Logic of Questions and Public Announcements

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Abstract. This paper aims to explore the role of questions in communication in a group of cooperative rational agents. Using epistemic representation of questions proposed in [6] we employ the framework of public announcement logic to explore the flow of information in the process of asking and replying questions in a group. We show that some of the erotetic notions we introduce nicely correspond to the standard epistemic ones.

Keywords: logic of questions, erotetic logic, epistemic logic, public announcement

1 What Is a Question?

The earliest attempts to treat questions as logical entities date back the late 1920' to the work of F. Cohen. Rudolf Carnap is considered to be the first author working with questions in a formalized manner (see [5]). Since then many logicians have been dealing with this topic (Åqvist, Belnap, Harrah, Hintikka, Kubiński, Prior, and others) and produced a variety of different approaches.

We shall apply an approach which concentrates on answers to a question rather than to the question itself. This approach, called set-of-answers methodology (SAM) has been one of the main streams in the literature on logical analysis of questions (Hamblin, Belnap) and it is influential in the recent literature as well. Our approach is based on the semantic approach to questions of [4] and Inferential Erotetic Logic of [11].

1.1 Set-of-answers Methodology

SAM identifies a question with the set of its answers. Formally a question Q is defined as a set of *direct answers* $\{\alpha_1, \alpha_2, \ldots\}$, where $\alpha_1, \alpha_2, \ldots$ are syntactically

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distinct formulas of a background logical calculus. The set of direct answers of a question Q will be denoted as dQ and it is supposed that dQ has at least two members. The question Q is (completely) answered if the questioner gets to know one of the direct answers. This approach is sometimes considered to be unintuitive as it does not pay any attention to the surface form of a question.[1] However in the case of propositional questions, to which we restrict our attention in this article, the connection to the surface structure can be restored. We can assume that every question can be without the loss of meaning reformulated as asking which of the direct answers is the case. For example our question Q can be reformulated as:

Is it the case that α_1 or is it the case that α_2 or ...?

As we confine ourselves to propositional questions it quite obviously follows that we shall work just with *finite* sets of direct answers. Not any set of statements corresponds to a question, at least not to a reasonable one. Direct answers of a question which can be 'reasonably' asked in a certain situation are constrained with respect to the epistemic state of the questioner in this situation. Following [6] we assume that the criteria for a reasonable set of answers are as follows:

1. **Non-triviality**—agent does not know any of the answers to the question in advance
2. **Admissibility**—agent considers all the answers to be possible
3. **Context**—agent knows that one of the direct answers is the case

All these conditions are of epistemic nature; it is natural to represent them in an epistemic framework. SAM does not provide any direct method for translating natural language interrogatives into a formal language. In some cases it is necessary to consider an epistemic background of the corresponding agent and semantic constraints. For example the following question

Is it raining or snowing?

allows for two readings (depending on which one from the following two background presuppositions the agent has in mind):

a) It can be *both* raining and snowing.
b) It can *either* be raining or snowing.

Using SAM we can formalize the question in both cases as $?\{\alpha, \beta\}$, but the proper epistemic model must reflect the expected answers. [6] provides a discussion on auxiliary background contexts and proposes the formalization of 'context condition' based on maximal presuppositions of questions.

[1] At the first view it might seem that in order to analyze a question we have to be able to 'look inside the questioner's head' and see what he/she actually wants to know. In some other areas dealing with representation of questions, most notably the theory of database queries, it is exactly the structure of interrogative sentences what is formalized.

