Consequence Relations in Inferential Erotetic Logic

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1 Introduction

The aim of this text is to study consequence relations where both declaratives and interrogatives appear together. That is, we will discuss inferential structures in the field of *logic of questions*. It will be seen in the next subsection that many theories belong to the range of logic of questions. Our interest is restricted to *inferential erotetic logic* (IEL, for short) and erotetic consequences introduced by it.

IEL was established by Andrzej Wiśniewski and his collaborators in the 1990s. The book [14] and article [16] contain a well-written presentation of Wiśniewski's approach. We utilize the framework of this theory and show some properties, relationships, and possible generalizations.

1.1 Logic and questions

Referring to Kubiński and Gornstein in [7], David Harrah mentions the long history of formal-logical approaches to interrogatives. F. Cohen (1929) and R. Carnap seem to be the first logicians working with the idea of interrogatives formalized inside logic. The real "boom" of such attempts appeared in 1950s and continued in 1960s. The reader can find very nice overviews of the history of erotetic logic in [7], [14, chapter 2], and [5].

Logic of questions is multiparadigmatic. There are many branches given by different ways of formalization of interrogatives. This is nicely illustrated by Harrah's examples of "(meta)axioms" (cf. [6, pp. 25–26]). He groups them into three sets according to the acceptance by erotetic logicians.

The first group includes (meta)axioms accepted in almost all systems. Harrah calls them *absolute axioms* and examples are:

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- Every question has at least one partial answer.
- (In systems with negation) For every statement P, there exists a question Q whose direct answers include P and the negation of P.
- Every question Q has a presupposition P such that: P is a statement, and if Q has any true direct answer, then P is true.

The second group—*standard axioms*—is often accepted, but not in all systems.

- Every question has at least one direct answer.
- Every direct answer is a statement.
- Every partial answer is implied by some direct answer.
- Every question is expressed by at least one interrogative.
- Each interrogative expresses exactly one question.
- Given an interrogative I there is an effective method for determining the direct answers to the question expressed by I.

The last group is called *excentric axioms*. Thus, the following examples of such axioms are accepted only in some interrogative systems.

- If two questions have the same direct answers, then the two questions are identical.
- Every question Q has a presupposition that is true just in case some direct answer to Q is true.

We will not attempt to provide an overview of interrogative theories, nor shall we discuss formal approaches to a question analysis. We entirely omit "pragmatical" approaches and theories based on logical analysis of natural language[1] as well as epistemic-imperative approaches.[2] We will, however, answer the following problems:

- What is the formal shape of questions?
- What kind of inferences are studied?

An important part of erotetic logic working with inferential structures consists of models of inquiry processes. Hintikka's *interrogative model of inquiry* is the best known case (cf. [8]). Also IEL is presented as an alternative to Hintikka's approach (cf. [16, 17]). Since we want to focus on consequence relations only, this topic will not be explicitely studied here.

Another important issue to solve is the *question—answer* relationship. Inferential erotetic logic is based on explicit *set-of-answers methodology* (SAM, for short), which seems to provide one of the possible solutions to this problem as well as to the problem of formal "shape" of questions.

[1]Such approach can be found in Pavel Tichý's transparent intensional logic; see [11] and [10].

[2]It must be emphasized that many terms have their origin just in these theories and we use them in an informal way.

1.2 Set-of-answers methodology

This kind of methodology is often connected with Hamblin's postulates:

1. An answer to a question is a statement.
2. Knowing what counts as an answer is equivalent to knowing the question.
3. The possible answers to a question are exhaustive set of mutually exclusive possibilities.

The vague formulation evokes discussions and enables to formulate various kinds of SAM (cf. [7]). Inferential erotetic logic accepts only the first two postulates and tries to keep maximum of the (classical) declarative logic and its consequence relation.[3]

First, let us define a (general) erotetic language $\mathcal{L}_Q$. A (general) declarative language $\mathcal{L}$ is extended by curly brackets ($\{, \}$) and question mark (?).[4]

Simultaneously, we will use the following metavariables:

- small Greek letters $\alpha, \beta, \varphi, \ldots$ for declarative sentences,
- $Q, Q_1, \ldots$ for questions,
- capital Greek letters $\Gamma, \Delta, \ldots$ for sets of declaratives, and
- $\phi, \phi_1, \ldots$ for sets of questions.

Second, a question Q is the following structure

$$?\underbrace{\{\alpha_1, \alpha_2, \ldots\}}_{dQ}$$

In the correspondence with the first and second postulate, a question is a set of declarative sentences, which are called *direct answers*. The symbol dQ is used for a set of direct answers to a question Q. In the case of a question $?\{\alpha_1, \alpha_2, \ldots\}$ we expect one of the following answers:

It is the case that α_1.
It is the case that α_2.
$\vdots$

Although we have not posed any restriction on dQ and, generally, every set of declaratives is a question, it is useful to postulate that a question should have length at least two, i.e., $|dQ| = 2$. In case of finite versions of questions $?\{\alpha_1, \ldots, \alpha_n\}$ we suppose that the listed direct answers are semantically non-equivalent. The class of finite questions corresponds to the class of *questions of*

[3] The full acceptance can be found in an intensional-semantic approach presented by Jeroen Groenendijk and Martin Stokhof (cf. [4, 5]).

[4] In the text we use only propositional examples in the language with common connectives ($\wedge, \vee, \rightarrow, \neg$).

the first kind in [14]. Some of them are important in our future examples and counterexamples. Let us mention two abbreviations and terms that are very frequent in this paper.

- *Simple yes-no questions* are of the form $?\alpha$, which is an abbreviation for $?\{\alpha, \neg\alpha\}$. If α is an atomic formula, then the term *atomic yes-no question* is used.

- A *conjunctive question* $?|\alpha, \beta|$ requires the answer whether α (and not β), or β (and not α), or neither α nor β, or both (α and β). It is an abbreviation for $?\{(\alpha \wedge \beta), (\neg\alpha \wedge \beta), (\alpha \wedge \neg\beta), (\neg\alpha \wedge \neg\beta)\}$. Similar versions are $?|\alpha, \beta, \gamma|$, $?|\alpha, \beta, \gamma, \delta|$, and so on.

By using the word *explicit* for set-of-answers methodology in inferential erotetic logic, we emphasize the role of listed direct answers.[5] Explicit SAM brings out the *answerhood conditions* of questions: the meaning of interrogatives resides in the background logic and in the structure of a set of direct answers.[6]

1.3 Consequence relations in IEL

Having answered the question *What is a question in our system?*, we have to move on to inferences. We believe that consequence relations are the central point of logic.[7] Inferential erotetic logic adds three new consequence relations (mixing declaratives and interrogatives) to the standard one.

- *Evocation* is a binary relation $\langle \Gamma, Q \rangle$ between a set of declaratives and a question.

- *Erotetic implication* is a ternary relation $\langle Q, \Gamma, Q_1 \rangle$ between an initial question Q and an implied question Q_1 with respect to a set of declaratives Γ.

- *Reducibility* is a ternary relation $\langle Q, \Gamma, \phi \rangle$ between an initial question Q and a set of questions ϕ with respect to a set of declaratives Γ.

Motivations and natural-language examples of both the evocation and erotetic implication can be found in [14, 16]. Reducibility is studied in [9, 14, 19].

[5] In the original version of IEL, questions are not identified with sets of direct answers: questions belong to an object-level language and are expressions of a strictly defined form, but the form is designed in such a way that, on the metalanguage level (and only here), the expression which occurs after the question mark designates the set of direct answers to the question. Questions are defined in such a way that sets of direct answers to them are explicitly specified. The general framework of IEL allows for other ways of formalizing questions. [personal communication with Andrzej Wiśniewski]

[6] This connection of the meaning of questions and answerhood conditions are stated in Belnap's *answerhood requirement*, cf. [4, p. 3].

[7] Declarative logic can be defined by its consequence relation as a set of pairs $\langle \Gamma, \Delta \rangle$, where Γ and Δ are sets of formulas and Δ is usually considered to be a singleton.

The chosen shape of questions in IEL makes it possible to compare questions in the sense of their *answerhood power*. Inspired by [4] and [14, section 5.2.3] we will examine the relationship of "giving an answer" of one question to another, which is a generalisation of Kubiński's term "weaker question".

Our aim is to study erotetic consequence relations in a very general manner, independently of the logic behind. The definitions of IEL consequences are based on the semantic entailment and the model approach relative to the chosen logical background.

1.4 Model-based approach

Let us introduce the set of all models for a declarative language as follows:

$$\mathcal{M}_{\mathsf{L}} = \{\mathbf{M} \mid \mathbf{M} \text{ is a (semantic) } \textit{model} \text{ for } \mathcal{L}\}.$$

The term *model* varies dependently on a background logic L. If L is classical propositional logic (CPL, for short), then $\mathcal{M}_{CPL}$ is a set of all *valuation*. In case of predicate logic it is a set of all *structures* with a realizations of non-logical symbols. Because of the possibility of adding some other constraints for models we will deal with (e.g., finiteness, preferred models, etc.), let us generally use a set $\mathcal{M} \subseteq \mathcal{M}_{\mathsf{L}}$. If necessary, the background logic and restrictions posed on models will be stated explicitely.

