A Definable Failure of the Singular Cardinal Hypothesis

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Abstract We show first that it is consistent that κ is a measurable cardinal where the GCH fails, while there is a lightface definable wellorder of $H(\kappa^+)$. Then with further forcing we show that it is consistent that GCH fails at $\aleph_\omega$, $\aleph_\omega$ strong limit, while there is a lightface definable wellorder of $H(\aleph_{\omega+1})$ ("definable failure" of the singular cardinal hypothesis at $\aleph_\omega$). The large cardinal hypothesis used is the existence of a κ^{++}-strong cardinal, where κ is κ^{++}-strong if there is an embedding $j : V \to M$ with critical point κ such that $H(\kappa^{++}) \subseteq M$. By work of M. Gitik and W. J. Mitchell [12], [20], our large cardinal assumption is almost optimal. The techniques of proof include the "tuning-fork" method of [10] and [3], a generalisation to large cardinals of the stationary-coding of [4] and a new "definable-collapse" coding based on mutual stationarity. The fine structure of the canonical inner model $L[E]$ for a κ^{++}-strong cardinal is used throughout.

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Contents

1 Introduction

In his paper [14], L. Harrington showed how to make the continuum arbitrarily large, while introducing a Δ^1_3 wellorder of $\mathscr{P}(\omega)$. Using a forcing with countable support, V. Fischer and S. D. Friedman showed in [4] how to violate CH with a Δ^1_3 wellorder of $\mathscr{P}(\omega)$ while obtaining non-trivial behaviour of cardinal characteristics of the continuum. A related construction with finite support and larger values of the continuum appears in [5].

In the first main result of this paper (see Theorem 4.1 below) we extend these results to a measurable cardinal which violates GCH, and then in the second part of this paper (see Theorem 5.1 below) we push this down to $\aleph_\omega$. Both results are a contribution to the area of set-theoretical research which studies the compatibility of large cardinals with definability in the presence of axioms which fail in core models (such as the failure of GCH).

In order to prove Theorem 4.1, one must add many new subsets of κ, wellorder them definably, and preserve the measurability of κ in the process. The preservation of measurability of κ which violates GCH, starting from a κ^{++}-strong cardinal (see Definition 2.1), is a non-trivial problem, first solved by W. H. Woodin in his "surgery argument" (see [2]). However, Woodin's argument relies heavily on the homogeneity properties of κ-Cohen forcing. In order to overcome this restriction, the paper [10] introduced a lifting argument based on the fusion property of κ-Sacks forcing, which is not subject to these restrictions. The use of κ-Sacks forcing let to a number of new results ([3, 8, 15, 9, 10, 11]), most notably to the solution of the old problem concerning the number of normal measures ([9]).

We prove in this paper the following two theorems:

Theorem 1.1 *(GCH) Starting from a κ^{++}-strong cardinal κ, it is consistent that GCH fails at κ, κ remains measurable, and there is a lightface definable wellorder of $H(\kappa^+)$.*

This theorem is proved by an iteration similar to [4], but applied to many inaccessible cardinals simultaneously and using fine-structural properties of the canonical inner model $L[E]$ for a κ^{++}-strong cardinal κ. Key steps are the preservation of measurability of κ, using methods of [10], [3], and the preservation of the desired stationary sets, based on ideas of [7].

Theorem 1.2 *(GCH) Starting from a κ^{++}-strong cardinal κ, it is consistent that GCH fails at $\aleph_\omega$, $2^{\aleph_n} < \aleph_\omega$ for every $n < \omega$, and there is a lightface definable wellorder of $H(\aleph_{\omega+1})$.*

The proof of this theorem starts with the model obtained in the first theorem and uses mutually stationary sets defined via the fine structure of $L[E]$ to introduce "definable" collapses which make κ equal to $\aleph_\omega$ in the generic extension. Since $2^{\aleph_\omega}$ equals $\aleph_{\omega+2}$ in this model, we obtain a definable failure of the Singular Cardinal Hypothesis at $\aleph_\omega$.

2 Preliminaries

2.1 Notation

The forcing notation is standard, following [16]. In particular, if P is a forcing notion and p, q are conditions in P, we write $p \leq q$ to express that p is a stronger condition than q. We say that P is κ-closed for an uncountable regular cardinal κ if all decreasing sequences of length $< \kappa$ have a lower bound (similarly for the notion of κ-distributivity).

2.2 Strong cardinals

Definition 2.1 *κ is called λ-strong for a cardinal $\lambda > \kappa$ if there exists an elementary embedding $j : V \to M$ with critical point κ such that $\lambda < j(\kappa)$ and $H(\lambda) \subseteq M$.*

In [8], the same property was called λ-hypermeasurable. In [8], you can also find some discussion concerning the exact form of the definition (some authors, such as [18] or [16], use the V-hierarchy instead of the H-hierarchy to gauge the strength of the embedding).

Under GCH, we can without loss of generality assume that if κ is κ^{++}-strong (which will be the strength required in our argument), then $j : V \to M$ is an extender ultrapower embedding in that

$$M = \{j(f)(\alpha) \mid f : \kappa \to V, \alpha < \kappa^{++}\}, \tag{2.1}$$

where it holds that $H(\kappa^{++}) \subseteq M$ and ${}^{\kappa}M \subseteq M$ (M is closed under κ-sequences in the universe). The handbook article [2] is a rich source of information on these notions.

2.3 Perfect tree forcing at an inaccessible

In [17], A. Kanamori generalizes the Sacks forcing at ω to a regular cardinal κ. Although κ can be any regular cardinal, we will deal here only with the case when κ is a (strongly) inaccessible cardinal. Assuming the reader's familiarity with this paper, or with some other exposition of the generalized Sacks forcing, we give here a quick review and point out some differences in our definition. There are two new non-trivial requirements which need to be added to the original definition of the perfect-tree forcing at ω so that the perfect-tree forcing at an inaccessible κ behaves properly (for instance is κ-closed). If $p \subseteq 2^{<\kappa}$ is a tree, then:

(2.2)

(i) If $s_0 \subseteq s_1 \subseteq \dots$ is a sequence of nodes in p indexed by $\alpha < \lambda < \kappa$ for some limit λ, then $\bigcup_{\alpha<\lambda} s_\alpha$ is a node in p;

(ii) If s is a node in p, the length of s is a limit ordinal, and splitting nodes $t \subsetneq s$ are unbounded in s, then s splits.

In fact, unlike in [17], we will use the following modification (ii*) of (ii) given in (2.3). The reason for this is to ensure that the lifting argument works more easily, see later in the text in Section 4.

(2.3)

(ii*) If s is a node in p, the length of s is a limit ordinal α, and splitting nodes $t \subsetneq s$ are unbounded in s, then:

(a) If α has countable cofinality, then s splits.

(b) If α has uncountable cofinality, then s does not split.

Definition 2.2 *Let κ be an inaccessible cardinal. We say that p is a* cof ω-splitting perfect κ-tree *if $p \subseteq 2^{<\kappa}$ is a tree of height κ closed under initial segments (i.e. if $s \in p$ and $s' \subseteq s$ then $s' \in p$), for each $s \in p$ there is $t \supseteq s$ in p which splits (i.e. both $t^\frown 0$ and $t^\frown 1$ are in p), and p satisfies the conditions (i) in (2.2) and (ii*) in (2.3).*

If p and q are cof ω-splitting perfect κ-trees, we write $p \leq q$ to denote that p is a stronger condition, where $p \leq q \leftrightarrow p \subseteq q$.

Let p be a cof ω-splitting perfect κ-tree. We define some notation which we find useful (the notation needs to take into account the cof ω splitting and so it will divert slightly from the usual notation). By induction on $\alpha < \kappa$ we define the α-th splitting level of p as follows. If the α-th splitting level of p is already defined, we define the $\alpha+1$-th splitting level of p as the collection of all splitting nodes $s \in p$ such that there is a $t \subsetneq s$ with t being on the α-th splitting level, and such that there is no splitting node between t and s. If α is a limit ordinal, then we define the α-th splitting level as the collection of all nodes $s \in p$ such that s is the union of nodes t_β for $\beta < \alpha$ such that t_β is on the β-th splitting level of p. Note that for a limit α, the α-th splitting level of p consists of splitting nodes if and only if α has countable cofinality. We denote the set $\{s \in p \,|\, s \text{ is on the } \alpha\text{-th splitting level of } p\}$ by $\mathrm{Split}_\alpha(p)$. We write $p \leq_\alpha q$ if $p \leq q$ and $\mathrm{Split}_\alpha(p) = \mathrm{Split}_\alpha(q)$. If σ is a node in p, we denote by $p|\sigma$ the restriction of the tree p to σ, i.e. $p|\sigma = \{t \in p \,|\, t \subseteq \sigma \vee \sigma \subseteq t\}$.

Finally, let $\mathrm{Succ}_\alpha(p)$ be equal to:

– If α is a successor ordinal, or a limit ordinal of countable cofinality, then $\mathrm{Succ}_\alpha(p) = \{t \in p \mid \exists s \in \mathrm{Split}_\alpha(p)\, t = s^\frown i \text{ for } i \in \{0,1\}\}$.
– If α is a limit ordinal of uncountable cofinality, then $\mathrm{Succ}_\alpha(p) = \mathrm{Split}_\alpha(p)$.

We view the nodes in $\mathrm{Succ}_\alpha(p)$ as the possible choices ("successors") at the α-th splitting level (in case α is a limit ordinal of uncountable cofinality, then there are no choices, and so $\mathrm{Succ}_\alpha(p) = \mathrm{Split}_\alpha(p)$). This notation will be useful in the fusion arguments in Sections 3.3 and 4.

As in [17] and in [9], where the trees with modifications as in (ii*) are discussed, one can show that the forcing consisting of the cof ω-splitting perfect κ-trees is κ-closed and satisfies κ-fusion (a decreasing sequence $\langle p_i \mid i < \kappa\rangle$ is a fusion sequence if $p_i \geq_i p_{i+1}$ for each $i < \kappa$; one can show that in this case $\bigcap_{i<\kappa} p_i$ is a cof ω-splitting κ-perfect tree). Under GCH this forcing has the κ^{++}-cc, and hence preserves all cofinalities.