1.2 Multi-agent Propositional Epistemic Logic

We start with a standard modal language with finitely many modalities K_i, 'the agent i knows', where $i \in \{1, \dots, m\}$, and dual operators M_i, 'the agent i admits', defined in a standard way $M_i\varphi \equiv \neg K_i \neg\varphi$, and the standard Kripke semantics. A *Kripke frame* is a relational structure $\mathcal{F} = \langle S, R_1, \dots, R_m \rangle$ with a set of states S and accessibility equivalence relations $R_i \subseteq S^2$. *Kripke model* $\mathbf{M}$ is a pair $\langle \mathcal{F}, v \rangle$ where v is a valuation of atomic formulas. The satisfaction relation $\models$ is defined in a standard way:

- $(\mathbf{M}, s) \models p$ iff $(\mathbf{M}, s) \in v(p)$
- $(\mathbf{M}, s) \models \neg\varphi$ iff $(\mathbf{M}, s) \not\models \varphi$
- $(\mathbf{M}, s) \models \psi_1 \wedge \psi_2$ iff $(\mathbf{M}, s) \models \psi_1$ and $(\mathbf{M}, s) \models \psi_2$
- $(\mathbf{M}, s) \models K_i\varphi$ iff $(\mathbf{M}, s_1) \models \varphi$, for each s_1 such that sR_is_1

The choice of a particular modal system is not essential for our approach, in this paper we shall work with the system S5, used in the majority of epistemic logics. Now we formalize the conditions 1–3 specified above in the epistemic framework.

1.3 Askable Questions

The language of our logic of questions is an extension of a standard multimodal language $\mathcal{L}^K$—we add the question mark $?_i$ and the brackets.

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid ?_i\{\varphi, \dots, \varphi\}$$

Let us stress that there is no restriction on the formulas within the scope of the question operator, in particular we can have questions about questions. The semantics of the modal part of our logic is standard (see above), what remains is to define the satisfaction relation for questions. To say that a question is true or false (in a certain state) does not seem to make a good sense. We can say instead that it is 'reasonable' (from the point of view of what agents knows) to ask a question. We already listed the conditions for a question to be reasonable, now we can formalize these conditions in the epistemic framework. We shall call the questions satisfying our conditions *askable*.

Definition 1 *It holds for a question* $Q^i = ?_i\{\alpha_1, \dots, \alpha_n\}$ *that* $(\mathbf{M}, s) \models Q^i$ *iff*

1. $(\forall \alpha \in dQ^i)((\mathbf{M}, s) \not\models K_i\alpha)$
2. $(\forall \alpha \in dQ^i)((\mathbf{M}, s) \models M_i\alpha)$
3. $(\mathbf{M}, s) \models K_i\big(\bigvee_{\alpha_j \in dQ^i} \alpha_j\big)$

We say that Q^i *is* askable *in the state* $(\mathbf{M}, s)$ *by an agent* i.

Questions are in our epistemic approach complex modal formulas. We can define implication between two questions as a classical implication between the corresponding modal formulas:

$$(\mathbf{M}, s) \models Q^i_1 \to Q^i_2 \text{ iff } (\mathbf{M}, s) \models Q^i_1 \text{ implies } (\mathbf{M}, s) \models Q^i_2$$

The askability of Q_1 (in s, by i) implies the askability of Q_2 (in s, by i). For example, it holds that

$$?_i\{\alpha_1, \ldots, \alpha_n\} \rightarrow ?_i\{\alpha_j, \neg\alpha_j\}$$

for each $j \in \{1, \ldots, n\}$. (See also [6].) If we return to our initial example:

Is it raining or snowing?

implies both questions

Is it snowing?

and

Is it raining?

2 Communication

As we said, our goal is to explore the role of questions in communication in a group of rational agents. In particular, to explore it as a tool of exchange of not only factual, but also epistemic information ('who knows what'). We shall employ the standard framework of public announcement logic (PAL) in the sense of [9].

2.1 Public Announcement Logic

Let us imagine a group of three card players: Trinity, Morpheus, and Neo. One of the important cards they play with is called *Mr. Smith*. After a deal no cards are left, so one of the players must have *Mr. Smith* and all of them are aware of this. In fact, Neo got *Mr. Smith*, but neither Morpheus nor Trinity know, which of the other two players has it. In particular, both of them consider possible both situations where Neo got *Mr. Smith* and where he did not. Or, in other words, they are not able to distinguish between these situations. If Neo announces in front of both Morpheus and Trinity (i.e., publicly from the viewpoint of the group of players)

"I got *Mr. Smith*.",

everybody in the group learns this fact. Morpheus and Trinity do not consider the situations, where Neo does not have *Mr. Smith*, as admissible any more.