Speaking about *tautologies* of a logic L we mean the set of formulas

$$\textsc{Taut}_{\mathsf{L}} = \{\varphi \mid (\forall \mathbf{M} \in \mathcal{M}_{\mathsf{L}})(\mathbf{M} \models \varphi)\}.$$

If a restricted set of models $\mathcal{M}$ is in use, we speak about $\mathcal{M}$-*tautologies*

$$\textsc{Taut}_{\mathsf{L}}^{\mathcal{M}} = \{\varphi \mid (\forall \mathbf{M} \in \mathcal{M})(\mathbf{M} \models \varphi)\}.$$

All semantic terms will be relativized to $\mathcal{M}$. Each declarative sentence φ (in $\mathcal{L}$) has its (restricted) set of models

$$\mathcal{M}^{\varphi} = \{\mathbf{M} \in \mathcal{M} \mid \mathbf{M} \models \varphi\}$$

and similarly for a set of sentences Γ

$$\mathcal{M}^{\Gamma} = \{\mathbf{M} \in \mathcal{M} \mid (\forall \gamma \in \Gamma)(\mathbf{M} \models \gamma)\}.$$

1.4.1 (Semantic) entailment

Let us recall the common (semantic) entailment relation. For any set of formulas Γ and any formula ψ:

$$\Gamma \models \psi \text{ iff } \mathcal{M}^{\Gamma} \subseteq \mathcal{M}^{\psi}.$$

In case $\Gamma = \{\varphi\}$ we write only $\varphi \models \psi$.

$$\varphi \models \psi \text{ iff } \mathcal{M}^{\varphi} \subseteq \mathcal{M}^{\psi}$$

Now, we introduce *multiple-conclusion* entailment (mc-entailment, for short).

$$\Gamma \mid\!\models \Delta \text{ iff } \mathcal{M}^\Gamma \subseteq \bigcup_{\delta \in \Delta} \mathcal{M}^\delta$$

If $\mathcal{M}^\Gamma = \mathcal{M}^\Delta$, let us write $\Gamma \equiv \Delta$.[8]

Mc-entailment is reflexive ($\Gamma \mid\!\models \Gamma$), but it is neither symmetric nor transitive relation:

Example 1 *Let* $\Gamma \subseteq \textsc{Taut}_{\mathsf{L}}$, Δ *be a set of sentences containing at least one tautology and at least one contradiction, and* Σ *be such that* $\bigcup_{\sigma \in \Sigma} \mathcal{M}^\sigma \subset \mathcal{M}_{\mathsf{L}}$. *Then* $\Gamma \mid\!\models \Delta$ *and* $\Delta \mid\!\models \Sigma$, *but* $\Gamma \mid\!\not\models \Sigma$.

Entailment is definable by mc-entailment:

$$\Gamma \models \varphi \text{ iff } \Gamma \mid\!\models \{\varphi\}$$

On the other hand, mc-entailment is not definable by entailment. In this context, the following theorem could be surprising at the first sight.[9]

Theorem 1 *Entailment (for* L*) is compact iff mc-entailment (for* L*) is compact.*

The proof can be found in [14, pp. 109–110].

1.5 Basic properties of questions

As soon as we have introduced the term *question* and the model-based approach, we can mention some basic properties of questions. The term *soundness* is one of the most important terms in IEL.

Definition 1 *A question* Q *is* sound in $\mathbf{M}$ *iff* $\exists \alpha \in dQ$ *such that* $\mathbf{M} \models \alpha$.

For all IEL consequnce relations, it is important to state the soundness of a question with respect to a set of declaratives.

Definition 2 *A question* Q *is* sound relative to Γ *iff* $\Gamma \mid\!\models dQ$.

The sum of all classes of models of each direct answer α, i.e., $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$, is called *semantic range* of a question Q. Considering semantic range, our liberal approach admits some strange questions; one of them is a *completely contradictory question* that has only contradictions in its set of direct answers, its semantic range beeing just $\emptyset$. Another type is a question with a tautology among its direct answers, then the semantic range expands to the whole $\mathcal{M}$. Questions with such a range are called *safe.*[10] Of course, it need not be any tautology among direct answers for to be a safe question.

[8] In case of the semantic equivalence of formulas φ and ψ it will be only written $\varphi \equiv \psi$. On the other hand, two different sets of models do not imply the existence of two different sets of sentences (in $\mathcal{L}$).

[9] We say that mc-entailment is compact iff for each $\Gamma \mid\!\models \Delta$ there are finite subsets $G \subseteq \Gamma$ and $D \subseteq \Delta$ such that $G \mid\!\models D$.

[10] This term originates from Nuel Belnap.

Definition 3 • *A question Q is* safe *iff* $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha = \mathcal{M}$.

- *A question Q is* risky *iff* $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subset \mathcal{M}$.

Questions $?\alpha$, $?|\alpha, \beta|$ are safe in CPL, but neither is safe in Bochvar logic. $?\{\alpha, \beta\}$ is risky in CPL. Neither $?\alpha$ nor $?|\alpha, \beta|$ are safe in intuitionistic logic, but there are safe questions in this logic; just each question with at least one tautology among direct answers. Simple yes-no questions are safe in logics that accept the law of excluded middle.

It is good to emphasize that the set of direct answers of a safe question is mc-entailed by every set of declaratives. On the other hand, knowing a question to be sound relative to every set of declaratives implies its safeness.

Fact 1 *Q is safe iff* $(\forall \Gamma)(\Gamma \mid\models dQ)$.

Specially, safe questions are sound relative to $\Gamma = \emptyset$.

1.6 The road we are going to take

Since a background logic determines the properties of entailment, we do not want to pose any restriction on it. This model-based approach was inspired by *minimal erotetic semantics* from [16].

After introducing evocation and presupposition in section 2, we will show the role of maximal and prospective presuppositions in the relationship to semantic range of questions. Some classes of questions will be based on it. One could be surprised that we are not going to discuss answers in this section; in fact, there is not much to say about them. It turns out that various types of answers do not play any special role in the inferential structures.

Section 3 is crucial from the chosen viewpoint. We investigate erotetic implication and reducibility there. An important part is devoted to a discussion of the role of an auxiliary set of declaratives. We will demonstrate some variants of erotetic implication and their properties. All three consequence relations will be studied under their mutual influence and the relation "giving answer" will be stressed.

Questions will be considered as independent structures not being combined by logical connectives. Going through the definitions of erotetic implication and reducibility we can recognize their "both-sidedness" and just reducibility can substitute such combination of questions.

2 Questions and declaratives

In this section, we introduce two terms: *evocation* and *presupposition*. The first one will provide a consequence relation between a set of declaratives and a question. The second one is an important term in almost all logics of questions. In IEL there are some classes of questions based on it.

2.1 Evocation

Consider the following example: after a lecture, we expect a lecturer to be ready to answer some questions that were *evoked* by his or her talk. Thus, evocation seems to be the most obvious relationship among declarative sentences and questions. (Of course, next to the connection *question—answer*.) Almost every information can give rise to a question. What is the aim of such a question?

First, it should complete our knowledge in some direction. Asking a question we want to get more then by the conclusion based on a background knowledge. A question Q should be *informative relative to* Γ, it means, there is no direct answer to Q which is a conclusion of Γ.

Second, after answering an evoked question, no matter how, the answer must be consistent with the evoking knowledge. Moreover, *transmission of truth into soundness* is required: if an evoking set of declaratives has a model, there must be at least one direct answer of the evoked question that is true in this model. An evoked question should be sound relative to an evoking set of declaratives (see Definition 2).[11]

The definition of *evocation* is based on the previous two points (cf. [14, 16]). A question Q is evoked by a set of declaratives Γ if Q is sound and informative relative to Γ.

Definition 4 *A set of declarative sentences* Γ evokes *a question* Q *(let us write* $\Gamma \models Q$*) iff*

1. $\Gamma \mid\models dQ$,
2. $(\forall \alpha \in dQ)(\Gamma \not\models \alpha)$.

In our model-based approach we can rewrite both conditions this way:

1. $\mathcal{M}^\Gamma \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$
2. $(\forall \alpha \in dQ)(\mathcal{M}^\Gamma \not\subseteq \mathcal{M}^\alpha)$

In some special cases (e.g., dQ is finite or entailment is compact) we can define evocation without the link to mc-entailment. The first condition is of the form: there are $\alpha_1, \ldots, \alpha_n \in dQ$ such that $\Gamma \models \bigvee_1^n \alpha_i$.

Evocation yields some clear and useful properties of both a set of declaratives and an evoked question. The following fact lists some of them.

Fact 2 *If* $\Gamma \models Q$*, then*

- Γ *is not a contradictory set,*
- *there is no tautology in* dQ*, and*
- Q *is not a completely contradictory question.*

[11] For now, as we do not discuss epistemic issues, we shall not use the word "knowledge" but use the phrases "set of declarative(s) (sentences)" or "database" instead.

However, by Fact 1, we obtain a less intuitive conclusion: every safe question is evoked by any Γ that does not entail any direct answer to it. It underlines the special position of safe questions and their semantic range. When we restrict the definition of evocation to risky questions only, we get the definition of *generation*. [14, chapter 6]

Generation does not solve all problems with irrelevant and inefficient evoked questions either. We can accept another restriction to avoid questions that have direct answers which are incompatible with declaratives in Γ. Borrowing an example from [2], $\Gamma = \{\alpha \vee \beta, \gamma\}$ evokes also $?\{\alpha, \beta, \neg\gamma\}$. To eliminate this, the consistency of each direct answer with respect to Γ could be required, i.e., we could add the third condition to Definition 4:

$$(\forall \alpha \in dQ)(\mathcal{M}^{\Gamma} \cap \mathcal{M}^{\alpha} \neq \emptyset)$$

Some solutions of the problem of irrelevant and inefficient questions based on a semantics in the background are discussed in the just mentioned paper [2]. For our purpose we keep Definition 4 unchanged.

Back to safe questions, let us mention the following fact:

Fact 3 *If* $\emptyset \models Q$, *then* Q *is safe.*

As a conclusion of semantic definition of evocation we have the following expected behavior of evocation: semantically equivalent databases evoke the same questions.

Fact 4 *For every* Γ, Δ *and* Q, *if* $\Gamma \equiv \Delta$, *then* $\Gamma \models Q$ *iff* $\Delta \models Q$.

If mc-entailment is compact and $\Gamma \models Q$, then we are still not allowed to conclude that there is a finite subset $\Delta \subseteq \Gamma$ such that $\Delta \models Q$. See the first item in the followin fact.

Fact 5 *If* $\Gamma \models Q$ *and* $\Delta \subseteq \Gamma \subseteq \Sigma$, *then*

- $\Delta \models Q$ *if* $\Delta \mid\models dQ$,
- $\Sigma \models Q$ *if* $(\forall \alpha \in dQ)(\Sigma \not\models \alpha)$.

The second item points out the non-monotonicity of evocation (in declaratives). Considering questions as sets of answers, evocation is non-monotonic in interrogatives as well, see section 3.3.

Fact 6 *If* $\Gamma \models Q$ *and the entailment is compact, then* $\Delta \models Q_1$ *for some finite subset* dQ_1 *of* dQ *and some finite subset* Δ *of* Γ.