One can naturally define a product and an iteration of the cof ω-splitting perfect κ-tree forcing, as in [17]. The support of the forcing is of size $\leq \kappa$. Our notational conventions need to be extended to include the supports: if p and q are conditions in the product or iteration of some length μ and F is a subset of μ of size $< \kappa$, then $p \leq_{\alpha,F} q$ means that for all $\xi \in F$, $p \restriction \xi \Vdash p(\xi) \leq_\alpha q(\xi)$. We naturally extend the notion of the restriction $p|\sigma$ by coordinates if we work with a product. In the context of an iteration, given $\sigma : F \to 2^{<\kappa}$, we need to determine first by induction on $\xi \in F$ whether the coordinate-wise restriction $p|\sigma$ is meaningful: given the restriction $(p \restriction \xi)|(\sigma \restriction \xi)$, $(p \restriction \xi+1)|(\sigma \restriction \xi+1)$ is defined whenever $(p \restriction \xi)|(\sigma \restriction \xi) \Vdash \sigma(\xi) \in p(\xi)$.

Product-style and iteration-style forcings based on the perfect-tree forcing were used in arguments where preservation of large cardinals was in focus, see for instance [10], [9], and [8] (a product), [3] (an iteration), or [15], where the cofinalities change. One can even formulate more general variants, such as the κ-Miller forcing and combine it with the Sacks forcing, see [11].

Remark 2.3 In [10] and other papers referenced in the previous paragraph, a perfect tree at an inaccessible κ is defined more restrictively: it is required that the splitting nodes are determined uniquely by a certain club in κ. It is easy to see that the forcing defined in [10] is a dense subforcing of our forcing, and hence they are equivalent.

2.4 The extender model $L[E]$

We will give here only a brief review of the properties which we will need in the proof.

We will work in an extender model $L[E]$ such that

$$L[E] \models \kappa \text{ is a } \kappa^{++}\text{-strong cardinal}, \tag{2.4}$$

where E is an indexed sequence of partial extenders of length less than κ^{+3}. The indexing of extenders is a somewhat delicate question. For our needs, it will suffice to say that the (full) extender at κ witnessing the κ^{++}-strength of κ is indexed at some limit ordinal $\mu \in (\kappa^{++}, \kappa^{+3})$. Also, for every regular cardinal $\lambda < \kappa^{+3}$, $E_\lambda = \emptyset$, i.e. the extender indexed by λ is always trivial. In general, the indexing is chosen to ensure that the extenders are added "amenably" (see below).

We say that a regular cardinal θ is *sufficiently large* if $\theta > 2^{2^\kappa}$. We will localize this to inaccessible cardinals $\alpha < \kappa$: here θ is sufficiently large (with respect to α) if $\theta > 2^{2^\alpha}$. As we will start with GCH, it will be the case that $2^{2^\kappa} = \kappa^{++}$ and $2^{2^\alpha} = \alpha^{++}$.

In order to analyze the structure $L[E]$, we will work with general structures are of the form:

$$\langle L_\alpha[E], \in, E \restriction \alpha, E_\alpha \rangle, \tag{2.5}$$

where $\alpha < \kappa^{+3}$, $L_\alpha[E] = L_\alpha[E \restriction \alpha]$ and E_α is the top extender indexed at α. We can assume that $E_\alpha \subseteq L_\alpha[E]$, and moreover that the extra sets are amenable in that for every $x \in L_\alpha[E]$, $E_\alpha \cap x \in L_\alpha[E]$, and $(E \restriction \alpha) \cap x \in L_\alpha[E]$.

If θ is a regular cardinal, we can restrict ourselves to simpler structures of the form

$$\langle L_\theta[E], \in, E \restriction \theta \rangle. \tag{2.6}$$

This is because of the fact mentioned above, namely that the top extender E_θ is always empty for a regular cardinal $\theta < \kappa^{+3}$. We will write L^E_θ to denote the whole structure $\langle L_\theta[E], \in, E \restriction \theta \rangle$.

For a limit ordinal α, we will identify $E \restriction \alpha$ with $E \cap L_\alpha[E]$; in particular $\langle L_\alpha[E], \in, E \restriction \alpha \rangle = \langle L_\alpha[E], \in, E \cap L_\alpha[E] \rangle = L^E_\alpha$.

The basic issue we need to review is *condensation*. We need some definitions first which will help us to formulate the condensation principle for $L[E]$ that we need.

For a set model $M = \langle M, \in, E^M \rangle$, where E^M is a unary predicate, we say that A is $\Sigma_n(M)$ in $p \in M$ if A is definable over M using a Σ_n formula with the parameter p.

Definition 2.4 *ϱ is the Σ_n-*projectum *of M if ϱ is the least ρ such that for some $p \in M$, there is a set A which is $\Sigma_n(M)$-definable in p such that $A \cap \omega\rho \notin M$. We say that p witnesses that ϱ is the Σ_n-projectum of M.*

Definition 2.5 *M is n-*sound *if M is the Σ_n-hull of $\varrho \cup \{p\}$ in M, where ϱ is the Σ_n-projectum of M and whenever p witnesses that ϱ is the Σ_n-projectum.*

Fact 2.6 (Condensation) *Suppose that M is Σ_n-elementary in $\langle L^E_\theta, E_\theta\rangle$, $\bar{M}$ is the transitive collapse of M and the Σ_n-projectum of $\bar{M}$ is a subset of M. Also suppose that $\bar{M}$ is n-sound and the cardinality of M is not the successor of an inaccessible cardinal. Then $\bar{M}$ is an initial segment of $L[E]$.*

Proof. See Theorem 8.2 and Pages 87-88 of [21]. Note that alternative (b) in the referred results cannot occur as by our smallness assumption about $L[E]$, the total extenders on the E-sequence have length in intervals $[\alpha^+, \alpha^{++})$ where α is inaccessible. □

The condensation properties articulated in Fact 2.6 of $L[E]$ allow us to prove the following facts.

Fact 2.7 *Let θ be a fixed regular cardinal $> \alpha^{++}$, where $\alpha \leq \kappa$ is an inaccessible cardinal. Then for every $n < \omega$, there exists a continuous sequence of Σ_n-elementary submodels $\langle M^{\alpha,n}_i \mid i < \alpha^+\rangle$ of L^E_θ such that for each $i < \alpha^+$ the following hold:*

(i) Each $M^{\alpha,n}_i$ has size α.
(ii) $\alpha + 1 \subseteq M^{\alpha,n}_i$.
(iii) For i of cofinality α, $M^{\alpha,n}_i$ is closed under $< \alpha$ sequences existing in $L[E]$.
(iv) Moreover each $M^{\alpha,n}_i$ has the property that its transitive collapse $\pi : M^{\alpha,n}_i \to \bar{M}^{\alpha,n}_i$ yields an initial segment of the $L[E]$-hierarchy:

$$(2.7) \qquad \langle \bar{M}^{\alpha,n}_i, \in, \pi''(E \cap M^{\alpha,n}_i) \cap \bar{M}^{\alpha,n}_i\rangle = L^E_{\bar{\alpha}} \text{ for some } \bar{\alpha} < \alpha^+.$$

(v) $\bar{M}^{\alpha,n}_i \in \bar{M}^{\alpha,n}_{i+1}$.

Proof. Let us write $N_\beta = \langle N_\beta, \in, E^{N_\beta}\rangle$, where $\beta < \alpha^+$, to denote the Σ_n-Skolem hull of $\beta \cup \{\alpha\}$ in L^E_θ. Let us define $C = \{\beta < \alpha^+ \mid \beta = \alpha^+ \cap N_\beta\}$. Then for $\beta \in \mathrm{Lim}(C)$, N_β satisfies the property of M in the hypothesis of Fact 2.6 as β is the Σ_n-projectum of N_β. Define $\langle M^{\alpha,n}_i \mid i < \alpha^+\rangle$ to be the enumeration of these N_β's. □

Note that Fact 2.7 readily generalizes to the following Facts:

Fact 2.8 *The sequence as in Fact 2.7 can be so chosen to satisfy:*

(i) $M^{\alpha,n}_0$ contains any previously fixed parameter p from $L_\theta[E]$.
(ii) The closure under $< \alpha$ sequences and the condensation property (iv) in Fact 2.7 hold relative to an additional parameter R, providing that

R is a generic filter for a "small" forcing: Assume that $R \subseteq \alpha$ is a generic filter over $L_\theta[E]$ for some cardinal-preserving forcing $P_R \in L_\theta[E]$ of size at most α^+ (in $L_\theta[E]$). Then there exists for each $n < \omega$ a continuous sequence $\langle M_i^{\alpha,n} \mid i < \alpha^+\rangle$ of Σ_n-elementary substructures of $L_\theta^{E,R} = \langle L_\theta[E][R], \in, E \cap L_\theta[E], R\rangle$ of size α whose elements satisfy:

(a) Each $M_i^{\alpha,n}$ contains R as an element.

(b) If i has cofinality α, the model $M_i^{\alpha,n}$ is closed under $< \alpha$-sequences in $L[E][R]$.

(c) The transitive collapse $\bar{M}_i^{\alpha,n}$ of $M_i^{\alpha,n}$ is a generic extension of an initial segment of the $L[E]$ hierarchy:

(2.8)

$$\langle \bar{M}_i^{\alpha,n}, \in, \pi''(E \cap M_i^{\alpha,n}) \cap \bar{M}_i^{\alpha,n}, R\rangle = L_{\bar{\alpha}}^{E,R} \text{ for some } \bar{\alpha} < \alpha^+.$$

(d) $\bar{M}_i^{\alpha,n} \in \bar{M}_{i+1}^{\alpha,n}$.

Proof. Ad (i). If p were the least parameter for which the sequence of $M_i^{\alpha,n}$'s could not be chosen with $p \in M_0^{\alpha,n}$, then p is Σ_m-definable for some larger m, contradicting Fact 2.7 for that m.

Ad (ii). Introduce the forcing P_R as a parameter into $M_0^{\alpha,n}$ and notice that R is $P_R \cap M_i^{\alpha,n}$-generic over $M_i^{\alpha,n}$ for club-many $i < \alpha^+$ such that $M_i^{\alpha,n}$ is Σ_n-elementary in L_θ^E relative to R. The desired sequence is obtained by adjoining R to the members of this club. □

Instead of having a distinct sequence for each $n < \omega$, we can argue that we can fix a single sequence which is fully elementary.

