:

(a) is not special, and
(b) is nowhere Suslin.

By (a), $\mathcal{S}(T^*)$ is not $\mathbb{R}$-embeddable. By (b), $\mathcal{S}(T^*)$ is nowhere Suslin.

The Trees T^* and $\mathcal{S}(T^*)$

The trees T^* and $\mathcal{S}(T^*)$ are witnesses to the following theorems.

Theorem (Baumgartner [B1970])

There exists a tree of height ω_1 which is $\mathbb{R}$-embeddable and not $\mathbb{Q}$-embeddable.

Theorem (Baumgartner [B1970])

There exists a tree of height ω_1 which is nowhere Suslin but not $\mathbb{R}$-embeddable.

These theorems prove that the implications

$$\text{special} \implies \mathbb{R}\text{-embeddable} \implies \text{nowhere Suslin}$$

cannot be reversed.

Results of Baumgartner's Dissertation

Theorem (Baumgartner [B1970])

The following statements are equivalent:

1. *Every Aronszajn tree is special;*
2. *Every Aronszajn tree is $\mathbb{R}$-embeddable.*

Proof.

(1) $\Rightarrow$ (2): Immediate because being special is equivalent to being $\mathbb{Q}$-embeddable.

(2) $\Rightarrow$ (1): Suppose that T is an Aronszajn tree which does not embed into $\mathbb{Q}$. Then $\mathcal{S}(T)$ is an Aronszajn tree satisfying that $\mathcal{S}(T) \restriction \{\alpha + 1 : \alpha < \omega_1\} = T$ is not special. So $\mathcal{S}(T)$ is not $\mathbb{R}$-embeddable. $\square$

Results of Baumgartner's Dissertation

Theorem (Baumgartner [B1970])

$MA + \neg CH$ implies that every tree with no uncountable chains and size less than 2^ω has a strictly increasing and continuous map into $\mathbb{Q}$, and in particular, that all Aronszajn trees are special.

Theorem (Baumgartner, Malitz, and Reinhardt [BMR1970])

$MA + \neg CH$ implies that every tree-like partial order of size less than 2^ω with no uncountable chains embeds into $\mathbb{Q}$.

As we described above, T^* is a tree of size 2^ω with no uncountable chains which is not special.

Results of Baumgartner's Dissertation

As we have discussed, in ZFC we can prove that the properties of being special, $\mathbb{R}$-embeddable, and nowhere Suslin are distinct for trees of height ω_1. For ω_1-trees, the best we can get is a consistency result.

Theorem (Baumgartner [B1970])

Assume that $V = L[A]$ for some set $A \subseteq \omega_1$. Then there exists an $\mathbb{R}$-embeddable Aronszajn tree which is not special.

Applying the shift operator to a tree as in the above theorem, it follows that under the same assumption there exists a nowhere Suslin Aronszajn tree which is not $\mathbb{R}$-embeddable. In addition, these facts remain true after forcing arbitrarily many Cohen reals ([B1970]).

This concludes the discussion of Baumgartner's dissertation.

The Diamond Principle

The constructibility assumption in the above theorem was soon replaced by the diamond principle and its variations. Starting with Jensen, it was recognized that a great variety of Aronszajn and Suslin trees can be constructed with diamond.

Theorem (Devlin [D1972])

Assume $\diamondsuit$. Then there exist 2^{ω_1} many pairwise non-isomorphic Aronszajn trees which are $\mathbb{R}$-embeddable but not special.

Theorem (Kunen, K., Larson, J., and Steprāns, J. [KLS2012])

Assume $\diamondsuit$. Then for any set $A \subseteq \mathbb{R}$ which contains no perfect subset, there exists a special Aronszajn tree which has no continuous strictly increasing map into A. In particular, this statement holds for $A = \mathbb{Q}$.

Stationary Antichains and Club Antichains

Definition

Let T be an ω_1-tree. An antichain $A \subseteq T$ is a *stationary antichain* if the set

$$\{\mathrm{ht}_T(x) : x \in A\}$$

is a stationary subset of ω_1.

An antichain $A \subseteq T$ is a *club antichain* if the set

$$\{\mathrm{ht}_T(x) : x \in A\}$$

is a club subset of ω_1.

Special Trees Have Stationary Antichains

Lemma

Let T be a special Aronszajn tree. Then T has a stationary antichain.

Proof.

Let $f : T \to \omega$ be a specializing function. For each $\alpha < \omega_1$ choose some $x_\alpha \in T_\alpha$.

By the pressing down lemma, there is a stationary set $X \subseteq \omega_1$ on which the map $\alpha \mapsto f(x_\alpha)$ is constant.