These and some more properties of evocation (and generation) are discussed in the book [14].

2.2 Presuppositions

Many properties of questions are based on the term *presupposition*. Everyone who has attended a basic course of research methods in social sciences has heard of importance to consider presuppositions of a question in questionaries.

What is presupposed must be valid under each answer to a question. Moreover, an answer to a question should bring at least the same information as presupposition does. The following definition (originally given by Nuel Belnap) is from [14].

Definition 5 *A declarative formula* φ *is a* presupposition *of a question* Q *iff* $(\forall \alpha \in dQ)(\alpha \models \varphi)$.

A presupposition of a question is entailed by each direct answer to the question. Let us write $\mathsf{Pres}Q$ for the set of all presuppositions of Q.

At the first sight, the set $\mathsf{Pres}Q$ could contain a lot of sentences. Let us have a question $Q = ?\{\alpha_1, \alpha_2\}$, the set of presuppositions (e.g., in CPL) contains $(\alpha_1 \vee \alpha_2)$, $(\alpha_1 \vee \alpha_2 \vee \varphi)$, $(\alpha_1 \vee \alpha_2 \vee \neg\varphi)$, etc. Looking at the very relevant member $(\alpha_1 \vee \alpha_2)$ it is useful to introduce the concept of *maximal presupposition*. Formula $(\alpha_1 \vee \alpha_2)$ entails each presupposition of the question Q.

Definition 6 *A declarative formula* φ *is a* maximal presupposition *of a question* Q *iff* $\varphi \in \mathsf{Pres}Q$ *and* $(\forall \psi \in \mathsf{Pres}Q)(\varphi \models \psi)$.

The model-theoretical view brings it about in a direct way. The definition of presupposition gives $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \mathcal{M}^\varphi$, for each $\varphi \in \mathsf{Pres}Q$, which means

$$\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcap_{\varphi \in \mathsf{Pres}Q} \mathcal{M}^\varphi = \mathcal{M}^{\mathsf{Pres}Q}$$

and the set $\mathcal{M}^{\mathsf{Pres}Q}$ is a model-based counterpart to the definition of maximal presuppositions.

If the background logic has tautologies, each of them is in $\mathsf{Pres}Q$.

$$\textsc{Taut}_{\mathsf{L}}^{\mathcal{M}} \subseteq \mathsf{Pres}Q$$

Considering safe questions we get

Fact 7 *If* Q *is safe, then* $\mathsf{Pres}Q = \textsc{Taut}_{\mathsf{L}}^{\mathcal{M}}$.

This fact says that if Q is safe, then $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha = \mathcal{M}^{\mathsf{Pres}Q}$. In classical propositional logic the disjunction of all direct answers of a question is a presupposition of this question and if $\mathsf{Pres}Q = \textsc{Taut}_{\mathsf{CPL}}^{\mathcal{M}}$, then Q is safe. This evokes a (meta)question whether the implication from right to left is valid. If Q is not safe, then we know that $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$ is a proper subset of $\mathcal{M}$. But what about $\mathcal{M}^{\mathsf{Pres}Q}$? After introducing a class of *normal questions* in section 2.2.1 it will be valid $\mathcal{M}^{\mathsf{Pres}Q} \subset \mathcal{M}$ as well as the implication from right to left (see Fact 9).

A presupposition can be seen as an information which is announced by asking a question, without answering it. Such information is relatively small. The semantic range of all maximal presuppositions is wider than the range of a question. Looking at finite CPL example where the disjunction of all direct answers forms just the semantic range of the question brings us to the idea of *prospective presupposition*. It is a presupposition which a question Q is sound relative to.

Definition 7 *A declarative formula* φ *is a* prospective presupposition *of a question* Q *iff* $\varphi \in \mathsf{Pres}Q$ *and* $\varphi \mid\models dQ$*. Let us write* $\varphi \in \mathsf{PPres}Q$*.*

All prospective presuppositions of a question are equivalent:

Lemma 1 *If* $\varphi, \psi \in \mathsf{PPres}Q$*, then* $\varphi \equiv \psi$*.*

Proof If $\mathbf{M} \models \varphi$, then there is $\alpha \in dQ$ such that $\mathbf{M} \models \alpha$. Since $\psi \in \mathsf{Pres}Q$, $\alpha \models \psi$ and it gives $\mathbf{M} \models \psi$. We got $\varphi \models \psi$.

The proof of $\psi \models \varphi$ can be done the same way. QED

A prospective presupposition forms exactly the semantic range of a question.

$$\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \mathcal{M}^{\mathsf{PPres}Q}$$

If Q has a prospective presupposition φ, it can be understood as the "strongest" presupposition because φ entails each presupposition of Q.

Two questions with the same sets of presuppositions have the same prospective presuppositions.

Lemma 2 *If* $\mathsf{Pres}Q = \mathsf{Pres}Q_1$ *and both* $\mathsf{PPres}Q$ *and* $\mathsf{PPres}Q_1$ *are not empty, then* $\mathsf{PPres}Q = \mathsf{PPres}Q_1$*.*

Proof We show that if $\varphi \in \mathsf{PPres}Q$ and $\psi \in \mathsf{PPres}Q_1$, then $\varphi \equiv \psi$.

$\varphi \in \mathsf{PPres}Q$ implies $\mathcal{M}^{\varphi} = \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$ and $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} \subseteq \mathcal{M}^{\psi}$, because $\psi \in \mathsf{Pres}Q$. It gives $\mathcal{M}^{\varphi} \subseteq \mathcal{M}^{\psi}$ and $\varphi \models \psi$.

The proof that $\psi \models \varphi$ is similar. QED

Presuppositions of evoked questions are entailed by the evoking set of declaratives.

Fact 8 *If* $\Gamma \models Q$*, then* $\Gamma \models \varphi$*, for each* $\varphi \in \mathsf{Pres}Q$*.*

The implication from right to left does not hold. If we only know $\mathcal{M}^{\Gamma} \subseteq \mathcal{M}^{\mathsf{Pres}Q}$, we are not sure about $\mathcal{M}^{\Gamma} \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$ as required by the first condition of evocation. Clearly, the informativeness must be ensured as well. Let us note that it cannot be improved by replacing of $\mathsf{Pres}Q$ by $\mathsf{PPres}Q$. We will return to this in the next subsection at the topic of normal questions. To sum up all general conditions of an evoked question (by Γ) and its presuppositions let us look at this diagram:

$$\boxed{\mathcal{M}^{\Gamma} \quad \subseteq \quad \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \quad \mathcal{M}^{\mathsf{PPres}Q} \quad \subseteq \quad \mathcal{M}^{\mathsf{Pres}Q}}$$

2.2.1 Classes of questions based on presuppositions

Using the term *presupposition* we can define some classes of questions.[12]

Normal questions A question Q is called *normal* if it is sound relative to its set of presuppositions ($\mathsf{Pres}Q \mid\models dQ$).

- $Q \in \textsc{normal}$ iff $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \mathcal{M}^{\mathsf{Pres}Q}$

Model-based approach introduces normal questions as questions with semantic range delimitated by models of maximal presuppositions. Working with finite sets of direct answers and in logical systems with the "classical" behavior of disjunction (each direct answer entails the disjunction of all direct answers) we do not leave the class NORMAL. Non-normal questions can be found in classical predicate logic.

Two announced facts follow. They continue on the discussions at Fact 7 and Fact 8.

Fact 9 *If* $\mathsf{Pres}Q = \textsc{Taut}_{\mathsf{L}}^{\mathcal{M}}$ *and* Q *is normal, then* Q *is safe.*

Let us only add the clear fact, that the class of safe questions is a subset of the class of normal questions ($\textsc{safe} \subseteq \textsc{normal}$).

Fact 10 *If* $\Gamma \models \varphi$*, for each* $\varphi \in \mathsf{Pres}Q$*, and* $\Gamma \not\models \alpha$*, for each* $\alpha \in dQ$ *of a normal question* Q*, then* $\Gamma \models Q$*.*

Regular questions Each question with the non-empty set of prospective presuppositions is *regular*.

- $Q \in \textsc{regular}$ iff $(\exists \varphi \in \mathsf{Pres}Q)(\varphi \mid\models dQ)$

Regularity of Q gives $\mathcal{M}^{\mathsf{Pres}Q} \subseteq \mathcal{M}^{\varphi} \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$ and it holds

$$\textsc{regular} \subseteq \textsc{normal}$$

If entailment is compact, both classes are equal.

Normal questions are sound relative to $\mathsf{Pres}Q$ and regular questions are sound relative to $\mathsf{PPres}Q$. The following example shows an expected fact that it is still not sufficient for evocation.

Example 2 (in CPL) *Let* $Q = ?\{(\alpha \vee \beta), \alpha\}$*. This question is normal and regular, the formula* $(\alpha \vee \beta)$ *is a prospective presupposition of* Q*, but* $\mathsf{Pres}Q \not\models Q$*.*

If there is a set of declaratives Γ such that $\Gamma \models Q$, then normal (regular) questions are sound as well as informative relative to $\mathsf{Pres}Q$ ($\mathsf{PPres}Q$). This is summed up by

Lemma 3 *Let* $\Gamma \models Q$*, for some set of declaratives* Γ*. Then*

[12] Names and definitions of the classes are from [14].

1. *$Q \in$ NORMAL implies* $\mathsf{Pres}Q \models Q$.

2. *$Q \in$ REGULAR implies $\varphi \models Q$, for $\varphi \in \mathsf{PPres}Q$.*

Proof For the first item, only informativeness (relative to $\mathsf{Pres}Q$) must be showed. But if it is not valid, then Fact 8 causes non-informativeness of Q relative to Γ.

The second item is provable by the same idea. QED

Self-rhetorical questions Another special class of questions are *self-rhetorical* questions. They have at least one direct answer entailed by the set of presuppositions.

- $Q \in$ SELF-RHETORICAL iff $(\exists\alpha \in dQ)(\mathsf{Pres}Q \models \alpha)$

From this definition, it is clear that self-rhetorical questions are normal. However, do we ask such questions? This class includes such strange questions as *completely contradictory* questions that have only contradictions in the set of direct answers, and questions with tautologies among direct answers.