Fact 2.9 *The sequences in Facts 2.7 and 2.8 can be chosen to satisfy that each M_i^α is fully elementary in L_θ^E, or $L_\theta^{E,R}$, respectively.*

Proof. Perform the above argument with L_θ^E replaced by $L_{\theta+1}^E$ and intersect the resulting models with L_θ^E. □

We close this section by a fact on the definability of the sequence E:

Fact 2.10 *Suppose V is a set-generic extension of $L[E]$. Then for every regular uncountable κ, the sequence $E \restriction \kappa$ is lightface definable over $H(\kappa)^V$.*

Proof. See Theorems 3.29 and 3.33 of [23]. □

2.4.1 Suitability

The key concept of the proof is "suitability" of certain transitive structures of the form $\langle M, \in, E^M\rangle$.

Definition 2.11 *Let E be the extender sequence fixed in Section 2.4. Given an inaccessible cardinal $\alpha \leq \kappa$, a model $M = \langle M, \in, E^M \rangle$ is called α-*suitable* if:*

(i) M is transitive, and satisfies ZF$^-$ *relative to E^M.*
(ii) M contains $\alpha + 1$ as a subset.
(iii) $E^M = E \cap M$.
(iv) M thinks that it has an α^{++}, and moreover it thinks that its α^+ and α^{++} are the α^+ and α^{++} of $(L[E^M])^M = (L[E \cap M])^M$.

Fact 2.12 *The property of being a suitable structure of size α is lightface expressible in $H(\alpha^+)$ of $L[E][G]$ for every G, which is a generic filter for a set forcing notion.*

Proof. By Fact 2.10. □

We give a couple of examples of suitable structures. The collapse of a structure M_i^α as in Fact 2.9 is an α-suitable structure. However, the intuition is to include more structures, specifically all cardinal-preserving set generic extensions of the initial segments of the $L[E]$ hierarchy which obey ZF$^-$. Notice that for $L_\beta[E][G]$, where G is a P-generic filter for some cardinal-preserving $P \in L_\beta[E]$ and $\alpha < \beta < \alpha^+$ such that $L_\beta^E \models$ ZF$^-$, the structure $\langle L_\beta[E][G], \in, E \cap L_\beta[E] \rangle$ is α-suitable. More to the point, for each M_i^α as in Fact 2.9, if P is a cardinal-preserving forcing notion (in V) which is an element of M_i^α and $p \in P$ is (M_i^α, P)-generic (see Section 2.5), then whenever G is a P-generic filter containing p, the transitive collapse of $M_i^\alpha[G]$ is a generic extension of an initial segment of the $L[E]$-hierarchy, and is thus α-suitable.

Suitable structures capture enough information to allow for a coherent definition of certain canonical almost disjoint stationary sets in $\mathrm{cof}(\alpha) \cap \alpha^+$ for an inaccessible $\alpha \leq \kappa$, which are the vehicle of the coding for the definable wellorder. For technical reasons, the stationary sets $\langle S_\xi^\alpha \mid \xi \in A_\alpha \rangle$ for a given inaccessible α will be indexed by members of the set $A_\alpha = \alpha^{++} \cup \{-1\}$, where -1 is construed as a formal symbol denoting a fixed set different from each ordinal in α^{++}, but available in every model of ZF$^-$ (for instance we can choose $-1 = \langle 0, 0 \rangle$). The purpose of the index -1 is to keep S_{-1}^α stationary, while the stationarity of the sets S_ξ^α for $\xi < \alpha^{++}$ may be explicitly killed by the coding forcing (see below in Section 3.2.1).

Lemma 2.13 *For each inaccessible $\alpha \leq \kappa$, there are:*

(i) A bookkeeping function $F^\alpha : \alpha^{++} \to L_{\alpha^{++}}[E]$, which enumerates $L_{\alpha^{++}}[E]$ with cofinally many repetitions, definable over $L_{\alpha^{++}}^E$ via a formula φ, and

(ii) For $A_\alpha = \alpha^{++} \cup \{-1\}$, there is a sequence $\langle S^\alpha_\xi \,|\, \xi \in A_\alpha\rangle$ of almost disjoint stationary subsets of $\alpha^+ \cap \operatorname{cof}(\alpha)$ definable over $L^E_{\alpha^{++}}$ via a formula ψ,

such that the following hold:

Whenever M, N are α-suitable models, $F^{\alpha,M}$, $F^{\alpha,N}$ denote the interpretations of φ in M, N, respectively, $\vec{S}^{\alpha,M} = \langle S^{\alpha,M}_\xi \,|\, \xi \in (A_\alpha)^M\rangle$, $\vec{S}^{\alpha,N} = \langle S^{\alpha,N}_\xi \,|\, \xi \in (A_\alpha)^N\rangle$ denote the interpretations of ψ in M, N, respectively, and $(\alpha^+)^M = (\alpha^+)^N$, then $F^{\alpha,M}, F^{\alpha,N}$ agree on $(\alpha^{++})^M \cap (\alpha^{++})^N$ and $\vec{S}^{\alpha,M}, \vec{S}^{\alpha,N}$ agree on $(A_\alpha)^M \cap (A_\alpha)^N$. In particular, if M is α-suitable and $(\alpha^+)^M = \alpha^+$, the $F^{\alpha,M}$ and $\vec{S}^{\alpha,M}$ equal the restrictions of F^α, $\vec{S}^\alpha$ up to the $(\alpha^{++})^M$ (including -1).

Proof. Define for all $\delta < \alpha^{++}$, $F(\delta) = a$ iff via Gödel pairing, δ codes a pair (α_0, α_1) where a has rank α_0 in the natural wellorder of $L^E_{\alpha^{++}}$ (so that the function F names a cofinally often). As regards the stationary sets, let $\langle D_i \,|\, i < \alpha^+\rangle$ be the canonical $L^E_{\alpha^+}$-definable $\Diamond$-sequence at α^+ (which exists by condensation, see Fact 2.6): by induction on $i < \alpha^+$, let D_i be the set D such that $\langle D, D'\rangle$ is the $L[E]$-least pair such that D is a subset of i, D' is closed unbounded in i and $D \cap \gamma \neq D_\gamma$ for all γ in D' (if no such pair exists, we let $D_i = i$).

For each $\xi < \alpha^{++}$, let A_ξ be the $L^E_{\alpha^{++}}$-least subset of α^+ coding ξ and define S^α_ξ, for $\xi < \alpha^{++}$ to be the set of all $i < \alpha^+$ such that $D_i = A_{\xi+1} \cap i$. Define S^α_{-1} similarly from A_0. □

Assume now that the generic R in Fact 2.8 preserves all stationary sets S^α_ξ, $\xi \in A_\alpha$. Then a sequence $\langle M^\alpha_i \,|\, i < \alpha^+\rangle$ given by Fact 2.9 is useful to argue that locally with respect to any fixed stationary set $S^\alpha_{\xi_0}$ in $\langle S^\alpha_\xi \,|\, \xi \in A_\alpha\rangle$, the sequence $\langle S^\alpha_\xi \,|\, \xi \in A_\alpha\rangle$ can be treated as really disjoint. Although it makes no difference here, we can assume that $\xi_0 \neq -1$ (because Lemma 2.14 will later be invoked only when $\xi_0 \neq -1$).

Let us denote as γ_i the ordinal $M^\alpha_i \cap \alpha^+$ for each $i < \alpha^+$. Let $\tilde{S}^\alpha_{\xi_0} = \{i \in S^\alpha_{\xi_0} \,|\, i = \gamma_i\} \subseteq S^\alpha_{\xi_0}$. Evidently, $\tilde{S}^\alpha_{\xi_0}$ is still a stationary subset of α^+.

Lemma 2.14 *Let $\langle S^\alpha_\xi \,|\, \xi \in A_\alpha\rangle$ be a sequence as in Lemma 2.13, $\langle M^\alpha_i \,|\, i < \alpha^+\rangle$ a sequence as in Fact 2.9, and let $\tilde{S}^\alpha_{\xi_0}$ be as above. Then there exists $i \in \tilde{S}^\alpha_{\xi_0}$ such that i does not belong to S^α_β for any $\beta \in M^\alpha_i$, $\beta < \alpha^{++}$, such that $\beta \neq \xi_0$.*

Proof. Otherwise there exists for each $i \in \tilde{S}^\alpha_{\xi_0}$ a $\beta \in M^\alpha_i$ such that $\beta \neq \xi_0$ and $i \in S^\alpha_\beta$. Because β is in M^α_i, it holds that $\beta < i$. Define a regressive function f which to each $i \in \tilde{S}^\alpha_{\xi_0}$ assigns some such $\beta < i$. By Fodor's theorem, there is a stationary set $T \subseteq \tilde{S}^{\alpha}_{\xi_0}$ and β_0 such that $f^{-1}(\beta_0) = T$. This implies that the intersection of T with $S^{\alpha}_{\beta_0}$, and so also the intersection of $S^{\alpha}_{\xi_0}$ with $S^{\alpha}_{\beta_0}$, contains a stationary set, and is therefore of size α^+. This contradicts the almost-disjointness of $\langle S^{\alpha}_{\xi} \mid \xi \in A_\alpha \rangle$. □

2.5 (N, P)-generic conditions

To verify suitability of different structures, we will review some of the well-known facts originating from the proper-forcing theory:

If $N = \langle N, \in \rangle$ is a (possibly non-transitive) model of ZF$^-$ which contains a partial order $P \in N$, then if G is a P-generic filter over V, we write

$$N[G] = \{\sigma^G \mid \sigma \text{ is a } P\text{-name in } N\}.$$

Fact 2.15 *Consider a structure of the form $\langle H(\theta)^{V[G]}, \in, H(\theta), G \rangle$, where θ is some large enough regular cardinal and G is a P-generic filter for some $P \in N$ for some set N. Assume*

$$\langle N, \in, N \cap H(\theta), N \cap G \rangle \prec \langle H(\theta)^{V[G]}, \in, H(\theta), G \rangle.$$

Then $N = N \cap H(\theta)[N \cap G]$ and $N \cap H(\theta) \cap \mathrm{ORD} = N \cap \mathrm{ORD}$.