Then

$$\{x_\alpha : \alpha \in X\}$$

is a stationary antichain. □

Results on Stationary Antichains

Theorem (Shelah [S1982])

Assuming $\diamondsuit$, there exists a special Aronszajn tree with no club antichain.

In comparison, MA_{ω_1} implies that every Aronszajn tree is special and has a club antichain.

Theorem (Shelah [S1982])

Assuming $\diamondsuit$, there exists an Aronszajn tree which is $\mathbb{R}$-embeddable, not special, and contains no club antichain.

Theorem (Shelah [S1982])

Assuming $\diamondsuit^*$, there exists an Aronszajn tree which is $\mathbb{R}$-embeddable and contains no stationary antichain.

Almost Suslin Trees

Definition

An ω_1-tree with no stationary antichain is called an *almost Suslin tree*.

Any special Aronszajn tree is not an almost Suslin tree. So MA_{ω_1} implies that there does not exist an almost Suslin tree.

Every Suslin tree is an almost Suslin tree, but the converse is provably false.

Theorem (Devlin and Shelah [DS1979])

If there exists a Suslin tree, then there exists an almost Suslin tree which is not a Suslin tree.

Almost Suslin trees do not have to be Aronszajn trees; see for example Todorčević [T1984, Section 4].

Generic Reals and Suslin Trees

Theorem (Shelah [S1984])

Cohen forcing $Add(\omega)$ forces that there exists a Suslin tree.

Theorem (Todorčević [T2007, page 39])

Cohen forcing $Add(\omega)$ forces that there exists an $\mathbb{R}$-embeddable Aronszajn tree with no stationary antichain.

Generic Reals and Suslin Trees

Laver proved that the same is not true for random reals.

Theorem (Laver [L1987])

Assuming MA_{ω_1}, forcing any number of random reals with the product measure will force that all Aronszajn trees are special.

So SH is consistent with an arbitrarily large continuum of any uncountable cofinality. In particular, SH is consistent with 2^ω being singular.

Special Subtrees

Theorem

Assuming $\diamondsuit^*$, for every stationary and costationary set $S \subseteq \omega_1$, there exists a non-special Aronszajn tree T such that $T \restriction S$ is special and $T \restriction (\omega_1 \setminus S)$ has no stationary antichain.

On the other hand, if T restricted to a club is special, then so is T.

Proposition

Suppose that T is an ω_1-tree, $C \subseteq \omega_1$ is a club, and $T \restriction C$ is special. Then T is special.

Special Subtrees

Proof.

Fix a specializing function $g : T \restriction C \to \omega$.

For each $x \in T \restriction C$, let $\beta_x = \min(C \setminus (\mathrm{ht}_T(x) + 1)$, and fix a bijection

$$h_x : \{y \in T \restriction [\mathrm{ht}_T(x), \beta_x) : x \leq_T y\} \to \omega.$$

Define $f : T \to \omega \times \omega$ as follows. Given $y \in T$, let α_y be the largest element of $C \cap (\mathrm{ht}_T(y) + 1)$, which exists because C is club. Define

$$f(y) := (g(y \restriction \alpha_y), g_{y \restriction \alpha}(y)).$$

Then $y <_T z$ implies $f(y) \neq f(z)$ as is easy to check. □

S-st-Special

Definition (Shelah [S1982])

Let $S \subseteq \omega_1$ be a stationary set of limit ordinals. An ω_1-tree T is *S-st-special* if there exists a function $f : T \restriction S \to \omega_1$ such that:

1. for all $x \in T \restriction S$, $f(x) < \mathrm{ht}_T(x)$;
2. for all $x <_T y$ in $T \restriction S$, $f(x) \neq f(y)$.

If T is $(\omega_1 \cap \mathrm{Lim})$-st-special, then T is special. But if $S \subseteq \omega_1$ is stationary and co-stationary, then T being S-st-special does not imply that $T \restriction S$ is special.

Lemma

If T is S-st-special then T is Aronszajn and has a stationary antichain (and hence is not Suslin).

Back to Suslin's Hypothesis

A natural question is whether SH is equivalent to the statement that all Aronszajn trees are special. All of the early models of SH satisfy that all Aronszajn trees are special (namely, any model of MA_{ω_1} and Jensen's model of CH $+ \neg$SH).

