

Lonely points in ω^*

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Abstract

It is long known that ω^* is not homogeneous. Results in this direction go back to W. Rudin, Z. Frolík and others. K. Kunen was the first to give a ZFC *topological* reason for this by finding a so called topological type in ω^* — the class of weak P-points. Finally J. van Mill has given a *topological* description of 16 distinct types in ω^* . Inspired by two of his types, we define a new type — lonely points. A point is lonely if there is essentially only a single countable set “converging” to it and if this set has no isolated points. In this paper we look at the question whether these points exist in ω^* without assumptions beyond ZFC. It is shown, that the question is equivalent to finding a countable OHI, extremally disconnected, zerodimensional space with a remote weak P-point. We also show that the question is relatively hard and answer a somewhat easier question by finding a large subspace of ω^* which does have lonely points and showing that these points are weakly lonely in ω^* .

Keywords: $\beta\omega$, irreducibility, weak P-point, remote point, topological type

1 Introduction and definitions

The motivation for this paper comes from homogeneity questions in topology.

Definition 1.1. *A topological space X is homogeneous if for any two points there is a homeomorphism of the space mapping one to the other. A subset $T \subset X$ of the space is a topological type if it is invariant under homeomorphisms, that is any homeomorphism maps points in T to points in T .*

It is immediately seen that if a space contains two distinct nonempty topological types, then it cannot be homogeneous. This observation is generally used to prove nonhomogeneity of topological spaces. Thus Rudin proved ([Rud56]) that, assuming MA, ω^* is not homogeneous by proving that P-points (which form a type) exist. A decade later Z. Frolík proved in ZFC ([Fro67]) that ω^* contains $2^{\mathfrak{c}}$ many distinct topological types which gave a final answer to the homogeneity question for ω^* . His proof was, however, slightly unsatisfactory because it was of combinatorial nature and did not give a “topological” description of even a single type. This led to the question whether one can prove the existence of some “topologically” defined type in ω^* . It took another decade for the first answer to appear. In 1978 Kunen proved ([Kun78]) the existence of weak P-points in ω^* :

Definition 1.2. *A point $p \in X$ is a weak P-point if it is not a limit point of a countable subset of X . A subset $P \subseteq X$ is a weak P-set if it is disjoint from the closure of any countable set disjoint from it.*

Four years later, van Mill in his celebrated paper ([vM82]) gave a “topological” description of 16 topological types and proved (in ZFC) that they exist in ω^* . We give a definition of two points closely related to van Mill’s work which are relevant to this paper:

Definition 1.3. *A point $p \in X$ is*

- (1) *Countably discretely untouchable if it is not a limit point of a countable discrete set.*
- (2) *Uniquely accessible (UA) if whenever it is a limit point of two countable sets it is a limit point of their intersection. It is weakly uniquely accessible (wUA) iff whenever it is a limit point of two countable sets, it is a limit point of the intersection of their closures.*

A natural question to ask is whether we can have both properties at the same time. This leads to the following definition and questions:

Definition 1.4. *A point $p \in X$ is lonely iff it satisfies:*

- (i) *it is not a limit point of a countable discrete set,*
- (ii) *it is a limit point of a countable crowded set and*
- (iii) *it is uniquely accessible*

If the point satisfies (i-ii) and is weakly uniquely accessible we call it weakly lonely.

Question 1.1. *Does ω^* contain weakly lonely points?*

Question 1.2. *Does ω^* contain lonely points?*

Question 1.3. *Is there at least a “large” subspace of ω^* which contains lonely points?*

The paper is organized as follows: In the second section we state some embedding theorems, the third is devoted to the characterization of lonely points, fourth introduces some facts about OHI spaces and the last section contains an answer to the first and third question and some comments on the potential solution to the second question. The first four sections contain preparatory material for the proof of the main theorem which is contained in the last section. I would like to thank my advisor P. Simon for reading the earlier versions of this paper and for his guidance while working on the problem.

We shall denote by X^* the Čech-Stone remainder of X , i.e. $\beta X \setminus X$, in particular ω^* is the space of free ultrafilters on ω with the Stone topology. All the spaces considered are, unless otherwise stated, assumed to be (at least) $T_{3\frac{1}{2}}$ and crowded (that is without isolated points).