Our example gives a typical situation represented in the public announcement logic—after a public announcement of a statement φ (I got *Mr. Smith*), some other statement ψ holds (e.g., Morpheus knows that Neo got *Mr. Smith* and Trinity knows that Neo got *Mr. Smith*). In fact, the author of an announced statement is irrelevant in our framework. The statement is understood as information received by each member of a group in the same way. From this point of view, Neo's announcement in our example has the same effect as if an external observer announced:

"Neo got *Mr. Smith.*"

In order to obtain the logic of public announcement we shall extend the language of the multiagent epistemic logic in two ways: we introduce a public announcement operator [ ] and group knowledge operators E_G and C_G:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid [\varphi]\psi \mid E_G\varphi \mid C_G\varphi$$

The intended meaning of $[\varphi]\psi$ is 'after the public announcement of φ, it holds that ψ'. The weaker of the group knowledge operators E_G simply means that each agent from a group G knows some proposition φ, while the stronger one requires not only that φ is a group knowledge, but also that this fact is reflected by everybody in the group G (everybody knows that everybody knows φ and everybody knows that everybody knows that everybody knows φ, etc).

The new operators are defined as follows:

- $(\mathbf{M}, s) \models E_G\varphi$ iff $(\mathbf{M}, s) \models K_i\varphi$ for each $i \in G \subseteq \{1, \dots, m\}$
- $(\mathbf{M}, s) \models C_G\varphi$ iff $(\mathbf{M}, s_1) \models \varphi$ for each s_1 such that $s \left(\bigcup_{i \in G} R_i\right)^* s_1$, where $\left(\bigcup_{i \in G} R_i\right)^*$ is a reflexive and transitive closure of $\bigcup_{i \in G} R_i$
- $(\mathbf{M}, s) \models [\varphi]\psi$ iff $(\mathbf{M}, s) \models \varphi$ implies $(\mathbf{M}|_\varphi, s) \models \psi$, where $\mathbf{M}|_\varphi = \langle\langle S', R'_1, \dots, R'_m\rangle, v'\rangle$ is defined as follows:

$$S' = \{s \in S \mid s \models \varphi\}$$
$$R'_i = R_i \cap S'^2$$
$$v'(p) = v(p) \cap S'$$

The model $\mathbf{M}|_\varphi$ is obtained from $\mathbf{M}$ by deleting all states where φ is not true and by the corresponding restrictions of accessibility relations and the valuation function.

$s \left(\bigcup_{i \in G} R_i\right)^* s_1$ in fact means that s_1 is accessible from s by each R_i $(i \in G)$ in k steps, for any $k \geq 0$. We can also see C_G as an infinite conjunction of all finite iterations of the group knowledge E_G:

$$C_G\varphi \leftrightarrow \varphi \wedge E_G\varphi \wedge E_G E_G\varphi \wedge E_G E_G E_G\varphi \wedge \dots$$

So we immediately see the hierarchy of the introduced notions: Formula

$$C_G\varphi \rightarrow E_G\varphi \rightarrow K_i\varphi$$

is valid for each $i \in G$.

We introduce also a dual operator $\langle\ \rangle$ defined in a standard way as $\langle\varphi\rangle\psi$ iff $\neg[\varphi]\neg\psi$. If we rewrite the corresponding semantic clause, we obtain

- $(\mathbf{M}, s) \models \langle\varphi\rangle\psi$ iff $(\mathbf{M}, s) \models \varphi$ and $(\mathbf{M}|_\varphi, s) \models \psi$

The intended meaning of the dual operator is 'after a *truthful* announcement of φ, it holds that ψ'. It is easy to see that the diamond-like operator is stronger:

Lemma 1. $\models \langle\varphi\rangle\psi \rightarrow [\varphi]\psi$

The following proposition shows that languages $\mathcal{L}^{K[]}$ and $\mathcal{L}^{K}$ have the same expressive power—it provides a reduction of formulas with the public announcement operator to the epistemic ones. The corresponding equivalences in the proposition give in fact an axiomatization of the announcement operator in PAL without common knowledge [9, p. 81].