An evoked question is not of this kind.

Lemma 4 *If there is Γ such that $\Gamma \models Q$, then Q is not self-rhetorical.*

Proof From Fact 8. QED

Proper questions Normal and not self-rhetorical questions are called *proper*. Proper questions are evoked by their set of presuppositions.

- $Q \in$ PROPER iff $\mathsf{Pres}Q \models Q$

Evoked normal questions are proper (compare both Lemma 3 and Lemma 4).

3 Questions and questions

This section is devoted to various inferential structures in which questions appear on both sides (erotetic implication, reducibility of questions to sets of questions) and to relations between two questions based on their sets of direct answers. This second point focuses on "answerhood power" of questions formalized by set-of-answers methodology.

3.1 Erotetic implication

Now, we extend the class of inferences by "implication" between two questions with a possible assistance of some set of declaratives. Let us start with an easy and a bit tricky example. If I ask

Q: What is Peter a graduate of: faculty of law or faculty of economy?

then I can be satisfied by the answer

He is a lawyer.

even if I did not ask

Q_1: What is Peter: lawyer or economist?

The connection between both questions could be shown by the following set of declaratives:

Someone is a graduate of a faculty of law iff he/she is a lawyer.
Someone is a graduate of a faculty of economy iff he/she is an economist.

The first question Q can be formalized by $?\{\alpha_1, \alpha_2\}$ and the latter one, speaking of Peter's position, can be $?\{\beta_1, \beta_2\}$. Looking at the questions there is no connection between them. The relationship is based on the set of declaratives $\Gamma = \{(\alpha_1 \leftrightarrow \beta_1), (\alpha_2 \leftrightarrow \beta_2)\}$. Now, we say that Q *implies* Q_1 on the basis of Γ and write $\Gamma, Q \models Q_1$.

This relation is called *erotetic implication* (*e-implication*, for short) and the following definition is from [14].

Definition 8 *A question Q implies a question Q_1 on the basis of a set of declaratives Γ iff*

1. $(\forall \alpha \in dQ)(\Gamma \cup \alpha \mid\models dQ_1)$,
2. $(\forall \beta \in dQ_1)(\exists \Delta \subset dQ)(\Delta \neq \emptyset$ *and* $\Gamma \cup \beta \mid\models \Delta)$.

Returning to the introductory example, both questions are even *erotetically equivalent* with respect to Γ: $\Gamma, Q \models Q_1$ as well as $\Gamma, Q_1 \models Q$.

The definition requires a little comment. The first clause should express the soundness of an implied question relative to each extension of Γ by $\alpha \in dQ$. This *transmission of truth/soundness into soundness* has the following meaning: if there is a model of Γ and a direct answer to Q, then there must be a direct answer to Q_1 that is valid in this model. If Q_1 is safe, then this condition is always valid (see Fact 1).

The second clause requires direct answers to Q_1 to be *cognitively useful* in restricting the set of direct answers of the implying question Q.

In comparison with evocation, the role of the set of declaratives is a bit different. Γ plays, especially, the auxiliary role; e-implication is monotonic in declaratives and it gives the following [14, p. 173]:

Fact 11 *Let $\Gamma, Q \models Q_1$. Then $\Delta, \Gamma, Q \models Q_1$, for any set of declaratives Δ.*

This could be called *weakening in declaratives*. From this, it is clear that $\bot, Q \models Q_1$, for each Q and Q_1.

We wil say a word or two about auxiliary sets of declaratives in the next subsection.

3.1.1 Pure erotetic implication

Pure e-implication is e-implication with the empty set of declaratives. In our semantic approach, Γ includes only tautologies of a chosen logical system. From Fact 11, whenever two questions are in the relation of pure e-implication, then they are in the relation of e-implication for each set of declaratives.

If $Q \models Q_1$, then (pure) e-implication says that both questions have the same semantic range.

Lemma 5 *If $Q \models Q_1$, then $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha = \bigcup_{\beta \in dQ_1} \mathcal{M}^\beta$.*

Proof From the first condition of Definition 8

$$\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_1} \mathcal{M}^\beta$$

and from the second one

$$\bigcup_{\beta \in dQ_1} \mathcal{M}^\beta \subseteq \bigcup_{\Delta} \bigcup_{\alpha \in \Delta} \mathcal{M}^\alpha \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha.$$

QED

From this we can conclude that classes of safe and risky questions are closed under pure e-implication for both implied and implying questions.

Fact 12 *If $Q \models Q_1$, then Q is safe (risky) iff Q_1 is safe (risky).*

The same semantic range of questions linked together by pure e-implication does not form an equivalence relation on questions (see non-symmetry in Example 4 and non-transitivity in Example 5 in subsection 3.1.3). On the other hand, pure e-implication has some important conclusions for classes of presuppositions. [14, p. 184]

Lemma 6 *If $Q \models Q_1$, then $\mathsf{Pres}Q = \mathsf{Pres}Q_1$.*

Proof First, let us prove $\mathsf{Pres}Q \subseteq \mathsf{Pres}Q_1$. Let $\varphi \in \mathsf{Pres}Q$, so $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \mathcal{M}^\varphi$. Simultaneously, we know that from the second condition of the definition of pure e-implication there is a non-empty $\Delta \subset dQ$, for each $\beta \in dQ_1$, such that $\mathcal{M}^\beta \subseteq \bigcup_{\alpha \in \Delta} \mathcal{M}^\alpha \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$. Thus, $\mathcal{M}^\beta \subseteq \mathcal{M}^\varphi$, for each $\beta \in dQ_1$.

Second, for proving $\mathsf{Pres}Q_1 \subseteq \mathsf{Pres}Q$ suppose $\varphi \in \mathsf{Pres}Q_1$. The following inclusions are valid $\mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_1} \mathcal{M}^\beta \subseteq \mathcal{M}^\varphi$, for each $\alpha \in dQ$. QED

The claim of Lemma 6 is not extendable to the general term of e-implication (cf. Example 3).

On this lemma we can base the following statement about an influence of pure e-implication on classes of normal and regular questions.

Theorem 2 *If $Q \models Q_1$, then Q is normal iff Q_1 is normal.*

Proof If Q is normal, then

$$\bigcup_{\beta \in dQ_1} \mathcal{M}^{\beta} = \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \mathcal{M}^{\mathsf{Pres}Q} = \mathcal{M}^{\mathsf{Pres}Q_1}$$

The first equation is from Lemma 5, the second one is from the normality of Q, and the third one is from Lemma 6. QED

If we recall Lemma 2, then it is easy to say a similar fact for regular questions.

Theorem 3 *If $Q \models Q_1$, then Q is regular iff Q_1 is regular.*

Both theorems have similar results we have got for safe (risky) questions in Fact 12. Classes of normal and regular questions are closed to pure e-implication. Normal (regular) questions purely imply only normal (regular) questions and they are purely implied by the same kind of questions. [14, pp. 185–186]

Concerning classes of questions in relationship with e-implication, let us add that whenever $Q \models Q_1$, then Q is completely contradicory question iff Q_1 is.

Note on auxiliary sets of declaratives in e-implication Let us remind the introductory example on page 13 to emphasize the importance of declaratives for e-implication. Similarly, the following example will point out the role of implicitly and explicitly expressed presuppositions.

Example 3 (in CPL) *Let $Q_1 = ?\{\alpha, \beta, \gamma\}$ and $Q_2 = ?\{\alpha, \beta\}$, then neither $Q_1 \models Q_2$ nor $Q_2 \models Q_1$ (see the different semantic ranges of both questions). On the other hand, if we would know that it must be $(\alpha \vee \beta)$, then $(\alpha \vee \beta), Q_1 \models Q_2$.*

Keeping the context of this example: the question Q_2 is normal as well as regular, then $\mathsf{PPres}Q_2 \mid\models dQ_2$ and, in addition, there is Δ, non-empty proper subset of dQ_1, such that $\mathsf{PPres}Q_2 \mid\models \Delta$. It gives $\mathsf{PPres}Q_2, Q_1 \models Q_2$. If the set $\mathsf{PPres}Q_2$ is *explicitly* expressed, the implication from Q_1 to Q_2 is justified.

But now, back to the general approach. In the following fact we summarize when we can say that two questions and a set of declaratives are in the relationship of e-implication.

Fact 13 *Let us have Γ and two questions Q_1 and Q_2. In order to conclude $\Gamma, Q_1 \models Q_2$ it is sufficient to have $\Gamma \mid\models dQ_2$ and $\Gamma \mid\models \Delta$, where Δ is a non-empty proper subset of dQ_1.*

This fact can be formulated in this form: if Q_2 is sound relative to Γ and Γ gives a partial answer to Q_1, then Q_1 implies Q_2 with respect to Γ. We will add some points to this discussion in sections 3.2 and 3.4.2.

3.1.2 Regular erotetic implication

A special kind of e-implication arises if there is exactly one direct answer in each Δ, then we say that Q *regularly* implies Q_1 (on the basis of Γ). The following definition originates from [16].

Definition 9 *A question Q* regularly implies *a question Q_1 on the basis of a set of declaratives Γ iff*

1. $(\forall\alpha \in dQ)(\Gamma \cup \alpha \mid\models dQ_1)$,
2. $(\forall\beta \in dQ_1)(\exists\alpha \in dQ)(\Gamma \cup \beta \models \alpha)$.

Because of the special importance of this relation let us use the symbol $\vdash$ for it (so we write $\Gamma, Q \vdash Q_1$).

In the case of *pure regular e-implication*, both conditions are changed into the form:

1. $(\forall\alpha \in dQ)(\alpha \mid\models dQ_1)$,
2. $(\forall\beta \in dQ_1)(\exists\alpha \in dQ)(\beta \models \alpha)$.

If $Q \vdash Q_1$ such that we can answer Q_1, then we have an answer to Q. The relationship of pure regular e-implication between two questions says that the implied question is "stronger" than the implying one in the sense of answerhood (see also section 3.3).

Regularity can be enforced by the minimal number of direct answers of an implying question: if $Q \models Q_1$ and $|dQ| = 2$, then $Q \vdash Q_1$.