Theorem (Shelah [S1982])

SH does not imply that every Aronszajn is special. Namely, it is consistent that there exists a stationary and costationary set $S \subseteq \omega_1$ such that:

1. *every Aronszajn tree is S-st-special (and hence not Suslin);*
2. *there exists an Aronszajn tree T such that $T \restriction (\omega_1 \setminus S)$ has no stationary antichain (and hence T is not special).*

Back to Suslin's Hypothesis

Shelah's proof used a complicated kind of forcing iteration called an "ω_1-free iteration," which is variation of a countable support forcing iteration. Both Shelah's theorem and the technique he used to prove it were improved by Chaz Schlindwein.

Theorem (Schlindwein [S1993])

Let T be an Aronszajn tree with no stationary antichain. There is a property of a forcing poset called T-proper which implies that the forcing is proper and does not add a stationary antichain to T, and moreover, being T-proper is preserved by any countable support forcing iteration.

Back to Suslin's Hypothesis

Theorem (Schlindwein [S1993])

It is consistent that SH holds and there exists a non-special Aronszajn tree with no stationary antichain.

Other applications of Schlindwein's forcing preservation theorem are given in K. "A forcing axiom for a non-special Aronszajn tree" ([K2020]).

Club Isomorphisms

Recall the result of Gaifman and Specker [GS1964] that there exist 2^{ω_1}-many pairwise non-isomorphic normal Aronszajn trees.

In 1985 Abraham and Shelah introduced the following weakening of the isomorphism relation on trees.

Definition (Abraham and Shelah [AS1985])

Let T and U be trees of height ω_1. Then T and U are *club isomorphic* if there exists a club $C \subseteq \omega_1$ and an isomorphism $f : T \upharpoonright C \to U \upharpoonright C$. In that case, f is called a *club isomorphism*.

The pairwise non-isomorphic Aronszajn trees given in Gaifman and Specker's article are all club isomorphic.

Club Isomorphisms

Lemma

Let T and U be normal ω_1-trees. Then T and U are club isomorphic iff there exists an unbounded set $X \subseteq \omega_1$ such that $T \restriction X$ and $U \restriction X$ are isomorphic.

Sketch.

Let $f : T \restriction X \to U \restriction X$ be an isomorphism. Let $C := X \cup \lim(X)$. Define $f^+ : T \restriction C \to U \restriction C$ extending f as follows.

Let $x \in T \restriction \lim(X)$ have height δ. By the normality of T pick some $z \geq_T x$ in $T \restriction X$. Define $f^+(x) := f(z) \restriction \delta$. Use the normality of T and U to show that this works. □

An Essentially Unique Aronszajn Tree

Abraham and Shelah [AS1985] introduced the hypothesis that there exists an *essentially unique Aronszajn tree*:

Any two normal Aronszajn trees are club isomorphic.

This hypothesis implies that all Aronszajn trees are special. Namely, there exists a special normal Aronszajn tree, and any other normal Aronszajn tree is club isomorphic to it. But if a tree is special on a club of levels, then it is special.

In particular, the hypothesis of an essentially unique Aronszajn tree is a logical strengthening of SH.

An Essentially Unique Aronszajn Tree

Theorem (Abraham and Shelah [AS1985])

Let T and U be normal Aronszajn trees. Then there exists a forcing poset of size ω_1 which is proper and adds a club isomorphism between T and U.

Corollary

The proper forcing axiom implies that any two normal Aronszajn trees are club isomorphic.

In fact, by a forcing iteration theorem of Shelah, any countable support forcing iteration of proper forcings of size ω_1 with length ω_2 is proper and ω_2-c.c. So a model of an essentially unique Aronszajn tree can be obtained without large cardinals.

Club Isomorphisms and Martin's Axiom

Theorem (Abraham and Shelah [AS1985])

MA_{ω_1} does not imply that any two normal Aronszajn trees are club isomorphic.

In particular, the hypothesis that all Aronszajn trees are special does not imply that there is an essentially unique Aronszajn tree.

Theorem (Abraham and Shelah [AS1985])

It is consistent to have Martin's axiom, any two normal Aronszajn trees are club isomorphic, and 2^ω arbitrarily large.

Families of Non-Club-Isomorphic Aronszajn Trees

Recall that neither SH nor the hypothesis that every Aronszajn tree is special have any impact on the value of 2^ω. In contrast:

Theorem (Abraham and Shelah [AS1985])

Suppose the weak diamond principle holds (equivalently, $2^\omega < 2^{\omega_1}$). Then there exists a family of 2^{ω_1} many normal pairwise non-club-isomorphic special Aronszajn trees.

Theorem (Todorčević [T1984])

Suppose that there exists an Aronszajn tree with no stationary antichain. Then there exists a family of 2^{ω_1} many normal pairwise non-club-isomorphic Aronszajn trees.

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