Proof. Let q be in N. It holds in $H(\theta)^{V[G]}$ that there is some $\dot{q}$ a name in $H(\theta)$ such that $\dot{q}^G = q$. By elementarity, it holds that there is some name $\dot{q}$ in $N \cap H(\theta)$ and $\dot{q}^{N \cap G} = q$. Similarly, since $H(\theta)$ and $H(\theta)^{V[G]}$ have the same ordinals, the same must hold by elementarity for N and $N \cap H(\theta)$. □

Given an elementary substructure $\langle N, \in \rangle \prec \langle H(\theta), \in \rangle$ and a forcing notion $P \in N$, we can ensure under some conditions a suitable converse to Fact 2.15. We say that a condition $p \in P$ is (N, P)-generic if every P-generic filter G containing p meets in N every dense open set D which is an element of N.

Note that if $P \cup \{P\}$ is a subset of N, then the weakest condition of P is (N, P)-generic.

Fact 2.16 *Assume $P \in N$, G is P-generic, and θ is a large enough regular cardinal (large with respect to P).*

(i) If $\langle N, \in \rangle \prec \langle H(\theta), \in \rangle$, then

$$\langle N[G], \in \rangle \prec \langle H(\theta)^{V[G]}, \in \rangle.$$

(ii) If moreover $p \in P$ is (N, P)-generic and G contains p, then

$$\langle N[G], \in, N, G \cap N \rangle \prec \langle H(\theta)^{V[G]}, \in, H(\theta), G \rangle.$$

Proof. For proofs of these facts, see [22], section III. □

As a corollary, if $P \in M_i^\alpha$ is a cardinal-preserving forcing notion, where M_i^α is as in Fact 2.9, and p is (M_i^α, P)-generic and G is a P-generic filter containing p, then the transitive collapse of $M_i^\alpha[G]$ is an α-suitable structure.

3 The forcing construction

3.1 The main idea

For any pair x, y of subsets of an inaccessible $\alpha \leq \kappa$ let $z = x * y$ be defined by $z = \{2\xi \mid \xi \in x\} \cup \{2\xi + 1 \mid \xi \in y\}$. We will define an iteration $\mathbb{P} = \langle (\mathbb{P}_\alpha, \dot{P}^\alpha) \mid \alpha \leq \kappa \rangle$ so that for each $\alpha \leq \kappa$ inaccessible and generic $G_{\alpha+1}$ for $\mathbb{P}_{\alpha+1}$ the following holds in $L[E][G_{\alpha+1}]$:

(i) $2^\alpha = \alpha^{++}$.

(ii) There exists a wellorder $<_\alpha$ in $L[E][G_{\alpha+1}]$ so that (where "W" is for "wellorder"):
(W*) $x <_\alpha y$ iff for some ordinal $\xi < \alpha^{++}$, $S^\alpha_{\alpha\xi+2\zeta}$ is nonstationary for ζ in $x * y$ and $S^\alpha_{\alpha\xi+2\zeta+1}$ is nonstationary for ζ not in $x * y$.

We will show that (W*) can be equivalently expressed by a statement in $H(\alpha^+)$ of $L[E][G_{\alpha+1}]$:

(W)** $x <_\alpha y$ iff it holds in $H(\alpha^+)$ of $L[E][G_{\alpha+1}]$ that there exists R a subset of α such that for any α-*suitable* model $M = \langle M, \in, E^M \rangle$ *of size* α containing R there exists an ordinal $\xi < (\alpha^{++})^M$ such that $S^{\alpha,M}_{\alpha\xi+2\zeta}$ is nonstationary in M for ζ in $x * y$ and $S^{\alpha,M}_{\alpha\xi+2\zeta+1}$ is nonstationary in M for ζ not in $x * y$.

Note that (W**) is really expressible in $H(\alpha^+)$ of $L[E][G_{\alpha+1}]$ by Fact 2.12.

Lemma 3.1 *The direction from right to left in (W**) is provable.*

Proof. Assume $R \subseteq \alpha$ satisfies the assumption on the right side of (W**), in particular $R \in L[E][G_{\alpha+1}]$. Because the forcing $\mathbb{P}_{\alpha+1}$ preserves cofinalities (see Theorem 3.8) and the forcing $\dot{P}^\alpha$ is an iteration of length α^{++} with support $\leq \alpha$ (see Section 3.2 for details), R is actually an element of $L[E][G_\alpha * g^\alpha_\xi]$, for some $\xi < \kappa^{++}$ and generic filter g^α_ξ for $\dot{P}^\alpha_\xi$. The size of the forcing notion $\mathbb{P}_\alpha * \dot{P}^\alpha_\xi$ is just α^+ in $L[E]$ (again see Section 3.2).

Now, there is a general theorem which says that if $V[G]$ is a generic extension for some complete Boolean algebra P, then all intermediate models $V \subseteq M \subseteq V[G]$ are obtained as generic models over some regular subalgebras of P. Since $L[E] \subseteq L[E][R] \subseteq L[E][G_\alpha * g^\alpha_\xi]$, this implies (when translated back to the language of posets) that R is actually a generic filter for some cofinality-preserving forcing of size less or equal α^+.

Consider the α-suitable model $\langle L_\theta[E][R], \in, E \cap L_\theta[E]\rangle$ for some large enough regular $\theta > \alpha^{++}$. By Fact 2.8 (ii), we can find an α-suitable $\langle N, \in, N \cap E\rangle$ of size α which contains R as an element, and which is a transitive collapse of an elementary substructure of $\langle L_\theta[E][R], \in, E \cap L_\theta[E]\rangle$. By our assumption, in N, there exists an ordinal $\xi < (\alpha^{++})^N$ such that $S^{\alpha,N}_{\alpha\xi+2\zeta}$ is nonstationary in N for ζ in $x * y$ and $S^{\alpha}_{\alpha\xi+2\zeta+1}$ is nonstationary in N for ζ not in $x * y$. By elementarity, there is an $\xi < (\alpha^{++})^{L_\theta[E][R]} = \alpha^{++}$ such that $S^{\alpha}_{\alpha\xi+2\zeta}$ is nonstationary in $L_\theta[E][R]$ for ζ in $x * y$ and $S^{\alpha}_{\alpha\xi+2\zeta+1}$ is nonstationary in $L_\theta[E][R]$ for ζ not in $x * y$.

Since non-stationarity is preserved upwards, these sets remain non-stationary in $L[E][G_{\alpha+1}]$ as required. $\square$

The forcing detailed below is set up to ensure the converse, that is, that we can witness $x <_\alpha y$ in every suitable M of size α.

3.2 Definition of the forcing

Our forcing $\mathbb{P} = \langle(\mathbb{P}_\alpha, \dot{P}^\alpha) \mid \alpha \leq \kappa\rangle$ will be an iteration with Easton support of forcings P^α defined in $V^{\mathbb{P}_\alpha}$ for each inaccessible $\alpha \leq \kappa$, where P^α is itself an iteration of length α^{++} with support of size $\leq \alpha$. Let G_α denote the generic for the iteration $\mathbb{P}_\alpha$, then the forcing P^α for the stage $\alpha \leq \kappa$ is defined in $L[E][G_\alpha]$ according to Definition 3.2 (for further reference, the P^α-generic over $L[E][G_\alpha]$ will be denoted as g^α):

Definition 3.2 $P^\alpha = \langle(P^\alpha_\xi, \dot{Q}^\alpha_\xi) \mid \xi < \alpha^{++}\rangle$ *is an iteration of length* α^{++} *with supports of size* $\leq \alpha$ *defined as follows (more details about the specific forcings are stated in subsequent Sections 3.2.1–3.2.3).*

- *Limit stage* ξ. For a limit ordinal ξ, define P^α_ξ as the iteration $\langle(P^\alpha_\zeta, \dot{Q}^\alpha_\zeta) \mid \zeta < \xi\rangle$ with supports of size $\leq \alpha$.
- *Successor stage* $\xi + 1$. Let g^α_ξ denote the P^α_ξ-generic. The forcing $P^\alpha_{\xi+1}$ is defined as $P^\alpha_\xi * \dot{Q}^\alpha_\xi$, where $\dot{Q}^\alpha_\xi$ is a name for the forcing Q^α_ξ defined in $L[E][G_\alpha][g^\alpha_\xi]$ as follows:

  Order subsets of α in $L[E][G_\alpha][g^\alpha_\xi]$ using an ordering $<^\alpha_\xi$ defined as follows: $x <^\alpha_\xi y$ iff the $L[E]$-least $\mathbb{P}_\alpha * P^\alpha_\xi$-name for x is less than the $L[E]$-least such name for y in the canonical wellorder of $L[E]$. We assume that for $\xi < \zeta$, the ordering $<^\alpha_\xi$ is an initial segment of $<^\alpha_\zeta$.

  - Q^α_ξ is the trivial forcing unless the bookkeeping function F^α denotes at ξ a $\mathbb{P}_\alpha * P^\alpha_\xi$-name for a pair of subsets of α such that $x_\xi <^\alpha_\xi y_\xi$.

– If the bookkeeping function F^α denotes at ξ a $\mathbb{P}_\alpha * P^\alpha_\xi$-name for a pair of subsets of α such that $x_\xi <^\alpha_\xi y_\xi$, we set $Q^\alpha_\xi = Q^{\alpha,0}_\xi * \dot{Q}^{\alpha,1}_\xi * \dot{C}^\alpha_\xi$ defined as follows:

(i) $Q^{\alpha,0}_\xi$ adds a club which will kill the stationary sets in the interval $[\alpha\xi, \alpha\xi + \alpha)$ according to $x_\xi * y_\xi$. See Section 3.2.1 for the precise definition.
Let H^α_ξ be generic for $Q^{\alpha,0}_\xi$. Let $X^\alpha_\xi \in L_{\alpha^{++}}[E][G_\alpha][g^\alpha_\xi][H^\alpha_\xi]$ be a subset of α^+ which codes $G_\alpha * g^\alpha_\xi * H^\alpha_\xi$ so that $L[E][G_\alpha][g^\alpha_\xi][H^\alpha_\xi] = L[E][X^\alpha_\xi]$ (this is possible because this set has hereditary cardinality $< \alpha^{++}$). For future convenience we assume that $X^\alpha_\xi \cap \alpha$ codes G_α. Let X^α_ξ also code a level $\langle L_\beta[E], \in, E \cap L_\beta[E]\rangle$ for some $\alpha^+ < \beta < \alpha^{++}$ such that $L_\beta[E] \models |\xi| \leq \alpha^+$.
We have (where "L" is for "localized"):
(L*) If M is α-suitable and X^α_ξ belongs to M, then ξ is less than $(\alpha^{++})^M$, and $S^{\alpha,M}_{\alpha\xi+2\zeta}$ is not stationary in M for $\zeta \in x_\xi * y_\xi$, and $S^{\alpha,M}_{\alpha\xi+2\zeta+1}$ is not stationary in M for $\zeta \notin x_\xi * y_\xi$.