3.1.3 Basic properties of erotetic implication

In this subsection, we are going to be interested in such properties as reflexivity, symmetry, and transitivity of e-implication.

Erotetic implication is a reflexive relation.

Fact 14 *$\Gamma, Q \models Q$, for each Γ and Q.*

Even if there are examples of the symmetric behavior of e-implication, it is not a symmetric relation, generally.

Example 4 (in CPL) *Let $Q_1 = ?\{(\alpha \vee \beta), \alpha\}$ and $Q_2 = ?\{\alpha, \beta\}$. Then $Q_1 \models Q_2$, but $Q_2 \not\models Q_1$.*

In this example there is no non-empty proper subset of dQ_2 for the formula $(\alpha \vee \beta)$ to fulfil the second condition in the definition of e-implication. Moreover, it is useful to add that Q_1 regularly implies Q_2.

Erotetic implication is not transitive either.

Example 5 (in CPL) *$?(\alpha \wedge \beta) \vdash ?|\alpha, \beta|$ and $?|\alpha, \beta| \models ?\alpha$, but $?(\alpha \wedge \beta) \not\models ?\alpha$.*

On the other hand, if we consider regular e-implication only, the following theorem is valid.

Theorem 4 *If* $Q_1 \vdash Q_2$ *and* $Q_2 \vdash Q_3$*, then* $Q_1 \vdash Q_3$*.*

Proof The first condition of Definition 9 is done by Lemma 5.

The second clause of this definition is based on regularity that gives $(\forall\gamma \in dQ_3)(\exists\alpha \in dQ_1)(\mathcal{M}^\gamma \subseteq \mathcal{M}^\alpha)$. QED

We can do a cautious strengthening by the following fact:

Fact 15 *If* $\Gamma, Q_1 \vdash Q_2$ *and* $Q_2 \vdash Q_3$*, then* $\Gamma, Q_1 \vdash Q_3$*.*

As a final remark, let us add that presuppositions of an implied question are entailed by each direct answer of an implying question (with respect to an auxiliary set of declaratives).

Fact 16 *Let* $\Gamma, Q \models Q_1$*. Then*

1. $(\forall\alpha \in dQ)(\forall\varphi \in \mathsf{Pres}Q_1)(\Gamma \cup \alpha \models \varphi)$
2. *If e-implication is regular, then* $(\forall\beta \in dQ_1)(\forall\varphi \in \mathsf{Pres}Q)(\Gamma \cup \beta \models \varphi)$*.*

3.2 Evocation and erotetic implication

Both types of inferential structures can appear together and we are going to investigate their interaction.

As shown in the next example, e-implication does not preserve evocation. If we know $\Gamma \models Q_1$ and $Q_1 \models Q_2$, it does not mean that it must be $\Gamma \models Q_2$.

Example 6 (in CPL)

- $\{(\alpha \vee \beta)\} \models ?|\alpha,\beta|$ *and* $?|\alpha,\beta| \models ?(\alpha \vee \beta)$*, but* $\{(\alpha \vee \beta)\} \not\models ?(\alpha \vee \beta)$*.*
- $\{\alpha\} \models ?|\alpha,\beta|$ *and* $\{\alpha\}, ?|\alpha,\beta| \models ?\alpha$*, but there is an answer to* $?\alpha$ *in* $\{\alpha\}$*.*

Of course, we do not see anything pathological in this example. Knowing $(\alpha \vee \beta)$, resp. α, it is superfluous to ask $?(\alpha \vee \beta)$, resp. $?\alpha$.

Generally speaking, this brought us back to the role of an auxiliary set of declaratives in e-implication. Due to the admissibility of *weakening in declaratives* (Fact 11) we can arrive at structures of e-implications with Γ containing (direct) answers to some of the two questions. On the other hand, there are some solutions of this problem proposed by erotetic logicians.[13]

In contrast to the previous example, we can prove that evocation carries over through a regular e-implication.

Lemma 7 *If* $\Gamma \models Q_1$ *and* $Q_1 \vdash Q_2$*, then* $\Gamma \models Q_2$*.*

[13] See, for example, the definition of *strong e-implication* given by Wiśniewski in [14]. Fact 13 was the inspiration for the definition of strong e-implication. The definition is the same as that of e-implication, but $\Gamma \not\models \Delta$ is added into the second clause.

Proof The first condition requires $\mathcal{M}^\Gamma \subseteq \bigcup_{\beta \in dQ_2} \mathcal{M}^\beta$. It is valid because of the same semantic range of both questions.

Let us suppose that there is $\beta \in dQ_2$ entailed by Γ. Then $\mathcal{M}^\Gamma \subseteq \mathcal{M}^\beta$ and, thanks to regularity of e-implication, $\mathcal{M}^\Gamma \subseteq \mathcal{M}^\alpha$, for some $\alpha \in dQ_1$. But it is in contradiction with $\Gamma \models Q_1$. QED

Lemma 7 can be formulated not only in the version of pure regular e-implication.

Theorem 5 *If* $\Gamma \models Q_1$ *and* $\Gamma, Q_1 \vdash Q_2$*, then* $\Gamma \models Q_2$*.*

Proof First, we prove $\Gamma \mid\models dQ_2$. Supposing it is not true, then there is a model $\mathbf{M}_0$ of Γ such that $\mathbf{M}_0 \not\models \beta$, for each $\beta \in dQ_2$. Because of $\Gamma \mid\models dQ_1$, there is $\alpha_0 \in dQ_1$ and $\mathbf{M}_0 \models \alpha_0$. From $\Gamma \cup \alpha \mid\models dQ_2$, for each $\alpha \in dQ_1$, there must be some $\beta_0 \in dQ_2$ such that $\mathbf{M}_0 \models \beta_0$ and that is a contradiction.

Secondly, let us suppose that there is $\beta_0 \in dQ_2$ and $\Gamma \models \beta_0$. Regularity and second condition of e-implication give $\Gamma \cup \beta_0 \models \alpha$ and it follows $\Gamma \models \alpha$ that is in contradiction with $\Gamma \models Q_1$. QED

Since regularity was used only in the second part of the proof, we get an expected fact that $\Gamma \models Q_1$ and $\Gamma, Q_1 \vdash Q_2$ gives soundness of an implied question Q_2 relative to Γ.[14]

At the first sight, it need not be $\Gamma, Q_1 \models Q_2$ or $\Gamma, Q_2 \models Q_1$ if we only know that $\Gamma \models Q_1$ as well as $\Gamma \models Q_2$.[15] Generally, neither evocation nor e-implication says something new about structures of engaged questions. Nevertheless, we can expect that some clearing up of the structure of sets of direct answers could be helpful for the study of inferences. This will be discussed in the next section.

3.3 Comparing questions—relations of questions based on direct answers

So far we have introduced inferences that can produce certain relations between questions. However, it would be useful to be able to compare questions with respect to their "answerhood power". The chosen set-of-answers methodology brings us to a natural approach.

Let us start with relations among questions based on pure comparison of sets of direct answers.[16]

Definition 10 • *Two questions are* equal *($Q_1 = Q_2$) iff they have the same set of direct answers ($dQ_1 = dQ_2$).*

[14]The second part of the proof of Theorem 5 could be slightly changed and we obtain that strong e-implication carries over as well.

[15]Let us take as an example (in CPL) the case $\{\varphi\} \models ?\alpha$ and $\{\varphi\} \models ?\beta$. Then neither $\{\varphi\}, ?\alpha \models ?\beta$ nor $\{\varphi\}, ?\beta \models ?\alpha$.

[16]The original definition refers to *equivalent* questions instead of *equal* (cf. [14]), but we use the first term for *erotetically equivalent* or semantically equivalent.

- *A question Q_1 is* included *in a question Q_2 ($Q_1 \subset Q_2$) iff $dQ_1 \subset dQ_2$.*

This approach could be extended in a semantic way. We say that (an answer) α *gives* an answer to a question Q iff there is $\beta \in dQ$ such that $\alpha \models \beta$. Having two questions Q_1 and Q_2 we can define a relationship of "giving answers":

Definition 11 *A question Q_1* gives *a (direct) answer to Q_2 iff $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\alpha \models \beta)$.*

In this definition the first question is considered as to be (semantically) "stronger" than the second one. For this we use the symbol $\geq$ ($Q_1 \geq Q_2$).

If $Q_1 = Q_2$ or $Q_1 \subset Q_2$, then $Q_1 \geq Q_2$ and, moreover, each direct answer to Q_1 not only gives an answer to Q_2 but also *is* a (direct) answer to Q_2, i.e., $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\alpha \equiv \beta)$.

The relation $\geq$ has a slightly non-intuitive consequence: a completely contradictory question is the strongest one. However, the class of evoked questions is not affected by this problem.

Let us note an expected fact—stronger questions presuppose more than weaker ones.

Fact 17 *If $Q_1 \geq Q_2$, then $\mathsf{Pres}Q_2 \subseteq \mathsf{Pres}Q_1$.*

This fact is not too useful. It is better to notice the relationship among maximal presuppositions. We have $\mathcal{M}^{\mathsf{Pres}Q_1} \subseteq \mathcal{M}^{\mathsf{Pres}Q_2}$. Each maximal presupposition of a stronger question entails a maximal presupposition of a weaker one, respectively, a prospective presuposition of a stronger question entails a prospective presupposition of a weaker question. The semantic range of a stronger question is included in the semantic range of a weaker question.

Fact 18 *If $Q_1 \geq Q_2$, then $\bigcup_{\alpha\in dQ_1} \mathcal{M}^\alpha \subseteq \bigcup_{\beta\in dQ_2} \mathcal{M}^\beta$.*

It follows that the set of safe questions is closed under weaker questions.

Fact 19 *If Q_1 is safe and $Q_1 \geq Q_2$, then Q_2 is safe.*

The next example shows that safeness of weaker questions is not transferred to stronger ones.

Example 7 (in CPL) $?\{\beta \wedge \alpha, \neg\beta\} \geq ?\beta$

3.3.1 Answerhood, evocation, and erotetic implication

We can show some results of evocation and e-implication based on properties of the $\geq$-relation. The first one is an obvious fact that an implied stronger question is regularly implied.