Proposition 1 *The following equivalences are valid:*

$$\begin{aligned}[\varphi]p &\leftrightarrow (\varphi \to p)\\ [\varphi]\neg\psi &\leftrightarrow (\varphi \to \neg[\varphi]\psi)\\ [\varphi](\psi \wedge \chi) &\leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)\\ [\varphi]K_i\psi &\leftrightarrow (\varphi \to K_i[\varphi]\psi)\\ [\varphi][\psi]\chi &\leftrightarrow [\varphi \wedge [\varphi]\psi]\chi\end{aligned}$$

For common knowledge there is no such reduction, the language $\mathcal{L}^{KC[]}$ is more expressive than $\mathcal{L}^{KC}$. We only have a rule describing the relationship between public announcement and common knowledge, see e.g. [9, p. 83]:

$$\frac{(\chi \wedge \varphi) \to [\varphi]\psi \wedge E_G\chi}{(\chi \wedge \varphi) \to [\varphi]C_G\psi} \tag{1}$$

2.2 Announcement and Updates

As we said in our card example, after Neo announced

“I've got *Mr. Smith.*”,

this fact became (common) knowledge in the group. This seems to suggest that a publicly announced proposition becomes common knowledge. But it might not work for epistemic propositions. Let Neo say

“You don't know it yet, but I've got *Mr. Smith.*”

Although this statement, which can be formalized as

$$S^{neo} \wedge \neg K_{morph}(S^{neo}) \wedge \neg K_{trin}(S^{neo}),$$

is true in the moment of announcement, it is evident that its epistemic part (‘you don't know it yet’) becomes invalid after it is announced. So the formula $S^{neo} \wedge \neg K_{morph}(S^{neo}) \wedge \neg K_{trin}(S^{neo})$ becomes false after the announcement.

Formulas, which became false after they were truthfully announced (as in our example), are in the public announcement logic called an *unsuccessful update*; if they are true, we call them a *successful update.*

Definition 2 – *Formula* φ *is a* successful update *in* $(\mathbf{M}, s)$ *iff* $(\mathbf{M}, s) \models \langle\varphi\rangle\varphi$.
– *Formula* φ *is an* unsuccessful update *in* $(\mathbf{M}, s)$ *iff* $(\mathbf{M}, s) \models \langle\varphi\rangle\neg\varphi$.

Example 1 $\top$ *is a successful update in any state (as* $\langle\top\rangle\top$ *is a valid formula).* $\bot$ *is neither successful nor unsuccessful update in any state.*

We define similar notions for the [ ] modality:

Definition 3 *A formula* φ *is a* successful formula *iff* $[\varphi]\varphi$ *is valid, otherwise it is an* unsuccessful formula*.*

If a formula is an unsuccessful update, it cannot be commonly known in the updated model. Using the soundness proof of the rule (1) we can prove that a formula is true after a public announcement if and only if it gets common knowledge after the announcement, see [9, p. 83 and 86].

Proposition 2 $[\varphi]\psi$ *is valid iff* $[\varphi]C_G\psi$ *is valid.*

As a consequence we get

Proposition 3 $[\varphi]\varphi$ *is valid iff* $[\varphi]C_G\varphi$ *is valid.*

Thus, publicly announced successful formulas are commonly known.[2]

3 Asking and Replying Questions

Trinity from our example does not know who has *Mr. Smith*, it is quite natural for her to ask. After she asks, the other players are informed about her lack of knowledge and about ways she thinks it is possible to complete it. From this point of view we can see posing a question as a public announcement. This announcement is of a special kind–the information transmitted to the group is not about facts, but about knowledge.