(ii) $Q^{\alpha,1}_\xi$ localizes the property (L*) of X^α_ξ to a subset $Y^\alpha_\xi \subseteq \alpha^+$ such that:
(L)** For any $\gamma < \alpha^+$ of cofinality α and an α-suitable M of size α containing $Y^\alpha_\xi \cap \gamma$ as an element: If $\gamma = (\alpha^+)^M$ then for some $\bar{\xi}$ less than $(\alpha^{++})^M$, $S^{\alpha,M}_{\alpha\bar{\xi}+2\zeta}$ is not stationary in M for $\zeta \in x_\xi * y_\xi$, and $S^{\alpha,M}_{\alpha\bar{\xi}+2\zeta+1}$ is not stationary in M for $\zeta \notin x_\xi * y_\xi$.
See Section 3.2.2 for the precise definition.

(iii) C^α_ξ is a perfect-tree coding, which codes Y^α_ξ into a subset of α. See Section 3.2.3 for the precise definition.

We aim to show that this forcing preserves cofinalities, and also the desired stationary sets (while killing some other desired stationary sets). We start by defining the forcings $Q^{\alpha,0}_\xi$, $Q^{\alpha,1}_\xi$ and C^α_ξ and analyzing their properties.

3.2.1 The killing stationarity forcing $Q^{\alpha,0}_\xi$

This is a standard forcing. A condition p in this forcing is a closed increasing sequence in α^+ of size $\leq \alpha$ such that p is disjoint from $S^\alpha_{\alpha\xi+2\zeta}$ for each $\zeta \in x_\xi * y_\xi$ and from $S^\alpha_{\alpha\xi+2\zeta+1}$ for each $\zeta \notin x_\xi * y_\xi$. Since all stationary sets in S^α_ξ for $\xi \in A_\alpha$ are composed of ordinals of cofinality α, this forcing is easily seen to be α-closed. A standard argument, which uses the special stationary set S^α_{-1} which we have set apart, shows that this forcing is also α^+-distributive. By GCH at α, this forcing has size α^+ and hence preserves cofinalities.

3.2.2 The localization forcing $Q^{\alpha,1}_\xi$

Work in $L[E][X^\alpha_\xi]$. A condition in $Q^{\alpha,1}_\xi$ is an α^+-Cohen condition $r : \alpha^+ \to 2$, $|r| < \alpha^+$, in $L[E][X^\alpha_\xi]$ with the following properties:

(i) The domain $\mathrm{dom}(r)$ is a limit ordinal.

(ii) $X^\alpha_\xi \cap \mathrm{dom}(r)$ is the even part of r, i.e. for $\gamma \in \mathrm{dom}(r)$, γ belongs to X^α_ξ iff $r(2\gamma) = 1$.

(iii) The property (L**) holds when restricted to $r \restriction \gamma$ for $\gamma \le \mathrm{dom}(r)$ of cofinality α, as in (r**):

(r)** For any limit $\gamma < \alpha^+$ of cofinality α, $\gamma \le \mathrm{dom}(r)$ and a suitable M of size α containing $r \restriction \gamma$ as an element: If $\gamma = (\alpha^+)^M$ then for some limit ordinal $\bar\xi < (\alpha^{++})^M$, $S^{\alpha,M}_{\alpha\bar\xi+2\zeta}$ is not stationary in M for $\zeta \in x_\xi * y_\xi$, and $S^{\alpha,M}_{\alpha\bar\xi+2\zeta+1}$ is not stationary in M for $\zeta \notin x_\xi * y_\xi$.

Lemma 3.3 *$Q^{\alpha,1}_\xi$ is α-closed in $L[E][X^\alpha_\xi]$.*

Proof. This is because (r**) requires some non-trivial behavior only at ordinals of cofinality α. □

The argument in Theorem 3.8 actually shows that this forcing is α^+-preserving (in fact contains an α^+-closed dense subsets). The forcing has size α^+ and so preserves cofinalities.

We state a technical lemma which will be useful later:

Lemma 3.4 *If q is a condition in $Q^{\alpha,1}_\xi$ and $\gamma \ge \mathrm{dom}(q) + \alpha$ is a limit ordinal, then q can be extended to q^* with $\mathrm{dom}(q^*) = \gamma$.*

Proof. Let q^* be any extension with domain γ which codes on the odd part of the interval $[\mathrm{dom}(q), \mathrm{dom}(q) + \alpha)$ the ordinal γ. Then no new instances of (iii) in (r**) arise because no new γ of cofinality α needs to be considered (for no M containing $q^* \restriction [\mathrm{dom}(q), \mathrm{dom}(q) + \alpha)$ can it happen that there is $\gamma \le \mathrm{dom}(q^*)$ with $\gamma = (\alpha^+)^M$), and so q^* is a condition. □

3.2.3 The perfect-tree coding forcing C^α_ξ

We describe the forcing C^α_ξ. Let $Y^\alpha_\xi \subseteq \alpha^+$ be as above. We wish to find $R \subseteq \alpha$ which codes Y^α_ξ in the sense that $L[E][Y^\alpha_\xi][R] = L[E][R]$. We assume (the actual proof of this is delayed to Theorem 3.8) that $L[E][Y^\alpha_\xi]$ has the same cardinals as $L[E]$.

The perfect-tree coding C^α_ξ will use as conditions perfect α-trees with cof ω-splitting, see Section 2.3, but we put extra restrictions on the branches in these trees.

Define a sequence of models $\langle L^E_{\mu_i}[G_\alpha][Y^\alpha_\xi \cap i] \mid i < \alpha^+\rangle$ such that: For each i, μ_i is the least ordinal $\mu < \alpha^+$ of cofinality α such that $\mu > \sup\{\mu_k \mid k < i\}$ (this condition is vacuous for $i = 0$) so that $L^E_{\mu_i}[G_\alpha][Y^\alpha_\xi \cap i]$ satisfies:

(i) It is a model of ZF^- (relative to E);
(ii) it satisfies "α is the largest cardinal";

We will denote the models $L^E_{\mu_i}[G_\alpha][Y^\alpha_\xi \cap i]$ as $\mathcal{A}_i$.

Note that by nature of conditions (i) and (ii), the models are closed under $< \alpha$-sequences existing in $L[E][Y^\alpha_\xi]$: Let $f : \beta \to \mu_i$ for some $\beta < \alpha$, then for some $\mu < \mu_i$, $f : \beta \to \mu$. Let π be an injection from μ into α, $\pi \in \mathcal{A}_i$. Then $\pi \circ f \colon \beta \to \alpha$ belongs to $L[E][G_\alpha]$ and therefore to $\mathcal{A}_i$. It follows that $f = \pi^{-1} \circ (\pi \circ f)$ belongs to $\mathcal{A}_i$.

We say that a set $R \subseteq \alpha$ codes Y^α_ξ below i iff for all $k < i$,

$$k \in Y^\alpha_\xi \Leftrightarrow L^E_{\mu_k}[G_\alpha][Y^\alpha_\xi \cap k, R] \models \mathrm{ZF}^-.$$

For $T \subseteq 2^{<\alpha}$ a perfect α-tree with cof ω-splitting, let $|T|$ denote the least i such that $T \in \mathcal{A}_i$. A condition in C^α_ξ is a perfect α-tree T with cof ω-splitting such that R codes Y^α_ξ below $|T|$ whenever R is a branch through T in a set generic extension. Note that this property is expressible in $\mathcal{A}_{|T|}$. Also note that C^α_ξ is non-empty: for instance the tree which splits everywhere except at limits of uncountable cofinality (in order to obey the condition of cof ω-splitting) is present in $\mathcal{A}_0$ where there are no requirements on coding.

The trees in C^α_ξ are ordered by inclusion : $T \leq T'$ iff T is a subtree of T'. All notational conventions set out in Section 2.3 apply.

We first show that the forcing C^α_ξ is α-closed.

Lemma 3.5 *The forcing C^α_ξ is α-closed in $L[E][G_\alpha][Y^\alpha_\xi]$.*

Proof. Given a decreasing sequence $T_0 \geq T_1 \geq \cdots$ of conditions of length $\gamma < \alpha$, it suffices to realise that if $\nu = \sup\{|T_i| \mid i < \gamma\}$, then the sequence of the trees $T_0 \geq T_1 \geq \cdots$ is a subset of $\mathcal{A}_\nu$, and by $< \alpha$-closure, it is also an element of $\mathcal{A}_\nu$. It follows that the greatest lower bound $\bigcap_{i<\gamma} T_i$ is an element of $\mathcal{A}_\nu$, and hence no new instances of coding occur, which makes $\bigcap_{i<\gamma} T_i$ a condition. $\square$

We show in Theorem 3.8 that C^α_ξ preserves also α^+ (and larger cardinals as well by α^{++}-cc, due to GCH at α), and so it preserves all cofinalities.

The following lemma shows that the splitting levels can be separated by an arbitrary gap. This will be used in the lifting argument, see Stage A on page 28. We say that a cof ω-splitting perfect α-tree T *does not split between* $\mathrm{Split}_k(T)$ *and* β for some $k < \alpha$ and some $\beta < \alpha$ if for every node

$s \in \mathrm{Split}_k(T)$ such that the length of s is less than β it holds that all nodes t in T of length $\leq \beta$ such that either $t \supseteq s^\frown 0$ or $t \supseteq s^\frown 1$ are non-splitting.