Lemma 8 *If $\Gamma, Q_1 \models Q_2$ and $Q_2 \geq Q_1$, then $\Gamma, Q_1 \vdash Q_2$.*

Recall what is required of the regular e-implication: $(\forall\beta \in dQ_2)(\exists\alpha \in dQ_1)(\Gamma, \beta \models \alpha)$. Then the lemma follows.

If the relation is converted, i.e., Q_1 gives an answer to Q_2, then whenever Q_1 implies Q_2, Q_2 regularly implies Q_1 (both with respect to Γ). Moreover, both questions are erotetically equivalent relative to Γ.

Theorem 6 *If* $\Gamma, Q_1 \models Q_2$ *and* $Q_1 \geq Q_2$*, then* $\Gamma, Q_2 \vdash Q_1$.

Proof First, we need to show that $\Gamma \cup \beta \mid\models dQ_1$, for each $\beta \in dQ_2$. But it is an easy conclusion from $\Gamma, Q_1 \models Q_2$ because there is a subset $\Delta \subseteq dQ_1$ for each $\beta \in dQ_2$ such that $\Gamma \cup \beta \mid\models \Delta$.

The second condition of regular e-implication is justified by the same way as it is in Lemma 8. QED

Now, as it was stated before, we are going to study the influence of "giving answers" on the relationship of evocation and e-implication. We know that, generally, if Γ evokes Q_1 and Q_2, it need not be that either Q_1 implies Q_2 or Q_2 implies Q_1 (with respect to Γ). If a stronger question is evoked by Γ, then every weaker question regularly implies this stronger one with respect to Γ.

Theorem 7 *If* $\Gamma \models Q_1$ *and* $Q_1 \geq Q_2$*, then* $\Gamma, Q_2 \vdash Q_1$.

Proof First, $\Gamma \cup \beta \mid\models dQ_1$ is required for each $\beta \in dQ_2$. We get $\Gamma \mid\models dQ_1$ from $\Gamma \models Q_1$.

Second, from $Q_1 \geq Q_2$ we have $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\alpha \models \beta)$ and it gives the second condition of regular e-implication $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\Gamma \cup \alpha \models \beta)$. QED

To digress for a moment, this repeated connection of $\geq$ and regular e-implication is not an accident. The definition of regular e-implication says that if $Q_1 \vdash Q_2$, then Q_2 gives an answer to Q_1, i.e., $Q_2 \geq Q_1$. However, "giving an answer" does not produce e-implication, see the next example.

Example 8 (in CPL) $?\{(\alpha\wedge\varphi), (\beta\wedge\psi)\} \geq ?\{\alpha, \beta\}$*, but neither* $?\{(\alpha\wedge\varphi), (\beta\wedge\psi)\} \models ?\{\alpha, \beta\}$ *nor* $?\{\alpha, \beta\} \models ?\{(\alpha \wedge \varphi), (\beta \wedge \psi)\}$.

To go back to evocation, it is clear that two equal questions are both evoked by a set of declaratives if one of them is evoked by this set. Generally, it is not sufficient to know $\Gamma \models Q_1$ and $Q_1 \geq Q_2$ to conclude $\Gamma \models Q_2$. An evoked stronger question only implies the soundness of weaker questions relative to Γ. Let us illustrate it in the case that the first question is included in the second one ($Q_1 \subset Q_2$); there could be a direct answer to Q_2 which is entailed by Γ. This reminds us of the non-monotonic behavior of evocation. Notice that $Q_2 \subset Q_1$ will not help us either. In the connection with the relation $\geq$ we have to require a version of an equality.[17]

[17] We are not going to introduce a special name for this relationship; it is included in the erotetic equivalence.

Fact 20 *Let* $Q_1 \geq Q_2$ *and* $Q_2 \geq Q_1$*. Then*

- $\Gamma \models Q_1$ *iff* $\Gamma \models Q_2$*,*
- $Q_1 \vdash Q_2$ *as well as* $Q_2 \vdash Q_1$*.*

3.3.2 Controlling the cardinality of sets of direct answers

The set of direct answers of a weaker question can be much larger than that of a stronger question. The book [14] introduces two relations originated from Tadeusz Kubiński that prevent this uncontrolled cardinality.

Definition 12 *A question* Q_1 *is stronger then* Q_2 *(*$Q_1 \succeq Q_2$*) iff there is a surjection* $j : dQ_1 \rightarrow dQ_2$ *such that for each* $\alpha \in dQ_1$*,* $\alpha \models j(\alpha)$*.*

The number of direct answers of the weaker question Q_2 does not exceed the cardinality of dQ_1, i.e., $|dQ_1| \geq |dQ_2|$. From the surjection, additionaly, we know that each direct answer of a weaker question is given by some direct answer to a stronger question. We have used the term *stronger* in a bit informal way for questions that "give an answer" to weaker ones. It is clear that if $Q_1 \succeq Q_2$, then $Q_1 \geq Q_2$. But, unfortunately, we cannot provide any special improvement of previous results for $\succeq$-relation. In particular, Examples 7 and 8 are valid for $\succeq$-relation as well.

The other definition corresponds to a both-way relationship of "being stronger".

Definition 13 *A question* Q_1 *is* equipollent *to a question* Q_2 *(*$Q_1 \equiv Q_2$*) iff there is a bijection* $i : dQ_1 \rightarrow dQ_2$ *such that for each* $\alpha \in dQ_1$*,* $\alpha \equiv i(\alpha)$*.*

In this case, both sets of direct answers have the same cardinality ($|dQ_1| = |dQ_2|$). Let us add expected results gained from equipollency.

Fact 21 *If* $Q_1 \equiv Q_2$*, then*

- *both* $Q_1 \succeq Q_2$ *and* $Q_2 \succeq Q_1$*,*
- $\Gamma \models Q_1$ *iff* $\Gamma \models Q_2$*,*
- $Q_1 \vdash Q_2$ *as well as* $Q_2 \vdash Q_1$*.*

Of course, two equal questions are equipollent.

3.3.3 Partial answerhood

We declared that the study of various types of answers (generally speaking, answerhood) is not the central point of this paper. However, we can utilize the idea evoked by the second clause of Definition 8. Narrowing down the set of direct answers of an implying question seems to be a good base for the term *partial answer*.

Definition 14 *A declarative* φ gives a partial answer *to a question* Q *iff there is a non-empty proper subset* $\Delta \subset dQ$ *such that* $\varphi \mid\models \Delta$.

This definition allows us to cover many terms from the concept of the answerhood. Every direct answer gives a partial answer. Whenever ψ gives a (direct) answer, then ψ gives a partial answer. As a useful conclusion we get a weaker version of Theorem 7:

Fact 22 *If* $\Gamma \models Q_1$ *and each* $\alpha \in dQ_1$ *gives a partial answer to* Q_2*, then* $\Gamma, Q_2 \models Q_1$.

3.4 Questions and sets of questions

Working in the classsical logic, let us imagine we would like to know whether it is the case that α or it is the case that β. The question $?\{\alpha, \beta\}$ is posed. But there could be a problem when a device, to which we are going to address this question, is not able to accept it. (This can be caused, e.g., by a restricted language-acceptability.) However, assume that there exist two devices such that: the first one can be asked by the question $?\alpha$, and the other one is able to work with the question $?\beta$. From both machines, independently, we can get the following pairs of answers: $\{\alpha, \beta\}$, $\{\neg\alpha, \beta\}$, $\{\alpha, \neg\beta\}$ or $\{\neg\alpha, \neg\beta\}$.

Posing the question $Q = ?\{\alpha, \beta\}$ we expect that if an answer to Q is true, then there must be a true answer to each question from the set $\{?\alpha, ?\beta\}$. Thus, we require soundness condition "from an initial question to a set of questions".

Generally speaking, let us suppose that there are a question Q and a set of questions $\phi = \{Q_1, Q_2, \ldots\}$. For each model of a direct answer to Q there must be a direct answer in each Q_i valid in this model.

$$(\forall \alpha \in dQ)(\forall Q_i \in \phi)(\alpha \mid\models dQ_i)$$

Possible states (of the world) given by answers to questions in the set ϕ must be in a similar relation to the initial question. Whenever we keep a model of the choice of direct answers from each question in ϕ, then there must be a direct answer to Q true in this model. For this, let us introduce a *choice function* ξ such that $\xi(Q_i)$ chooses exactly one direct answer from dQ_i. For each set of questions ϕ and a choice function ξ there is a *choice set* $A^{\phi}_{\xi} = \{\xi(Q_i) \mid Q_i \in \phi\}$.[18] The soundness condition in the other direction ("from a set to initial question") will be expressed, generally, by $(\forall A^{\phi}_{\xi})(A^{\phi}_{\xi} \mid\models dQ)$.

Back to our example, there are four choice sets:

$$A_{\xi_1} = \{\alpha, \beta\}$$

$$A_{\xi_2} = \{\neg\alpha, \beta\}$$

$$A_{\xi_3} = \{\alpha, \neg\beta\}$$

$$A_{\xi_4} = \{\neg\alpha, \neg\beta\}$$

[18] If the set ϕ is clear from the context, we will write only A_{ξ}.

But the fourth one is not in compliance with the second soundness requirement, it is in contradiction with our (prospective) presupposition $(\alpha \vee \beta)$. If we admit the additional answer $(\neg\alpha \wedge \neg\beta)$ and a question in the form $?\{\alpha, \beta, (\neg\alpha \wedge \neg\beta)\}$, mutual soundness of this question and the set of questions $\{?\alpha, ?\beta\}$ will be valid. But this solution seems to be rather awkward. A questioner posing the question $?\{\alpha, \beta\}$ evidently presupposes $(\alpha \vee \beta)$. This will bring us to the definition of reducibility with respect to an auxiliary set of declaratives and the *mutual soundness* will be required in the following forms:

$$(\forall \alpha \in dQ)(\forall Q_i \in \phi)(\Gamma \cup \alpha \mid\models dQ_i)$$

and

$$(\forall A^{\phi}_{\xi})(\Gamma \cup A^{\phi}_{\xi} \mid\models dQ).$$

Our example produces more than soundness of Q relative to each A^{ϕ}_{ξ} (with respect to Γ), also *efficacy* of each A^{ϕ}_{ξ} with respect to a question Q is valid:

$$(\forall A^{\phi}_{\xi})(\exists \alpha \in dQ)(\Gamma \cup A^{\phi}_{\xi} \models \alpha).$$

It will be reasonable to keep this strengthening. We require to obtain at least one answer to an initial question from a choice set. Whenever Γ and A^{ϕ}_{ξ} describe the state of the world, there must be a direct answer to a question Q that does the same job.