If Neo says "I got *Mr. Smith*", then Trinity gets to know one of the direct answers and the question is answered for her. Naturally it means that the question is not reasonable for her any more. From the formal point of view it is not askable (in the corresponding state) because of the violation of the first condition from Definition 1. It might also happen that Morpheus says "I did not get *Mr. Smith*". Now the question becomes unreasonable again, but for different reasons. Trinity does not have a direct answer, but she can exclude one of them (Morpheus got *Mr. Smith*). We say that the question is *partially answered* for her (in the corresponding state). In fact, the question is inaskable because of the violation of the second condition (Definition 1). Let us define the terms we have mentioned:

Definition 4 *Let* Q^i *be a question* $?_i\{\alpha_1, \ldots, \alpha_n\}$*.*

– Q^i *is* answered *in* $(\mathbf{M}, s)$ *(for an agent i) iff* $(\mathbf{M}, s) \models \bigvee_{\alpha_j \in dQ^i}(K_i\alpha_j)$*. We write* $(\mathbf{M}, s) \models A_iQ$*.*
– Q^i *is* partially answered *in* $(\mathbf{M}, s)$ *(for an agent i) iff* $(\mathbf{M}, s) \models \bigvee_{\alpha_j \in dQ^i}(K_i\neg\alpha_j)$*. We write* $(\mathbf{M}, s) \models P_iQ$*.*

[2] Atoms, $K_i\varphi$, and $\neg K_i\varphi$ (for every φ) are examples of successful formulas.

To complete the picture we should also discuss inaskability caused by the last condition—the question where an agent does not believe that at least one of $\alpha \in dQ$ is the right answer.[3] As this kind of questions is not essential for the purposes of this paper, we shall not discuss the violation of the context condition here. We restrict ourselves to a special class of questions—*safe questions*, for which it is trivially satisfied.[4]

Definition 5 *A question Q^i is (for an agent i)*

- safe in a state $(\mathbf{M}, s)$ *iff* $(\mathbf{M}, s_1) \models \left(\bigvee_{\alpha_j \in dQ^i} \alpha_j\right)$, *for each s_1 such that $sR_i s_1$.*
- safe *iff* $\left(\bigvee_{\alpha_j \in dQ^i} \alpha_j\right)$ *is tautology.*

The class of safe questions includes, among others, the well known yes-no questions, i.e., questions of the form $?_i\{\alpha, \neg\alpha\}$. Obviously a safe question does not need to be askable, but an askable question in a state s (by an agent i) is safe in s. This is not surprising as the safeness condition coincides with the context condition in our definition of askable questions.

There is one more reason to restrict ourselves to the class of safe questions. One would expect that if a question is answered, it is also partially answered, but this is in general not the case. It holds, however, for *safe questions with pairs of mutually exclusive direct answers.*

Definition 6 *Q is a* question with pairs of mutually exclusive direct answers *in a state $(\mathbf{M}, s)$ iff for any $\varphi \in dQ$ if $(\mathbf{M}, s) \models \varphi$ then $(\mathbf{M}, s) \not\models \psi$ for some $\psi \in dQ - \{\varphi\}$.*

Fact 1 *If Q^i is a question with pairs of mutually exclusive direct answers in a state $(\mathbf{M}, s)$, then $(\mathbf{M}, s) \models A_i Q \rightarrow P_i Q$.*

The following proposition, proved in [6], puts both condition together.

Proposition 4 *Let Q^i be a safe question in $(\mathbf{M}, s)$ with pairs of mutually exclusive direct answers. Then the following conditions are equivalent:*

- $(\mathbf{M}, s) \models \neg Q^i$
- $(\mathbf{M}, s) \models P_i Q$
- *There is a formula φ such that $(\mathbf{M}, s) \models A_i?\{\varphi, \neg\varphi\}$ and $Q^i \rightarrow ?_i\{\varphi, \neg\varphi\}$ is valid.*

Partial answerhood of a question Q^i in some state is equivalent to the existence of a yes-no question, which is answered at that state and implied by Q^i. From the validity of $Q^i \rightarrow ?_i\{\varphi, \neg\varphi\}$ we know that inaskability of $?_i\{\varphi, \neg\varphi\}$[5] implies inaskability of Q^i and, therefore, φ (as well as $\neg\varphi$) imply either some $\alpha \in dQ^i$ or $\neg\alpha$ (for $\alpha \in dQ^i$).