2 Finding points in ω^*

Working in ω^* is sometimes difficult and it can be easier to work outside and then embed the resulting situation into ω^* . This was a technique van Mill used in his [vM82]. We use a theorem of Simon:

Definition 2.1. *A space is extremally disconnected if the closure of an open subset of the space is open.*

Theorem 2.1 ([Sim85]). *Every extremally disconnected compact space of weight $\leq \mathfrak{c}$ can be embedded into ω^* as a closed weak P -set.*

An easy observation shows that this theorem is adequate for our needs, since the embedding preserves lonely points:

Observation 2.2. *If p is a (weakly) lonely point in Y and if Y is a weak P -set in X then p is a (weakly) lonely point in X .*

Thus the plan is to build a countable extremally disconnected space X such that βX contains a lonely point. Since X is extremally disconnected iff βX is we can then use theorem 2.1 to transfer the point into ω^* .

3 Characterization of lonely points

In this section we take a closer look at the properties of lonely points. We first introduce some definitions connected to these properties and list some standard theorems for later reference. Then we prove a characterization theorem for the existence of lonely points.

Suppose $p \in \omega^*$ is lonely and S is a countable crowded set whose limit point p is.

Observation 3.1. *The following is immediate:*

- (a) p is not a limit point of a nowhere dense subset of S .
- (b) p is not a limit point of a countable subset of $\overline{S} \setminus S$. (Points with this property are called weak P -points).
- (c) Whenever D, H are two dense subsets of S then their intersection is nonempty.

We shall see that these properties in fact characterize lonely points in ω^* . We now introduce the notions hinted at by the previous observation and some facts about them.

3.1 Remote points

Definition 3.1 ([FG62]). *A point $p \in X^*$ is a remote point iff it is not a limit point (in βX) of a nowhere dense subset of X . A closed filter \mathcal{F} on X is remote if for any nowhere dense subset $N \subseteq X$ there is an $F \in \mathcal{F}$ which is disjoint from N .*

Remote points were investigated by several people (Chae and Smith, van Douwen and others). Here we mention a theorem of van Douwen:

Theorem 3.2 ([vD81],5.2). *βX is extremally disconnected at each remote point of X . As a corollary a remote point cannot be in the closure of two disjoint open subsets of X .*

Theorem 3.3. *Each remote point of an extremally disconnected space is weakly lonely.*

Proof. This easily follows from 3.2 and the fact that countable subsets of extremally disconnected spaces are C^* -embedded in them (see e.g. [[TopEnc],g-1]). □

3.2 Irresolvable spaces

Definition 3.2 ([Hew43]). *A crowded topological space is irresolvable if it does not contain disjoint dense subsets, otherwise it is resolvable. It is open hereditarily irresolvable (OHI for short) if every open subspace is irresolvable.*

Lemma 3.4. *The union of resolvable spaces is resolvable.*

Proof. Let $\{X_\alpha : \alpha < \kappa\}$ be an enumeration of the resolvable topological spaces and for $\alpha < \kappa$ let D_α^i , $i = 0, 1$ be disjoint sets dense in X_α . Define $D_0^i = D_0^i$ and for $i = 0, 1$ let

$$D_\alpha^i = D_\alpha^i \setminus \overline{\bigcup_{\beta < \alpha} D_\beta^i}$$

Let $D^i = \bigcup_{\alpha < \kappa} D_\alpha^i$. Since the D_α^i 's are dense in X_α necessarily

$$\overline{\bigcup_{\beta < \alpha} D_\beta^0} = \overline{\bigcup_{\beta < \alpha} D_\beta^1}$$

for every $\alpha < \kappa$ and we can conclude that D^0 is disjoint from D^1 . Both are dense in every X_α so also in their union, so their union is resolvable. \square

Corollary 3.5. *Any irresolvable, not OHI space contains a maximal (w.r.t. inclusion) resolvable subspace. Its complement is open hereditarily irresolvable.*

Proof. Suppose X is non OHI, there is an subset A of X which is resolvable. By the previous lemma the union R of all resolvable subspaces of X containing A is a resolvable proper (since X is irresolvable) subspace of X . Notice that R must be closed because the closure of a resolvable space is resolvable. Suppose $B \subseteq X \setminus R$ is resolvable. Then $R \cup B$ is resolvable, a contradiction with the definition of R . \square

3.3 Characterization theorems

We shall now make precise the statement from the beginning of this section that certain properties in fact characterize lonely points in ω^* :