3.4.1 Reducibility of questions to sets of questions

We can take advantage of the previous discussion for the direct definition of reducibility of a question to a set of questions. Now, we introduce *pure reducibility* that does not use any auxiliary set of declaratives.

Definition 15 *A question Q is* purely reducible *to a non-empty set of questions ϕ iff*

1. $(\forall \alpha \in dQ)(\forall Q_i \in \phi)(\alpha \mid\models dQ_i)$
2. $(\forall A^{\phi}_{\xi})(\exists \alpha \in dQ)(A^{\phi}_{\xi} \models \alpha)$
3. $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$

First two conditions express mutual soundness, the second one adds efficiacy, as it was discussed, and the last one requires *relative simplicity.* If Q is reducible to a set ϕ, we will write $Q \gg \phi$. The definition of pure reducibility was introduced by Andrzej Wiśniewski in [13].

Example 9 (in CPL)

- $?\{\alpha, \beta, (\neg\alpha \wedge \neg\beta)\} \gg \{?\alpha, ?\beta\}$
- $?|\alpha, \beta| \gg \{?\alpha, ?\beta\}$
- $?(\alpha \circ \beta) \gg \{?\alpha, ?\beta\}$*, where $\circ$ is any of the connectives: $\wedge, \vee, \rightarrow$*

In the first item, there is the "pure" version from the introductory discussion. All items display reducibilities between initial safe questions and sets of safe questions. The following theorem shows that it is not an accident.

Theorem 8 *If $Q \gg \phi$, then Q is safe iff each $Q_i \in \phi$ is safe,*

Proof The first condition of Definition 15 can be rewritten as $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^\beta$, for each $Q_i \in \phi$, and it gives the implication from left to right.

For the proof of the other implication, let us suppose $Q \gg \phi$ and that each $Q_i \in \phi$ is safe, but Q is not. It implies the existence of model $\mathbf{M}_0 \in \mathcal{M}$ such that $\mathbf{M}_0 \not\models \alpha$, for each $\alpha \in dQ$. The safeness of all Q_i gives $(\forall Q_i \in \phi)(\exists \beta \in dQ_i)(\mathbf{M}_0 \models \beta)$. Thus, there is A^ϕ_ξ made from these βs and $\mathbf{M}_0 \models A^\phi_\xi$. But it is in contradiction with the second condition of the definition of $Q \gg \phi$, which gives the existence of some $\alpha \in dQ$ such that $\mathbf{M}_0 \models \alpha$. QED

From this we know that if there is a risky question among questions in ϕ and $Q \gg \phi$, then Q must be risky too. [14, p. 197]

The rewritten first condition of Definition 15 is of the form

$$\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcap_i \bigcup_{\beta \in dQ_i} \mathcal{M}^\beta$$

and it brings out the relationship of semantic ranges. The semantic range of a reduced question is bounded by the intersection of all semantic ranges of Q_is.

The relation of pure reducibility is reflexive ($Q \gg \{Q\}$) and we can prove the following version of transitivity:

Theorem 9 *If $Q \gg \phi$ and each $Q_i \in \phi$ is reducible to some set of questions ϕ_i, then $Q \gg \bigcup_i \phi_i$.*

Proof The third condition of Definition 15 is clearly valid.

The first one is easy to prove. From $Q \gg \phi$ we get $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^\beta$, for each $Q_i \in \phi$, and from the reducibility of all Q_i in ϕ to an appropriate ϕ_i we have $\bigcup_{\beta \in dQ_i} \mathcal{M}^\beta \subseteq \bigcup_{\gamma \in dQ_j} \mathcal{M}^\gamma$, for each $dQ_j \in \phi_i$. It gives together $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcup_{\gamma \in dQ_j} \mathcal{M}^\gamma$, for each $Q_j \in \bigcup_i \phi_i$.

For the second one we require the existence of $\alpha \in dQ$ such that $A^{\cup_i \phi_i}_\xi \models \alpha$, for each $A^{\cup_i \phi_i}_\xi$. From the reducibility of all Q_i in ϕ to an appropriate ϕ_i we have that each $A^{\phi_i}_{\xi'}$ is a subset of some $A^{\cup_i \phi_i}_\xi$. It implies that if there is any model $\mathbf{M}$ of $A^{\cup_i \phi_i}_\xi$, it must be a model of some $A^{\phi_i}_{\xi'}$. From $Q \gg \phi$ we know that there is $\alpha \in dQ$ for each choice set $A^\phi_{\xi''}$ on ϕ. This choice set is made by elements of all $dQ_i \in \phi$ which are valid in $\mathbf{M}$. It means that $A^\phi_{\xi''}$ is valid in $\mathbf{M}$ as well as $\alpha \in dQ$. QED

Now let us look at the relationship of pure reducibility and pure e-implication. The following example shows that it need not be that e-implication causes reducubility. Both definitions have the same first conditions, but the second condition of reducibility can fail.

Example 10 (in CPL) *$?|\alpha, \beta| \models ?(\alpha \wedge \beta)$, but $?|\alpha, \beta| \not\gg \{?(\alpha \wedge \beta)\}$.*

On the other hand, we can prove that regular e-implication implies reducibility.

Lemma 9 *Let ϕ be a set of questions such that $Q \vdash Q_i$, for each $Q_i \in \phi$. If $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$, then $Q \gg \phi$.*

Proof Let us prove the second condition of Definition 15 that requires existence of $\alpha \in dQ$ such that $\mathcal{M}^{A_\xi^\phi} \subseteq \mathcal{M}^\alpha$, for each A_ξ^ϕ. It is known that $\mathcal{M}^{A_\xi^\phi} \subseteq \mathcal{M}^\beta$, for each $\beta \in A_\xi^\phi$. If $Q \vdash Q_i$, then for each $\beta \in dQ_i$ there is $\alpha \in dQ$ such that $\mathcal{M}^\beta \subseteq \mathcal{M}^\alpha$. Thus, $\mathcal{M}^{A_\xi^\phi} \subseteq \mathcal{M}^\alpha$. QED

What about if we know $Q_i \models Q$ or, even, $Q_i \vdash Q$, for each $Q_i \in \phi$, and $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$, can we conclude that $Q \gg \phi$? Example 10 gives the negative answer to this question as well as $?(\alpha \wedge \beta) \vdash ?|\alpha, \beta|$.

Not even reducibility produces e-implication.

Example 11 (in CPL) *$?(\alpha \wedge \beta) \gg \{?\alpha, ?\beta\}$, but $?(\alpha \wedge \beta)$ is not implied neither by $?\alpha$ nor by $?\beta$ and $?(\alpha \wedge \beta)$ does not imply neither $?\alpha$ nor $?\beta$.*

In the next subsection we will study some special cases of links between reducibility and e-implication.

So far we have worked only with the pure reducibility. It could be useful to introduce the general term of *reducibility* as it was seen in the introductory example. The definition is almost the same as Definition 15, but the mutual soundness and efficacy conditions are supplemented by an auxiliary set of declaratives Γ (cf. [9]). We will write $\Gamma, Q \gg \phi$.

Definition 16 *A question Q is* reducible *to a non-empty set of questions ϕ with respect to a set of declaratives Γ iff*

1. $(\forall \alpha \in dQ)(\forall Q_i \in \phi)(\Gamma \cup \alpha \mid\models dQ_i)$
2. $(\forall A_\xi^\phi)(\exists \alpha \in dQ)(\Gamma \cup A_\xi^\phi \models \alpha)$
3. $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$

The introductory discussion is displayed in this example:

Example 12 (in CPL) $(\alpha \vee \beta), ?\{\alpha, \beta\} \gg \{?\alpha, ?\beta\}$

As it is expected, the role of Γ is similar to the role of an auxiliary set of declaratives in e-implication:

Fact 23 *If $Q \gg \phi$, then $\Gamma, Q \gg \phi$, for each Γ.*

So we can speak of *weakening in declaratives* and it enables us to generalize Lemma 9.

Theorem 10 *If* $\Gamma, Q \vdash Q_i$, *for each* $Q_i \in \phi$, *and* $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$, *then* $\Gamma, Q \gg \phi$.

Proof The third and the first conditions of Definition 16 are obvious.
The second one requires that for each A_ξ^ϕ there is $\alpha \in dQ$ such that $\mathcal{M}^{\Gamma \cup A_\xi^\phi} \subseteq \mathcal{M}^\alpha$. From the construction of choice sets we know that for each A_ξ^ϕ and $Q_i \in \phi$ there is $\beta \in dQ_i$ (member of A_ξ^ϕ) such that $\mathcal{M}^{\Gamma \cup A_\xi^\phi} \subseteq \mathcal{M}^{\Gamma \cup \beta}$. The regular e-implication provides that there is $\alpha \in dQ$ for each $\beta \in dQ_i$ such that $\mathcal{M}^{\Gamma \cup \beta} \subseteq \mathcal{M}^\alpha$. QED

We close this subsection by reversing the "direction" of the reducibility relation. Let us suppose that we have generated a set of questions ϕ that are evoked by a set of declaratives Γ. Can we conclude that Γ evokes such a complex question which is reducible to the set ϕ? Generally, not. But if we know that the complex question gives an answer to some question from ϕ, the answer is positive.

Theorem 11 *If* Γ *evokes each question from a set* ϕ, $Q \gg \phi$, *and there is a question* $Q_i \in \phi$ *such that* $Q \geq Q_i$, *then* $\Gamma \models Q$.