[3] These questions are called *weakly presupposed*—see [6].
[4] The term *safe question* originates from Belnap, we use it here in a sense closed to Wiśniewski's approach.
[5] $(\mathbf{M}, s) \models \neg ?_i\{\varphi, \neg\varphi\}$ iff $(\mathbf{M}, s) \models A_i?\{\varphi, \neg\varphi\}$.

4 Public Announcement Updates and Questions

We shall discuss now in what kind of information is transmitted when an agent asks a question. If a question is askable, then it gets commonly known and hence it is successful. If our background epistemic logic is multi-modal S5, then formula $[Q^i]Q^i$ is valid, i.e.,

Fact 2 *Questions are successful formulas.*

In S5-models a question Q^i askable in a state s is askable in all states from the equivalence class sR_i. No 'cutting' of states in the model $\mathbf{M}$ forced by the public announcement of Q^i results in $(\mathbf{M}, s) \models Q^i$ and $(\mathbf{M}|_{Q^i}, s) \not\models Q^i$. Thus, a publicly announced question is commonly known (see Proposition 3). In other words there is no model and state such that $(\mathbf{M}, s) \not\models [Q^i]Q^i$.

Successful formulas have an important property: they do not bring anything new if they are announced repeatedly.

Fact 3 *Let* φ *be a successful formula.* $[\varphi][\varphi]\psi \leftrightarrow [\varphi]\psi$ *is valid.*

The proof is straightforward: $[\varphi][\varphi]\psi$ is equivalent to $[\varphi \wedge [\varphi]\varphi]\psi$ (Proposition 1), which is equivalent to $[\varphi]\psi$, because of the validity of $[\varphi]\varphi$ (φ is successful).

It is no surprise that askable questions are successful updates.

Fact 4 $(\mathbf{M}, s) \models Q^i$ *iff* $(\mathbf{M}, s) \models \langle Q^i \rangle Q^i$.

Whenever an agent publicly asks a question, it does not cause any change in her epistemic model, it remains askable until she gets some new information.

The last point we are going to talk about is the relationship of public announcement and answerhood. Whenever a question is (partially) answerable in a state, there is a formula φ such that after a public announcement of φ the question becomes inaskable there. Let us return to our group of players. Neo got *Mr. Smith*. Neither Trinity nor Morpheus know it. If Trinity publicly asks

> "Who got *Mr. Smith*?",

Morpheus can infer:

> "I did not get *Mr. Smith* and Trinity does not know who got it, therefore Neo got it."

Trinity's question was *informative* for Morpheus. The question "Who got *Mr. Smith*?", which was askable for Morpheus became inaskable after Trinity asked it, even if her question was not (partially) answered. This leads us to the definition of *informative formula.*

Definition 7 *A formula* φ *is* informative *(for an agent* i*) with respect to* Q *in* $(\mathbf{M}, s)$ *iff* $(\mathbf{M}, s) \models Q^i \wedge \langle \varphi \rangle \neg Q^i$.

Contrary to the partial answerhood (see the commentary below the Proposition 4) there need not be any direct logical connection between an informative formula and direct answers to the question. The informativeness can be forced by background knowledge of agents.

5 Questions and Answers in a Group of Agents

We said that we can consider single-agent questions as tasks to be solved by the corresponding agent. From this point of view it makes perfect sense to introduce group questions with the same motivation. Let us imagine a scientific team, which collaboratively seeks answers to some collective questions.

As in the case of individual questions we have to define askability and answerhood for the group case (to specify, which questions are 'reasonable' for a group).