Theorem 3.6. *If X is a countable OHI space with $p \in X^*$ a remote point which is a weak P-point of X^* , then p is a lonely point in βX .*

Proof. We first prove that the point is uniquely accessible. Suppose $D_0, D_1 \subseteq \beta X$ are two countable sets with $p \in \overline{D_0} \cap \overline{D_1}$. Then, because p is a weak P-point of X^* $p \in \overline{D_0} \cap \overline{X} \cap \overline{D_1} \cap \overline{X}$. Because p is remote, $p \in \text{int} \overline{D_0} \cap \overline{X} \cap \text{int} \overline{D_1} \cap \overline{X}$. Again, because p is remote it cannot be in the closure of two disjoint open sets (3.2), so $\text{int} \overline{D_0} \cap \overline{X} \cap \text{int} \overline{D_1} \cap \overline{X} = G \neq \emptyset$, but now, since X is OHI and D_0, D_1 are both dense in G , we have that $D_0 \cap D_1 \neq \emptyset$.

An OHI space is crowded so condition (ii) is also satisfied and condition (i) follows from the fact that discrete subsets of OHI spaces are nowhere dense (see [[vD93], 1.13]). \square

On the other hand:

Theorem 3.7. *If $p \in \omega^*$ is a lonely point then there is a countable, extremally disconnected OHI space X with a remote point which is a weak P-point of X^* .*

Proof. Let $X \subseteq \omega^* \setminus \{p\}$ be a countable set with $p \in \overline{X}$. Since p is a lonely point X must be irresolvable. By 3.5 we may assume X is OHI (since p cannot be in the closure of any resolvable subspace of X). Since X is a countable subset of ω^* it is extremally disconnected. By ([HSTT], 1.5.2) it is C^* -embedded in ω^* so $\overline{X} \approx \beta X$. Because p is a lonely point, it must be a weak P-point of $\overline{X} \setminus X \approx X^*$ and a remote point of X . \square

Theorems 3.6, 3.7 and 2.1 together give the following characterization:

Theorem 3.8. *There is a lonely point in ω^* iff there is a countable extremally disconnected OHI space X with a remote point p which is a weak P -point of X^* .*

From the proofs we see that the following is also true:

Theorem 3.9. *If there is a countable extremally disconnected OHI space X with a remote point then there is a countable set $S \subseteq \omega^*$ and $p \in \overline{S}$ such that $(\omega^* \setminus (\overline{S} \setminus S)) \cup \{p\}$ contains a lonely point.*

Theorem 3.9 will be used in the last section to answer the third question.

4 Constructing OHI spaces

In view of the characterization theorem we are interested in extremally disconnected OHI spaces. Here we present a tool for constructing such spaces based on an idea of Hewitt ([Hew43]).

Definition 4.1. *If P is a property of a space we say that (X, τ) is maximal P if it has P and for any topology σ finer than τ the space (X, σ) does not have P .*

Since a space is extremally disconnected iff the algebra of its clopen sets is the following proposition from boolean algebras will be relevant:

Theorem 4.1. *If \mathbb{B} is a maximal atomless subalgebra of a complete Boolean algebra \mathbb{C} , then \mathbb{B} is complete.*

Proof. (due to Balcar) Let \mathbb{D} be the completion of \mathbb{B} . Since \mathbb{B} is atomless so is \mathbb{D} . Now by the universal property of complete boolean algebras we can extend the (inclusion) embedding $\mathbb{B} \hookrightarrow \mathbb{C}$ to an embedding $\mathbb{D} \hookrightarrow \mathbb{C}$. But since \mathbb{B} was a maximal atomless subalgebra of \mathbb{C} and since \mathbb{D} is atomless we conclude that $\mathbb{D} = \mathbb{B}$ so \mathbb{B} is complete. \square

We will need the topological reformulation of the previous theorem:

Theorem 4.2. *Every maximal crowded and zerodimensional space is extremally disconnected.*

Proof. Note that a zerodimensional space is crowded iff the algebra of its clopen sets is atomless. It is extremally disconnected iff this algebra is complete (see e.g. [[TopEnc],g-1]). Now it is easy to see that a maximal crowded and zerodimensional topology on X corresponds to a maximal atomless subalgebra of the powerset of X . By 4.1 this algebra is complete so the space is extremally disconnected. \square

Coupled with a theorem of Hewitt:

Theorem 4.3 ([Hew43]). *Any maximal crowded and zerodimensional topology is OHI.*

we get:

Theorem 4.4. *Any zerodimensional crowded topology can be refined to an OHI extremally disconnected, zerodimensional crowded topology.*

5 The Main Theorem

To prove the existence of lonely points in ω^* we would need an extremally disconnected space with a remote point which is a weak P-point. Since the weak P-point property is hard to achieve we want at least remoteness to be able to use 3.9. The theorem 4.4 from the previous section suggests that we build a space with a remote closed filter and then refine the topology to make the space extremally disconnected OHI. Unfortunately when refining the topology new n.w.d. sets could appear and kill the remoteness of the filter. We need a stronger version of remoteness:

Definition 5.1. *A closed filter \mathcal{F} on X is strongly remote if for any set $A \subseteq X$ with empty interior there is an $F \in \mathcal{F}$ which is disjoint from A .*

It is easy to see that a strongly remote filter is a remote filter and also that a strongly remote filter is strongly remote in any finer topology. This is the key property, since if we build a strongly remote filter we can then use the theorem 4.4 without losing remoteness. This section will be devoted to a single theorem which will give us a strongly remote filter on a suitable space:

Theorem 5.1. *There is a crowded, T_2 , zerodimensional topology τ on ω and a strongly remote filter on ω with this topology.*

Proof. The proof of the theorem will come in several steps. First, we state a standard definition and lemma from boolean algebras.

Definition 5.2. *A boolean algebra \mathbb{B} has hereditary independence κ iff any for any $b \in \mathbb{B}$ and any subset $B \in [\mathbb{B} \upharpoonright b]^{<\kappa}$ there is an element $c \in \mathbb{B} \upharpoonright b$ which as well as its complement intersects any elementary meet over B . $\mathbb{B} \upharpoonright b$ is the factoralgebra $\{c \wedge b : c \in \mathbb{B}\}$. Furthermore if I is an ideal on an algebra, let $I^* = \{-b : b \in I\}$.*

Lemma 5.2. *There is an ideal I on ω , extending FIN and such that $\mathcal{P}(\omega)/I$ has hereditary independence \mathfrak{c} .*

Proof. The complete Boolean algebra $\mathbb{B} = \text{Compl}(\text{Clopen}(2^{\mathfrak{c}}))$ has hereditary independence \mathfrak{c} and is σ -centered so there is an ideal I on ω such that \mathbb{B} is isomorphic to $\mathcal{P}(\omega)/I$. \square

By the previous lemma fix an ideal $I \supseteq \text{FIN}$ on ω such that $\mathcal{P}(\omega)/I$ has hereditary independence \mathfrak{c} . Throughout this proof we will adopt the following notation:

Notation 5.1. *If I is an ideal on ω and \mathcal{CO} is a system of subsets of ω let $\tau_I(\mathcal{CO})$ denote the topology generated by $\{U, \omega \setminus U : U \in \mathcal{CO}\} \cup I^*$.*

Now let $\langle A_\alpha : \omega \leq \alpha < \mathfrak{c}, \alpha \text{ even} \rangle$ be an enumeration of $\mathcal{P}(\omega)$, $\langle (G_\alpha, n_\alpha) : \omega \leq \alpha < \mathfrak{c}, \alpha \text{ odd} \rangle$ an enumeration of $\{(G, n) : G \in I^*, n \in G\}$ and $\langle K_n : n < \omega \rangle$ an enumeration of $[\omega]^2$. Let $F_0 = \mathcal{F}_0 = \mathcal{CO}_0 = \emptyset$.

Proceed by induction constructing F_α (a closed filterbase), \mathcal{F}_α (a closed filter), \mathcal{CO}_α (clopen sets), τ_α (topology) for $\alpha < \mathfrak{c}$ such that the following is satisfied:

- (i) $|\mathcal{CO}_\alpha| \cdot |F_\alpha| \leq \alpha \cdot \omega$ for each $\alpha < \mathfrak{c}$.
- (ii) If $\alpha < \mathfrak{c}$ is limit, then $F_\alpha = \bigcup_{\beta < \alpha} F_\beta$, $\mathcal{CO}_\alpha = \bigcup_{\beta < \alpha} \mathcal{CO}_\beta$.
- (iii) $\tau_\alpha = \tau_I(\mathcal{CO}_\alpha)$, \mathcal{F}_α is the filter generated by F_α , $F_\alpha \subseteq \mathcal{CO}_\alpha$ for $\alpha < \mathfrak{c}$
- (iv) The family $\{\{U\}_I : U \in \mathcal{CO}_\alpha\}$ is independent in $\mathcal{P}(\omega)/I$ for each $\alpha < \mathfrak{c}$. (to make τ crowded).