Proof Soundness of Q relative to Γ requires the existence of an answer $\alpha \in dQ$ for each model $\mathbf{M} \models \Gamma$. From the evocation of each $Q_i \in \phi$ we have $(\forall \mathbf{M} \models \Gamma)(\forall Q_i \in \phi)(\exists \beta \in dQ_i)(\mathbf{M} \models \beta)$. So, each model of Γ produces some choice set such that $(\forall \mathbf{M} \models \Gamma)(\exists A_\xi^\phi)(\mathbf{M} \models A_\xi^\phi)$. Together with reducibility, where it is stated that $(\forall A_\xi^\phi)(\exists \alpha \in dQ)(A_\xi^\phi \models \alpha)$, we get $\Gamma \mid\models dQ$.
Informativness of Q with respect to Γ is justified by $\geq$-relation for some question $Q_i \in \phi$. If $\Gamma \models \alpha$, for some $\alpha \in dQ$, then it gives a contradiction with $\Gamma \models Q_i$. QED

Given the conditions of Theorem 11 are met, we obtain:

- $\Gamma, Q \models Q_i$, for each $Q_i \in \phi$, and
- $\Gamma, Q_i \vdash Q$, for $Q \geq Q_i$.

The first item is based on Fact 13 and the second one is given by the help of Theorem 7.

3.4.2 Reducibility and sets of yes-no questions

The concept of reducibility is primarily devoted to a transformation of a question to a set of "less complex" questions. The introductory discussion and its formalization in Example 12 evoke interesting questions:

- If we have an initial question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ with, at worse, a countable list of direct answers, is it possible to reduce it to a set of yes-no questions based only on direct answers of Q?

- Moreover, could we require the e-implication relationship between Q and questions in the set ϕ?

We can find an easy solution to these problems under condition that yes-no questions are safe and we have an appropriate set of declaratives. We will require Q to be sound with respect to Γ.

Lemma 10 *Let us suppose that yes-no questions are safe in the background logic. If a question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ is sound with respect to a set Γ, then there is a set of yes-no questions ϕ such that $\Gamma, Q \gg \phi$ and $\Gamma, Q \models Q_i$, for each $Q_i \in \phi$.*

Proof First, we define the set of yes-no questions ϕ based on the initial question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ such that

$$\phi = \{?\alpha_1, ?\alpha_2, \ldots\}.$$

Secondly, we prove the condition that is common for both reducibility and e-implication. The safeness of members of ϕ implies that $\mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^\beta$, for each $\alpha \in dQ$ and $Q_i \in \phi$. This gives $\mathcal{M}^{\Gamma \cup \alpha} \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^\beta$.

To prove reducibility we have to justify the second condition of Definition 16. We need to find an $\alpha \in dQ$ for each A_ξ^ϕ such that $\Gamma \cup A_\xi^\phi \models \alpha$. Two cases will be distinguished.

(a) If there is α from both A_ξ^ϕ and dQ, then choose this direct answer.

(b) If there is no direct answer $\alpha \in dQ$ in A_ξ^ϕ, then $\mathcal{M}^{\Gamma \cup A_\xi^\phi} = \emptyset$ and we can take any α from dQ.

The final step is the proof of e-implication. We have to show that for each Q_i and each direct answer $\beta \in dQ_i$ there is a non-empty subset $\Delta \subset dQ$ such that $\Gamma \cup \beta \mid\models \Delta$. For this, we utilize the shape of questions in ϕ.

(a) If $\beta \in dQ$, then Δ could be $\{\beta\}$ and $\Gamma \cup \beta \mid\models \{\beta\}$.

(b) If $\beta \notin dQ$ and $\Gamma \cup \beta$ has at least one model, we recognize that $\mathcal{M}^{\Gamma \cup \beta} \subseteq \mathcal{M}^\Gamma$. Simultaneously, β must be of the form $\neg \alpha_j$ and Δ can be defined as $dQ \setminus \{\alpha_j\}$. Together with soundness of initial question Q with respect to Γ, which means $\mathcal{M}^\Gamma \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$, we get $\mathcal{M}^{\Gamma \cup \beta} \subseteq \bigcup_{\alpha \in \Delta} \mathcal{M}^\alpha$.

QED

This lemma enables us to work with classes of questions which are known to be sound relative to sets of their presuppositions. (Normal and regular questions are the obvious example.) Whenever we know that the initial question is evoked by a set of declaratives, we get the following conclusion.

Fact 24 *Working in logics where yes-no questions are safe, if a question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ is evoked by a set of declaratives Γ, then there is a set of yes-no questions ϕ such that $\Gamma, Q \gg \phi$ and $\Gamma, Q \models Q_i$, for each $Q_i \in \phi$.*

This fact corresponds to Lemma 1 in the paper [9] where a bit different definition of reducibility is used, but the results are the same.

If dQ is finite or the entailment is compact, the set ϕ is finite set of yes-no questions. Simultaneously, it is useful to emphasize that the proof of Lemma 10 shows how to construct such a set.[19]

In logics with risky yes-no questions, the first condition of reducibility as well as e-implication can fail. It need not be $\mathcal{M}^{\Gamma,\alpha} \subseteq (\mathcal{M}^{\beta} \cup \mathcal{M}^{\neg\beta})$, for each $\alpha \in dQ$ and each $?\beta \in \phi$. More generally, we can ask for a help the auxiliary set of declaratives again. Let us remind Fact 13 and put soundness of each $Q_i \in \phi$ with respect to Γ. Going through the proof of Lemma 10, the rest is valid independently of safeness of yes-no questions. As a conclusion we get

Fact 25 *If a question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ is sound relative to a set Γ, and there is a set of yes-no questions $\phi = \{?\alpha_1, ?\alpha_2, \ldots\}$ (based on Q) such that $\Gamma \mid\models dQ_i$, for each $Q_i \in \phi$, then $\Gamma, Q \gg \phi$ and $\Gamma, Q \models Q_i$, for each $Q_i \in \phi$.*

The construction of yes-no questions provided by Lemma 10 does not prevent the high complexity of such yes-no questions. Observing the last item of Example 9, it seems worthwhile to enquire whether it is possible to follow this proces and to reduce a question (with respect to an auxiliary set of declaratives) to a set of atomic yes-no questions based on subformulas of the initial question (cf. [19]). The first restriction is clear, yes-no questions must be safe. But it is not all, the second clause of pure reducibility (Definition 15) requires the *truth-functional* connection of subformulas. Then the answer is positive. We can use repeatedly a cautious extension of Theorem 9:

Fact 26 *If $\Gamma, Q \gg \phi$ and each $Q_i \in \phi$ is reducible to some set of questions ϕ_i, then $\Gamma, Q \gg \bigcup_i \phi_i$.*

There is a similar concept based on properties of the classical logic and e-implication: *erotetic search scenarios* (see [16, 17, 18]). If we recall Example 5, we can recognize the "truth-functional" role of the question $?|\alpha, \beta|$ as an interlink between $?(\alpha \wedge \beta)$ and $?\alpha$, resp. $?\beta$.

References

[1] L. Běhounek. Fuzzification of Groenendijk-Stokhof propositional erotetic logic. *Logique & Analyse*, 185–188:167–188, 2004.

[2] K. De Clercq and L. Verhoeven. Sieving out relevant and efficient questions. *Logique & Analyse*, 185–188:189–216, 2004.

[3] J. Groenendijk. Questions and answers: Semantics and logic. In R. Bernardi and M. Moortgat, editors, *Proceedings of the 2nd CologNET-ElsET Symposium. Questions and Answers: Theoretical and Applied Perspectives*, pages 16–23. Universiteit Utrecht, 2003. http://dare.uva.nl/document/7521 (downloaded by April 2007).

[19] The same result is provided by theorems 7.49–7.51 in [14].

[4] J. Groenendijk and M. Stokhof. Partitioning logical space. Annotated handout, ILLC, Department of Philosophy, Universiteit van Amsterdam, 1990. Second European Summerschool on Logic, Language and Information; Leuven, August 1990; version 1.02.

[5] J. Groenendijk and M. Stokhof. Questions. In J. van Benthem and A. ter Meulen, editors, *Handbook of Logic and Language*, pages 1055–1125. Elsevier, Amsterdam, 1997.

[6] D. Harrah. On the history of erotetic logic. In A. Wiśniewski and J. Zygmunt, editors, *Erotetic Logic, Deontic Logic, and Other Logical Matters*, number 1890 in Acta Universitatis Wratislaviensis, pages 19–27. Wydawnictwo Uniwersytetu Wrocławskiego, Wrocław, 1997.

[7] D. Harrah. The logic of questions. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 8, pages 1–60. Kluwer, 2002.

[8] J. Hintikka, I. Halonen, and A. Mutanen. Interrogative logic as a general theory of reasoning. In D. Gabbay, R. Johnson, H. Ohlbach, and J. Woods, editors, *Handbook of the Logic of Argument and Inference*, pages 295–337. Elsevier, 2002.

[9] P. Leśniewski and A. Wiśniewski. Reducibility of questions to sets of questions: Some feasibility results. *Logique & Analyse*, 173–175:93–111, 2001.

[10] M. Peliš. Questions and logical analysis of natural language: the case of transparent intensional logic. *Logique & Analyse*, 185–188:217–226, 2004.

[11] P. Tichý. Questions, answers, and logic. *American Philosophical Quarterly*, 15:275–284, 1978.

[12] M. Urbański. Synthetic tableaux and erotetic search scenarios: Extension and extraction. *Logique & Analyse*, 173–175:69–91, 2001.

[13] A. Wiśniewski. On the reducibility of questions. *Erkenntnis*, 40:265–284, 1994.

[14] A. Wiśniewski. *The Posing of Questions: Logical Foundations of Erotetic Inferences*. Kluwer, 1995.

[15] A. Wiśniewski. Erotetic logic and explanation by abnormic hypotheses. *Synthese*, 120:295–309, 1999.

[16] A. Wiśniewski. Questions and inferences. *Logique & Analyse*, 173–175:5–43, 2001.

[17] A. Wiśniewski. Erotetic search scenarios. *Synthese*, 134:389–427, 2003.

[18] A. Wiśniewski. Erotetic search scenarios, problem solving, and deduction. *Logique & Analyse*, 185–188:139–166, 2004.

[19] A. Wiśniewski. Reducibility of safe questions to sets of atomic yes-no questions. In J. Jadacki and J. Paśniczek, editors, *Lvov-Warsaw School: The New Generation*, pages 215–236. Rodopi, Amsterdam/NY, 2006.