If we understand a group of agents as a cooperative team it seems natural to say that the team knows an (partial) answer to a question if at least one of its members does.[6]

Definition 8 *Let* Q^G *be a group question* $?_G\{\alpha_1, \ldots, \alpha_n\}$ *for a group* $G \subseteq \{1, \ldots, m\}$*.*

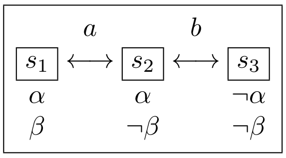
- Q^G *is* answered *in* $(\mathbf{M}, s)$ *(for a group G) iff* $(\mathbf{M}, s) \models \bigvee_{i \in G} \bigvee_{\alpha \in dQ^G} (K_i \alpha)$.
- Q^G *is* partially answered *in* $(\mathbf{M}, s)$ *(for a group G) iff* $(\mathbf{M}, s) \models \bigvee_{i \in G} \bigvee_{\alpha \in dQ^G} (K_i \neg\alpha)$.

Conversely if no one knows the answer to some question, it is reasonable to consider it askable for the whole group.

Definition 9 *A question Q is an* askable group question *at* $(\mathbf{M}, s)$ *(for a group of agents G) iff* $(\forall i \in G)((\mathbf{M}, s) \models Q^i)$.
Let us write $(\mathbf{M}, s) \models Q^G$.

It is evident in the case of individual askable questions that the corresponding agent is not able to find an answer (even a partial one) without an external information (otherwise the question would not be askable). The situation in group questions is different. Information necessary to reply a question might be scattered among the members of the group and so the group might reach an answer in the process of communication. Consider the following example: Let us have a group of two agents a and b. The following figure shows their knowledge structure.

Example 2



Agent a can 'see' s_1 and s_2 and knows α, agent b can 'see' s_2 and s_3 and knows $\neg\beta$. However, neither of them is able to (partially) answer the yes-no question $?(\alpha \to \beta)$, so this question is askable for the group $G = \{a, b\}$.

We can imagine their cooperative communication as follows: After asking the question $?(\alpha \to \beta)$ and recognizing that it is their group question, agents will cooperate and a announces he knows α and b learns it, so the fact α is common knowledge in the group G.

[6] We could also require that *every* member be able to reply a question, so a question is not replied by a group unless the reply is communicated among all its members.

$[K_a\alpha]C_G\alpha$

b will do the same with the fact $\neg\beta$.

$[K_b\neg\beta]C_G\neg\beta$

As our background epistemic logic is $\mathsf{S5}$, our agents are logically omniscient, so from $C_G\alpha$ and $C_G\neg\beta$ we can infer

$C_G\neg(\alpha \to \beta)$,

which is a commonly known direct answer to the question $?(\alpha \to \beta)$.

Our group was successful in seeking an answer using just 'internal' resources. They reached a direct answer after a series of announcements of facts they know. We can record it as

$[K_a\alpha][K_b\neg\beta]\neg(\alpha \to \beta)$

In general we can say that a question is *solvable*, if there is a series of announcements after which a (partial) answer is obtained.

Definition 10 *A question* $Q = ?\{\alpha_1, \ldots, \alpha_n\}$ *is*

- solvable *for the group* G *in a state* $(\mathbf{M}, s)$ *iff there are formulas* $\varphi_1, \ldots, \varphi_k$ *such that* $(\mathbf{M}, s) \models [K_{i_1}\varphi_1]...[K_{i_k}\varphi_k]\alpha$ *where* $i_j \in G$ *and* $\alpha \in dQ$*,*
- partially solvable *for the group* G *in a state* $(\mathbf{M}, s)$ *iff there are formulas* $\varphi_1, \ldots, \varphi_k$ *such that* $(\mathbf{M}, s) \models [K_{i_1}\varphi_1]...[K_{i_k}\varphi_k]\neg\alpha$ *where* $i_j \in G$ *and* $\alpha \in dQ$*.*

The scattered group knowledge from our example, i.e., a situation where a group is not aware of consequences of information known just to some of its individual members, is also represented in the framework of multiagent epistemic logic. It is called *distributed knowledge*. Formally we add to the framework of public announcement logic a new operator D^*_G with the following meaning (see [2]):