- (v) For each $n < \omega$ there is an $U \in \mathcal{CO}_{n+1}$ such that $|U \cap K_n| = 1$ (to make τ T_2).
- (vii) If $\omega < \alpha < \mathfrak{c}$ is odd then there is $U \in \mathcal{CO}_{\alpha+1}$ with $n \in U \subseteq G_\alpha$ (to make τ zerodimensional).
- (viii) If $\omega < \alpha < \mathfrak{c}$ is even then either $\text{int}_{\tau_{\alpha+1}} A_\alpha \neq \emptyset$ or there is $F \in \mathcal{F}_{\alpha+1}$ which misses A_α (to make \mathcal{F} strongly remote).

Suppose, that the construction can indeed be carried out. Let \mathcal{F} be the filter generated by $\bigcup\{F_\alpha : \alpha < \mathfrak{c}\}$, $\mathcal{CO} = \bigcup\{\mathcal{CO}_\alpha : \alpha < \mathfrak{c}\}$ and $\tau = \tau_I(\mathcal{CO})$. Then (ω, τ) with the closed filter \mathcal{F} satisfy the conclusion of the proof:

The topology is zerodimensional (the definition takes care of the sets from \mathcal{CO} , condition (vii) takes care of the sets from I^*).

The topology is also T_2 because if $x \neq y \in \omega$ then there is $n < \omega$, such that $K_n = \{x, y\}$ and by (v) there is $U \in \mathcal{CO}_n \subseteq \mathcal{CO}$ such that $|U \cap K_n| = 1$. This U is τ -clopen and separates x from y .

To show that τ is crowded it is sufficient to consider its basis, which consists of elements of the form:

$$\bigcap_{U \in P} U \cap \bigcap_{V \in N} (\omega \setminus V) \cap G \quad (1)$$

where $P, N \in [\mathcal{CO}]^{<\omega}$, $G \in I^*$. Now, by (iv) the family $\{[U]_I : U \in \mathcal{CO}\}$ is independent in $\mathcal{P}(\omega)/I$ with $FIN \subseteq I$ and $G \in I^*$ so (1) is finite iff there is some $U \in N \cap P$. But then (1) must be a subset of $U \cap (\omega \setminus U)$ so it must be empty. Thus the basis does not contain any finite sets beyond the empty set, so it is crowded as is the whole topology.

To prove that \mathcal{F} is strongly remote, choose $O \subseteq \omega$ such that $\text{int}_\tau O = \emptyset$. There is an $\alpha < \mathfrak{c}$, such that $O = A_\alpha$. Then $\text{int}_{\tau_{\alpha+1}} A_\alpha = \emptyset$, so there is $F \in \mathcal{F}_{\alpha+1} \subseteq \mathcal{F}$ such that $F \cap A_\alpha = \emptyset$.

So it remains to be shown that the inductive construction can be carried out all the way up to \mathfrak{c} . Suppose that we are at stage $\alpha < \mathfrak{c}$. If α is limit, we can let $\mathcal{F}_\alpha = \bigcup\{\mathcal{F}_\beta : \beta < \alpha\}$ and the conditions will be satisfied. Otherwise $\alpha = \beta + 1$. There are three cases:

Case $\beta = n < \omega$. Let $K_n = \{x, y\}$. Then by (iv) the subset $\{[U]_I : U \in \mathcal{CO}_n\}$ of $\mathcal{P}(\omega)/I$ is independent. Since $\mathcal{P}(\omega)/I$ has independence \mathfrak{c} and since $|\mathcal{CO}_n| \leq \omega < \mathfrak{c}$, there is an $U' \in \mathcal{P}(\omega)$ such that $\{[U]_I : U \in \mathcal{CO}_n\} \cup \{[U']_I\}$ is still independent. Then let $U = (U' \cup \{x\}) \setminus \{y\}$. We have that $[U']_I = [U]_I$ so (i,iv,v) are satisfied if we let $\mathcal{CO}_\alpha = \mathcal{CO}_n \cup \{U\}$.