Lemma 3.6 *If T belongs to C^α_ξ and $k, \beta < \alpha$ are two ordinals, then there is $T^* \leq_k T$ in C^α_ξ such that T^* does not split between* $\mathrm{Split}_k(T^*)$ *and β.*

Proof. If $|T| = i$ for some $i < \alpha^+$, then because $\mathcal{A}_i$ satisfies ZF^-, there is a thinning $T^* \leq_k T$ in $\mathcal{A}_i$ which does not split between $\mathrm{Split}_k(T)$ and β, and so T^* is a condition in C^α_ξ. □

As with the localization forcing, we need to show that the conditions can be extended arbitrarily.

Lemma 3.7 *If T belongs to C^α_ξ and $|T| \leq i < \alpha^+$, then there is $T^* \leq T$ in C^α_ξ such that $|T^*| = i$.*

Proof. We prove this by induction on i. We may assume $|T| < i$, otherwise there is nothing to prove. If $i = k+1$, then we may also assume by induction that $|T| = k$ and hence that T belongs to $\mathcal{A}_k$.

If $k \in Y^\alpha_\xi$, then we take $T^* \leq T$ to have the property that R is P_T-generic over $\mathcal{A}_k$ whenever R is branch through T^*, where P_T is the forcing whose conditions are the elements of T, ordered by extension (this forcing is isomorphic to the α-Cohen forcing). More precisely, the structure $\mathcal{A}_k$ has size α in $\mathcal{A}_i$, and so we can list all dense open sets of P_T in α-many steps. By induction on $k < \alpha$ construct $T^* \leq T$ so that the nodes on level $k+1$ hit the k-th dense open set. It follows that, $L^E_{\mu_k}[G_\alpha][Y^\alpha_\xi \cap k, R] \models \mathrm{ZF}^-$ for every branch R of T^*, by the P_T-genericity of R. So T^* is a condition and $|T^*| = i$.

If $k \notin Y^\alpha_\xi$, then choose a subset $R_0 \subseteq \alpha$ in $\mathcal{A}_i$ coding a wellordering of α of order type μ_k and take $T^* \leq T$ to be the tree whose branches are exactly the branches R through T such that for all ζ, $\zeta \in R_0$ iff R goes right at the 2ζ-th splitting level of T. Then T^* belongs to $\mathcal{A}_i$ and for R a branch through T^*, (R, T) computes R_0 and hence $L^E_{\mu_k}[G_\alpha][Y^\alpha_\xi \cap k, R]$ is not a model of ZF^-, since it contains R_0 as an element.

If i is a limit ordinal of cofinality less than α, use the α-closure of the forcing. If i has cofinality α, argue as follows: Choose $|T| = i_0 < i_1 < \ldots$ to be an α-sequence cofinal in i which belongs to $\mathcal{A}_i$. Let $T_0 = T$ and for each $k < \alpha$ let $T_{k+1} \in C^\alpha_\xi$ be least in $\mathcal{A}_{i_{k+1}}$ such that $|T_{k+1}| = i_{k+1}$ and $T_{k+1} \leq_k T_k$. Such T_k's exist by induction. If $T^* = \bigcap_{k<\alpha} T_k$, then $T^* \leq T$ belongs to $\mathcal{A}_i$ and satisfies the requirement for belonging to C^α_ξ. So $T^* \leq T$, $|T^*| = i$, as desired. □

3.3 Preservation of the desired stationary sets

Theorem 3.8 $\mathbb{P}$ *preserves the desired stationary subsets of* α^+ *for each inaccessible* $\alpha \leq \kappa$*. In particular* $\mathbb{P}$ *preserves cofinalities and hence cardinals.*

Proof. Let G_α be a generic for $\mathbb{P}_\alpha$. For an inaccessible $\alpha \leq \kappa$, and $\xi < \alpha^{++}$, let g^α_ξ be the generic for P^α_ξ over $L[E][G_\alpha]$. If p is a condition in P^α, we will write $(p(\xi)^0, p(\xi)^1, p(\xi)^2)$ to denote the forcing condition $p(\xi)$ which is an element of the forcing $Q^{\alpha,0}_\xi * \dot{Q}^{\alpha,1}_\xi * \dot{C}^\alpha_\xi$ in $L[E][G_\alpha * g^\alpha_\xi]$.

First notice that for each inaccessible $\alpha \leq \kappa$, the forcing $\mathbb{P}_\alpha$ has size α, and so all stationary subsets of α^+ are preserved in $L[E][G_\alpha]$. Also, if S is a stationary subset of α^+ in $L[E][G_{\alpha+1}]$, i.e. in the extension of $L[E][G_\alpha]$ by P^α, S will remain stationary in further extensions, and in particular in $L[E][G_{\kappa+1}]$, because the iteration $\mathbb{P}(> \alpha)$ is α^{++}-closed.

It follows that it suffices to check that for an inaccessible $\alpha \leq \kappa$, P^α over $L[E][G_\alpha]$ preserves the desired stationary sets. By this we mean the following: Assume that for some $p \in g^\alpha$, and $\xi < \alpha^{++}$, the condition $p \restriction \xi$ forces that the stationary set $S^\alpha_{\alpha\xi+\xi_0} = S$ for some $\xi_0 < \alpha$ should remain stationary. In other words the bookkeeping function F^α at stage ξ either does not denote a name for a pair $x_\xi * y_\xi$ of subsets of α in $L[E][G_\alpha * g^\alpha_\xi]$, or it denotes a name for a pair $x_\xi * y_\xi$ such that $x_\xi <^\alpha_\xi y_\xi$ and the forcing $Q^{\alpha,0}_\xi$ in Definition 3.2 does not explicitly kill the stationarity of S. Then we wish to show that S remains stationary in the whole extension $L[E][G_\alpha * g^\alpha]$.

Our strategy is as follows:

(†) Assume that our p also forces that $\dot{C}$ is a club subset of α^+. We wish to find and extension $q \leq p$ such that q forces that $\dot{C} \cap S$ is non-empty. The condition q will be the limit of a decreasing sequence $\langle p_k \,|\, k < \alpha \rangle$ of conditions in P^α.

We now show how to ensure (†).

Choose a continuous sequence of submodels $M^\alpha_i[G_\alpha]$ of $L_\theta[E][G_\alpha]$ as in Fact 2.8 for some large enough regular $\theta > \alpha^{++}$, with the full elementarity as is ensured by Fact 2.9, with the parameter $R = G_\alpha$ (here we identify G_α with a subset of α), such that $M^\alpha_0[G_\alpha]$ contains p, P^α, ξ, and $\dot{C}$ and other relevant parameters. Since in the generic extension by G_α all stationary sets in $\langle S^\alpha_\xi \,|\, \xi \in A_\alpha \rangle$ remain stationary (see Lemma 2.13 for the definition of these stationary sets), we can apply Lemma 2.14, and conclude that:

(3.9)

There exists i in $S = S^\alpha_{\alpha\xi+\xi_0}$ such that $i = M^\alpha_i[G_\alpha] \cap \alpha^+$ which does not belong to any S^α_β for any $\beta < \alpha^{++}$ in $M^\alpha_i[G_\alpha]$ such that $\beta \neq \alpha\xi + \xi_0$. Fix this $M[G_\alpha] = M^\alpha_i[G_\alpha]$. Let $\langle i_k \,|\, k < \alpha \rangle$ be a sequence cofinal in this i. In preparation for an argument concerning the coordinates in the perfect-tree coding forcing (see paragraph (C) on page 24), we choose this sequence to belong to the least $L_\mu[E][G_\alpha]$, in which the transitive collapse $\bar{M}[G_\alpha]$ of $M[G_\alpha]$ has size α (note that if $\bar{M}[G_\alpha]$ has size α in $L_\mu[E][G_\alpha]$, then i, which is the α^+ of $\bar{M}[G_\alpha]$, has also size α here).

The sequence $\langle p_k \mid k < \alpha\rangle$ will be constructed in $M[G_\alpha]$ in the sense that every proper initial segment of the sequence will be in $M[G_\alpha]$, and the limit q (not in $M[G_\alpha]$) will force $i \in \dot{C}$. Since $i \in S$, this will achieve the desired goal. To make sure that the limit of $\langle p_k \mid k < \alpha\rangle$ is correctly defined, we will also need to ensure that q is $M[G_\alpha]$-generic with respect to restrictions of P^α to $\xi < \alpha^{++}$. Let $\langle D_k \mid k < \alpha\rangle$ be an enumeration of all dense-open subsets of P^α in $M[G_\alpha]$. The next lemma shows that this sequence implicitly contains all relevant dense open sets for the restricted forcings:

Lemma 3.9 *For each $\xi \in \alpha^{++} \cap M[G_\alpha]$, the sequence $\langle D_k \restriction P^\alpha_\xi \mid k < \alpha\rangle$, where $D \restriction P^\alpha_\xi = \{p \restriction \xi \in P^\alpha_\xi \mid p \in D\}$, enumerates all dense open sets in the forcing P^α_ξ which belong to $M[G_\alpha]$.*

Proof. Clearly, if D is dense open in P^α_ξ and an element of $M[G_\alpha]$, then $\bar{D} = \{p \in P^\alpha \mid p = r^\frown q, q \in P^\alpha(> \xi) \text{ and } r \in D\}$ is definable, and hence an element of $M[G_\alpha]$. $\bar{D}$ is also a dense open set in P^α, and so occurs in the sequence $\langle D_k \mid k < \alpha\rangle$, and so $\bar{D} = D_{k_0}$ for some k_0 and $D = D_{k_0} \restriction P^\alpha_\xi$. $\square$

In constructing sequences below, we will need to build a fusion sequence on the coordinates with the perfect-tree coding forcing. Since it is an iteration, we will need to define a suitable form of fusion for names. Although for the proof that the desired stationary sets are preserved a simpler form of fusion suffices, we will formulate and use a more robust fusion which will anticipate the lifting context in Section 4.