- $(\mathbf{M}, s) \models D^*_G\varphi$ iff there is a set $\Psi = \{\psi \mid (\exists i \in G)((\mathbf{M}, s) \models K_i\psi)\}$ such that $\Psi \models \varphi$

There is another semantic definition of distributed knowledge often quoted in the literature cf. [1, 9]:

- $(\mathbf{M}, s) \models D_G\varphi$ iff $(\mathbf{M}, s_1) \models \varphi$ for all s_1 such that $s\left(\bigcap_{i\in G} R_i\right)s_1$

The relationship of both definitions is studied in [2]. Let us only note that if $(\mathbf{M}, s) \models D^*_G\varphi$, then $(\mathbf{M}, s) \models D_G\varphi$, but not vice versa. As the last remark of this section we show the relationship between distributed knowledge and the notion of solvability of a question. At the first view they might look equivalent, however a closer observation shows, that the correspondence is not that straightforward.

Firstly, the set Ψ might not be finite. If we are working within a compact logic, this is not a problem. However, in this article we use the epistemic logic $S5$ with common knowledge, which is not compact (see, e.g., [9]). Instead of restricting ourselves to the compact logics we introduce a finitely generated distributed knowledge D^+:

– $(\mathbf{M}, s) \models D^{+}_{G}\varphi$ iff there is a set $\Psi = \{\psi_1, ..., \psi_k\}$ such that $(\mathbf{M}, s) \models K_{i_1}\psi_1, ..., (\mathbf{M}, s) \models K_{i_k}\psi_k$, $i_j \in G$ and $\Psi \models \varphi$.

The following statement says that answerability is a weaker notion than that of (finitely generated) distributed knowledge. If a (partial) answer to a question Q is finitely generated distributed knowledge in a group G, then the question is (partially) solvable for the group G, but not vice versa.

Fact 5 *Let Q be a question.*

– *If $(\exists\alpha \in dQ)((\mathbf{M}, s) \models D^{+}_{G}\alpha)$ then Q is solvable for the group G in a state $(\mathbf{M}, s)$.*
– *If $(\exists\alpha \in dQ)((\mathbf{M}, s) \models D^{+}_{G}\neg\alpha)$ then Q is partially solvable for the group G in a state $(\mathbf{M}, s)$.*

Proof The proof follows from the definitions of solvability and finitely generated distributed knowledge D^{+}_{G}. Let us prove the first implication. From $(\mathbf{M}, s) \models D^{+}_{G}\alpha$ we obtain that each $K_{i_j}\psi_j$ is successful formula true in $(\mathbf{M}, s)$. It is easy to show that successful formulas true in a state are successful updates there and, simultaneously, they are commonly known. Let us write $\mathbf{M}|_{...}$ for the updated model after the series of public announcements $[K_{i_1}\psi_1] \ldots [K_{i_k}\psi_k]$. It follows that $(\mathbf{M}|_{...}, s) \models \psi_j$ for each $j \in \{1, \ldots, k\}$. As $\{\psi_1, \ldots, \psi_k\} \models \alpha$ we have $(\mathbf{M}|_{...}, s) \models \alpha$. The proof of the second implication is the same. QED

The reverse implication does not hold as the solvability of a question forces just a local entailment between ψ_i and φ, while the notion of distributed knowledge requires logical entailment between them.

6 Conclusion

In this article we made first steps towards exploring questions as a means of communication in a cooperative group of rational agents. There are still many things to be done. We worked with the 'classical' public announcement logic based on the modal logic S5. We shall explore weaker systems of modal logic, in particular we plan to work with a 'belief' version of public announcement. We would also like to study more deeply the relationship between group knowledge modalities and the problem of a search for an answer in a group of agents. Our formalization of questions does not deal with the predicate case; this is not a big disadvantage from the point of view of applications we considered in this paper—PAL is typically studied just for the propositional case—nevertheless, the extension of our basic definitions to the predicate case is on our agenda as well.

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