Case $\omega < \beta < \mathfrak{c}$, β odd. Since $|\mathcal{CO}_\beta| \leq \beta < \mathfrak{c}$ and because $\mathcal{P}(\omega)/I$ has hereditary independence \mathfrak{c} there is $U' \subseteq \omega$ such that $\mathcal{CO}_\beta/I \cup \{[U']_I\}$ is still independent. Let $U = (U' \cap G_\beta) \cup \{n_\beta\}$. Because $G_\beta \in I^*$ we have that $[U']_I = [U]_I$ so we can let $\mathcal{CO}_\alpha = \mathcal{CO}_\beta \cup \{U\}$ and (vii) with all other conditions is satisfied.

Case $\omega < \beta < \mathfrak{c}$, β even. If $\{[U]_I : U \in \mathcal{CO}_\beta\} \cup \{[\omega \setminus A_\beta]_I\}$ is independent in $\mathcal{P}(\omega)/I$, then we can let $\mathcal{CO}_\alpha = \mathcal{CO}_\beta \cup \{\omega \setminus A_\beta\}$ and again all conditions are satisfied. So suppose otherwise.

If we let $B = \omega \setminus A_\beta$, necessarily $B \notin I$ (otherwise already $\text{int}_{\tau_0} A_\beta \neq \emptyset$). We claim, that $\{[U \cap B]_I : U \in \mathcal{CO}_\beta\}$ is independent in $\mathcal{P}(\omega)/I \upharpoonright [B]_I$: If it were not, then for some elementary meet M over \mathcal{CO}_β we would have that $M \cap B \in I$ but then, since $M \subseteq_I A_\beta$, $\text{int}_{\tau_\beta} A_\beta \neq \emptyset$ a contradiction. Now, since $\mathcal{P}(\omega)/I$ has hereditary independence \mathfrak{c} , $\{[U \cap B]_I : U \in \mathcal{CO}_\beta\}$ is not maximal independent in $\mathcal{P}(\omega)/I \upharpoonright [B]_I$ (by (i) $|\mathcal{CO}_\beta| \leq \beta < \mathfrak{c}$), so there is $F \subseteq B$ such that $\{[F \cap B]_I : U \in \mathcal{CO}_\beta\} \cup \{[F]_I\}$ is independent in $\mathcal{P}(\omega)/I \upharpoonright [B]_I$ so, a fortiori, $\{[U]_I : U \in \mathcal{CO}_\beta\} \cup \{F\}$ is independent in $\mathcal{P}(\omega)/I$ and if we let $F_\alpha = F_\beta \cup \{F\}$ and $\mathcal{CO}_\alpha = \mathcal{CO}_\beta \cup \{F\}$ all conditions are satisfied and we are done. \square

Together the theorems 4.4, 3.9, 3.3 and 5.1 give us the following:

Theorem 5.3. *There is a countable set $S \subseteq \omega^*$ and $p \in \overline{S}$ such that $(\omega^* \setminus (\overline{S} \setminus S)) \cup \{p\}$ contains a lonely point. This point is weakly lonely in ω^* .*

To get a lonely point in ω^* we would need the filter from 5.1 to be a weak P-point. However the only known constructions of weak P-points yield variants of OK points. These points cannot be limit points of ccc sets. Countable OHI spaces (i.e. without isolated points) are nowhere locally compact, so their remainder is ccc and therefore these constructions are of no use. This leads to the following question:

Question 5.4. *Is there a countable (or at least separable) nowhere locally compact space X with a weak P-point of X^* .*

It is easy to construct a ccc space with a weak P-point: take $\mathbb{R} \cup \infty$ where the topology is generated by the standard topology of \mathbb{R} and the neighbourhoods of ∞ , which are of the form $(a, \infty] \setminus F$ for some countable F . However this space is not even regular. Finding a nonseparable ccc compact space is very hard and it is consistent that any ccc, compact, hereditarily normal space is hereditarily separable (see [[Tod00]]). This rules out a lot of candidates for a positive answer to 5.4, however it does not give a consistent negative answer to our original question either, since our space cannot be hereditarily normal (any extremally disconnected space of big weight contains a copy of ω^* , e.g. [[TopEnc],g-1]).

So the main question remains open:

Question 5.5. *Is there a lonely point in ω^* ?*

I do not have even a consistent answer.

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