The iteration P^α is composed of three types of forcings (when it is nontrivial): the stationarity killing forcing, the localization forcing, and the perfect tree coding. Abusing the notation a little, we will refer to the stages of the forcing where the perfect tree coding forcing C^α_ξ occurs as *fusion coordinates*. We will denote these fusion coordinates as ξ^2, where $\xi < \alpha^{++}$. (We view the coordinate $\xi < \alpha^{++}$ as a triple (ξ^0, ξ^1, ξ^2) with the corresponding condition $(p(\xi)^0, p(\xi)^1, p(\xi)^2)$ so that $p(\xi^0) = p(\xi)^0, p(\xi^1) = p(\xi)^1$, and $p(\xi^2) = p(\xi)^2$.) The collection of all fusion coordinates in α^{++} is denoted as $\mathscr{F}$. This convention will extend naturally to all concepts concerning fusion type arguments.

If P is a forcing notion and $\dot{x}$ a name for a ground model object, we say that $a \in P$ *determines* $\dot{x}$ if there is $y \in V$ such that $a \Vdash \dot{x} = y$.

The condition q desired in (†) above will be the limit of a sequence $\langle p_k \mid k <$

$\alpha\rangle$, decreasing relative to a $\subseteq$-increasing and continuous sequence of supports $\langle F_k \mid k < \alpha\rangle$:

$$p = p_0 \geq_{0,F_0\cap\mathscr{F}} p_1 \geq_{1,F_1\cap\mathscr{F}} \cdots .$$

The sequence $\langle p_k \mid k < \alpha\rangle$ will be constructed to ensure that q (the limit of p_k's) satisfies the following:

(*) For every $k < \alpha$ there is $\bar{k} < \alpha$ such that for every $r \leq q$ if for each $\xi^2 \in F_k \cap \mathscr{F}$, $r \restriction \xi^2$ forces that $r(\xi^2)$ has a stem $\dot{x}$ of length at least $\bar{k}$, and moreover $r \restriction \xi^2$ determines $\dot{x}$ up to $\bar{k}$, then r forces $\zeta \in \dot{C}$ for some $i_k \leq \zeta < i$.

In particular q forces that $i \in \dot{C}$ (because such r's are dense below q).

The sequence $\langle p_k \mid k < \alpha\rangle$ will be constructed by induction. The definition of F_k's needs only to ensure that these sets are $\subseteq$-increasing, continuous, and eventually capture the whole support of its limit q: $F_k \subseteq F_{k+1}$, $F_l = \bigcup\{F_k \mid k < l\}$ for a limit l, and $\bigcup\{F_k \mid k < \alpha\} = \mathrm{supp}(q)$. At limit stages $l < \alpha$, p_l will be defined to be the greatest lower bound of p_k's for $k < l$. Note that the limits are correctly defined and stay within $M[G_\alpha]$ because $M[G_\alpha]$ is closed under all $< \alpha$ sequences from $M[G_\alpha]$ existing in the universe, i.e. in $L[E][G_\alpha]$, and all forcings are α-closed.

Given p_k, we will describe now how p_{k+1} is to be constructed. The construction has two stages.

Stage 0. Determination. Using the construction detailed in the paragraph *Construction, Stage 0* below, construct $p^*_{k+1} \leq_{k,F_k\cap\mathscr{F}} p_k$ (the determined version of p_k) to satisfy the points (i)–(iii) below. See the paragraphs below Definition 2.2 in Section 2.3 to review the relevant notation.

(i) If σ is a function from $F_k \cap \mathscr{F}$ to $2^{<\alpha}$ then σ is determined in the sense that:
   - Either for every $\xi^2 \in F_k \cap \mathscr{F}$, $p^*_{k+1} \restriction \xi^2 | (\sigma \restriction \xi^2)$ is a condition and forces that $\sigma(\xi^2)$ is in $\mathrm{Succ}_k(p_k(\xi^2))$ (we call such σ's *suitable for p^*_{k+1} on the domain $F_k \cap \mathscr{F}$*),
   - Or there is $\zeta^2 \in F_k \cap \mathscr{F}$ such that $p^*_{k+1} \restriction \zeta^2 | (\sigma \restriction \zeta^2)$ is a condition and forces that $\sigma(\zeta^2)$ is not in $\mathrm{Succ}_k(p_k(\zeta^2))$;

(ii) Moreover there is some $\bar{k} < \alpha$ such that whenever σ is suitable for p^*_{k+1} on the domain $F_k \cap \mathscr{F}$ then σ is a function from $F_k \cap \mathscr{F}$ to $2^{\bar{k}+1}$.

(iii) If σ is suitable for p^*_{k+1} on the domain $F_k \cap \mathscr{F}$ and $s = p^*_{k+1}|\sigma$, then $s \restriction \xi^2$ determines $\mathrm{Split}_k(p_k(\xi^2))$ (and hence also $\mathrm{Succ}_k(p_k(\xi^2))$) for each ξ^2 in $F_k \cap \mathscr{F}$.

Construction, Stage 0.

Let $\langle \xi_l \mid l < |F_k \cap \mathscr{F}| \rangle$ be the increasing enumeration of $F_k \cap \mathscr{F}$. We abuse here notation and write ξ_l instead of $(\xi^2)_l$ to avoid over-indexing. We will build inductively a decreasing sequence of conditions $\langle p^{\xi_l} \mid l < |F_k \cap \mathscr{F}| \rangle$ with the greatest lower bound p^*_{k+1}. In order to verify the condition (ii) in the above list, we also identify a sequence of ordinals $\langle \bar{\xi}_l \mid l < |F_k \cap \mathscr{F}| \rangle$ so that: If $\sigma : \xi_l \cap F_k \cap \mathscr{F} \to 2^{<\alpha}$ is suitable on the domain $\xi_l \cap F_k \cap \mathscr{F}$, then its range is included in $2^{\bar{\xi}_l + 1}$.

Such sequences are constructed as follows. Set $p^{\xi_0} = p_k$ and $\bar{\xi}_0 = 0$.

If l is a limit ordinal, set p^{ξ_l} to be the greatest lower bound $\bigwedge_{m<l} p^{\xi_m}$ and $\bar{\xi}_l = \sup\{\bar{\xi}_m \mid m < l\}$.

To construct the condition $p^{\xi_{l+1}}$ if we already have p^{ξ_l}, let $\langle \sigma_m \mid m < \mu \rangle$ be an enumeration of all σ's suitable for $p^{\xi_l} \restriction \xi_l$ on the domain $\xi_l \cap F_k \cap \mathscr{F}$. Note that μ can be taken to be smaller than α because of the bound $\bar{\xi}_l$.

Build a sequence $\langle t^m \mid m < \mu \rangle$ of conditions in the restricted forcing $P^\alpha \restriction \xi_l$ with the following properties:

(3.10)

- (i) $t^0 = p^{\xi_l} \restriction \xi_l$;
- (ii) $t^0 \geq_{k,\xi_l \cap F_k \cap \mathscr{F}} t^1 \geq_{k,\xi_l \cap F_k \cap \mathscr{F}} t^2 \geq_{k,\xi_l \cap F_k \cap \mathscr{F}} \cdots$
- (iii) If m is a limit ordinal, let $t^m = \bigwedge_{n<m} t^n$;
- (iv) Set t^{*m+1} to be a condition below $t^m | \sigma_m$ which determines $\mathrm{Split}_k(p_k(\xi_l))$. To ensure $t^{m+1} \leq_{k,\xi_l \cap F_k \cap \mathscr{F}} t^m$, we will define $t^{m+1}(\xi_q)$ for every $q < l$ from t^{*m+1} using the Maximal Principle as follows (at coordinates $\zeta \notin F_k \cap \mathscr{F}$, we keep $t^{m+1}(\zeta) = t^{*m+1}(\zeta)$):
  - The condition $t^{*m+1} \restriction \xi_q$ forces that $t^{m+1}(\xi_q)$ is a name for the tree obtained from the tree $t^m(\xi_q)$ by replacing the subtree $t^m(\xi_q) | \sigma_m(\xi_q)$ by the tree $t^{*m+1}(\xi_q)$;
  - For any other suitable σ, $\sigma \neq \sigma_m$, the condition $(t^m | \sigma) \restriction \xi_q$ forces that $t^{m+1}(\xi_q)$ is a name for the tree $t^m(\xi_q)$.
    Since the collection of conditions
    $$\{(t^m | \sigma) \restriction \xi_q \mid \sigma \text{ is suitable for } p^{\xi_l} \restriction \xi_l\}$$
    is a maximal antichain below $t^m \restriction \xi_q$, this implies that $t^{m+1} \restriction \xi_q$ forces $t^{m+1}(\xi_q) \leq_k t^m(\xi_q)$. Since this holds for every $q < l$, this implies $t^{m+1} \leq_{k,\xi_l \cap F_k \cap \mathscr{F}} t^m$.

Set $p^{\xi_{l+1}} = (\bigwedge_{m<\mu} t^m)^\frown(p_k(\geq \xi_l))$, and $\bar{\xi}_{l+1} = \max(\bar{\xi}_l, \tau)$, where τ is the supremum of the lengths of nodes in $\mathrm{Split}_k(p_k(\xi_l))$ determined with respect to all suitable σ_m's.

Set $p^*_{k+1} = \bigwedge_{l<|F_k\cap\mathscr{F}|} p^{\xi_l}$ and $\bar{k} = \sup\{\bar{\xi}_l \,|\, l < |F_k \cap \mathscr{F}|\}$. By nature of the construction, we have that $p^*_{k+1} \leq_{k,F_k\cap\mathscr{F}} p_k$, $\bar{k} < \alpha$, and the points (i)–(iii) above are satisfied.

Stage 1. Construct $p_{k+1} \leq_{k,F_k\cap\mathscr{F}} p^*_{k+1}$ as the greatest lower bound of a $\leq_{k,F_k\cap\mathscr{F}}$-decreasing sequence $\langle p^m \,|\, m < \mu'\rangle$, built using the sequence $\langle \sigma_m \,|\, m < \mu'\rangle$ of σ's suitable for p^*_{k+1} on the domain $F_k \cap \mathscr{F}$ identified in Stage 0. Note again that μ' can be taken to be smaller than α.

At the successor step, let first ${}^*p^{m+1}$ be an extension of $p^m|\sigma_m$ which hits the dense open set D_k (by a condition in $M[G_\alpha]$) and determines the least $\zeta \in \dot{C}$ above i_k (and below i), and define p^{m+1} to be the condition which is obtained from ${}^*p^{m+1}(\xi^2)$ for each $\xi^2 \in F_k \cap \mathscr{F}$ by amalgamating ${}^*p^{m+1}|\sigma_m(\xi^2)$ and $p^m(\xi^2) \setminus p^m|\sigma_m(\xi^2)$, in order to ensure $p^{m+1} \leq_{k,F_k\cap\mathscr{F}} p^m$. See (3.10)(iv) above for details.

At limits, define p^m to be the greatest lower bound.

Also make sure that at non-trivial coordinates ξ^0 and ξ^1 in F_k, the condition $p_{k+1}\restriction\xi^l$ for $l \in \{0,1\}$ forces that $p_{k+1}(\xi^l)$ extends past i_k (and stays below i) (this can be done by elementarity of $M[G_\alpha]$, and by invoking Lemma 3.4 for the coordinate ξ^1). Similarly, make sure that for each $\xi^2 \in F_k \cap \mathscr{F}$, the condition $p_{k+1}\restriction\xi^2$ forces that $|p_{k+1}(\xi^2)| \geq i_k$ (by invoking Lemma 3.7).

Verification. Let q be the greatest lower bound of $\langle p_k \,|\, k < \alpha\rangle$. We need to verify that this is a condition in the forcing P^α. We do this by induction on $\xi \in \mathrm{supp}(q) \subseteq M[G_\alpha]$. Assume $q\restriction\xi$ is a condition. Recall that the ordinal i and the sequence $\langle i_k \,|\, k < \alpha\rangle$ cofinal in i were chosen with some care, see (3.9).

(A) Clearly, $q \restriction \xi^0$ forces that $q(\xi^0)$ is a condition because it is forced to be an increasing closed sequence of ordinals with i on top. By our assumption on p, it is forced below p that S (which contains i) is a stationary set which the conditions in ξ^0 do not need to avoid.

(B) We need to verify that $q\restriction\xi^1$ forces that $q(\xi^1)$ is a valid condition. Fix a generic $g^\alpha_\xi * H^\alpha_\xi$ for $P^\alpha_\xi * \dot{Q}^{\alpha,0}_\xi$ which contains $q \restriction \xi^1$. It suffices to verify that the definition of (r**) on page 16 holds for $\gamma = i$. By the construction of q, and by application of Lemma 3.9, the condition $q\restriction\xi^1$ is $(M[G_\alpha], P^\alpha_\xi * \dot{Q}^{\alpha,0}_\xi)$-generic, which implies:

$$\begin{aligned}(3.11)\quad & \langle M[G_\alpha][g^\alpha_\xi * H^\alpha_\xi], M[G_\alpha], M, \in, (g^\alpha_\xi * H^\alpha_\xi) \cap M[G_\alpha]\rangle \prec \\ & \prec \langle L_\theta[E][G_\alpha][g^\alpha_\xi * H^\alpha_\xi], L_\theta[E][G_\alpha], L_\theta[E], \in, g^\alpha_\xi * H^\alpha_\xi\rangle.\end{aligned}$$

In particular, $\mathrm{ORD} \cap M[G_\alpha] = \mathrm{ORD} \cap M[G_\alpha][g^\alpha_\xi * H^\alpha_\xi]$. Moreover, we know that that the domain of the transitive collapse $\pi : M[G_\alpha][g^\alpha_\xi * H^\alpha_\xi] \to \bar{M}[G_\alpha][\bar{g}^\alpha_\xi * \bar{H}^\alpha_\xi]$ is a generic extension, by virtue of (3.11), of an initial segment of the $L[E]$ hierarchy and is therefore an α-suitable structure, so that for some $\bar{\alpha} < \alpha^+$:

$$\bar{M}[G_\alpha][(\bar{g}^\alpha_\xi * \bar{H}^\alpha_\xi) \cap \bar{M}[G_\alpha]] = L_{\bar{\alpha}}[E][G_\alpha][(\bar{g}^\alpha_\xi * \bar{H}^\alpha_\xi)]. \tag{3.12}$$

The set $X^\alpha_\xi \subseteq \alpha^+$ which codes $G_\alpha * g^\alpha_\xi * H^\alpha_\xi$ has a $P^\alpha_{\xi^1}$-name τ in $M[G_\alpha]$ (because P^α and ξ are in $M[G_\alpha]$ and so the name for X^α_ξ is definable in $M[G_\alpha]$). It follows that $X^\alpha_\xi = \tau^{(g^\alpha_\xi * H^\alpha_\xi)}$ is an element of $M[G_\alpha][g^\alpha_\xi * H^\alpha_\xi]$, and by the properties of the collapsing map π we can conclude that $X^\alpha_\xi \cap i$ belongs to $\bar{M}[G_\alpha][(\bar{g}^\alpha * \bar{H}^\alpha_\xi)]$. As X^α_ξ codes the generic $g^\alpha_\xi * H^\alpha_\xi$ (besides G_α), it ensures the nonstationarity of $S^\alpha_{\alpha\xi+2\zeta}$ for $\zeta \in x_\xi * y_\xi$ and of $S^\alpha_{\alpha\xi+2\zeta+1}$ for $\zeta \notin x_\xi * y_\xi$ in all suitable models containing X^α_ξ as an element. By elementarity and the fact that $\pi(X^\alpha_\xi) = X^\alpha_\xi \cap i$, $X^\alpha_\xi \cap i$ ensures the nonstationarity of $S^{\alpha,\bar{M}[G_\alpha]}_{\alpha\xi+2\zeta}$ for $\zeta \in x_\xi * y_\xi$ and of $S^{\alpha,\bar{M}[G_\alpha]}_{\alpha\xi+2\zeta+1}$ for $\zeta \notin x_\xi * y_\xi$ in all suitable models containing $X^\alpha_\xi \cap i$ as an element. Now if N is any suitable model containing $(q(\xi^1))^{g^\alpha_\xi * H^\alpha_\xi}$ as an element such that $(\alpha^+)^N = i$, then N also contains $X^\alpha_\xi \cap i$ as an element (because $(q(\xi^1))^{g^\alpha_\xi * H^\alpha_\xi}$ codes it on its even part) and as $\pi(\alpha^+) = i = (\alpha^+)^N = (\alpha^+)^{\bar{M}[G_\alpha]}$, we have $S^{\alpha,N}_{\alpha\xi+\zeta} = S^{\alpha,\bar{M}[G_\alpha]}_{\alpha\xi+\zeta}$ for each $\zeta < \alpha$, which concludes the argument and shows that $q(\xi^1)$ is indeed forced to be a condition.

(C) We need to verify that $q \restriction \xi^2$ forces that $q(\xi^2)$ is a condition in the coding forcing $\dot{C}^\alpha_\xi$. Fix a generic $g^\alpha_\xi * H^\alpha_\xi * Y^\alpha_\xi$ which contains $q \restriction \xi^2$. We need to show that $T = (q(\xi^2))^{g^\alpha_\xi * H^\alpha_\xi * Y^\alpha_\xi}$ is an element of $\mathcal{A}_i$ (because if this is the case, $|T| = i$ and no new instances of the coding occur).

Because we have chosen at the beginning $\langle i_k \,|\, k < \alpha\rangle$ in the least $L_\mu[E][G_\alpha]$ where $\bar{M}[G_\alpha]$ has size α, we know that T belongs to $L_\mu[E][G_\alpha][Y^\alpha_\xi \cap i]$. By our choice of μ_i's it must be that $\mu \leq \mu_i$, and so T belongs to $\mathcal{A}_i$ as required.

This shows that only the stationary sets which are killed explicitly by the forcings $\dot{Q}^{\alpha,0}_\xi$ are killed, while other stationary sets in $\langle S^\alpha_\xi \,|\, \xi \in A_\alpha\rangle$ are preserved.

In order to show that P^α preserves cofinalities, it suffices to show that α^+ remains a cardinal (preservation of other cofinalities follows by combination of the α-closure and α^{++}-cc of P^α). Modify the above proof: if $\dot{f} : \alpha \to \alpha^+$ is a name for a function, then run the above argument with $\dot{f}$ instead of $\dot{C}$ and find a bound on the range of $\dot{f}$ in $L[E][G_\alpha]$.

This finishes the proof of Theorem 3.8. □

3.4 A definable failure of GCH at every inaccessible less or equal to κ

The argument above actually yields the following corollary.

Corollary 3.10 *The model $L[E][G_{\kappa+1}]$ satisfies for each inaccessible $\alpha \leq \kappa$ that $2^\alpha = \alpha^{++}$ and there is a wellorder of the subsets of α lightface definable in $H(\alpha^+)$.*

Proof. Under GCH, the forcing $\mathbb{P}$ preserves all cardinals (and cofinalities). The wellorder is given by (W**) on page 13. The complexity of the wellorder is Σ_2 (equivalently Δ_2) over $H(\alpha^+)$. $\square$

In the next section, we show that κ remains measurable after forcing with $\mathbb{P}$, which is one of the main theorems in our paper. Note that we needed to iterate the forcing at every $\alpha \leq \kappa$ in order to make allowances for the reflection properties at κ, which is supposed to stay measurable after forcing with $\mathbb{P}$.

4 A definable failure of GCH at a measurable

In this section, we will prove the following theorem:

Theorem 4.1 *(GCH) Starting from a κ^{++}-strong cardinal κ, it is consistent that GCH fails at κ, κ remains measurable, and there is a lightface definable wellorder of $H(\kappa^+)$.*

The proof will be given in a sequence of lemmas. Background material on preservation of measurability, in particular on lifting of the embeddings to generic extensions (e.g. Silver's lemma), can be found in [2].

Using the facts developed in Section 2.4, fix a κ^{++}-strong extender ultrapower embedding $j : V = L[E] \to M$ with critical point κ. Let $G = G_{\kappa+1} = G_\kappa * g^\kappa$ be $\mathbb{P}$-generic, where $\mathbb{P}$ is the forcing defined in Section 3.2.

By standard arguments (see for instance [10]) lift in $L[E][G_{\kappa+1}]$ to $j : L[E][G_\kappa] \to M[G_\kappa * g^\kappa * F]$, using the fact that $H(\kappa^{++}) = L_{\kappa^{++}}[E]$ is included in M, and so the forcing $(P^\kappa)^M$ is identical to P^κ. The generic filter F for the iteration $j(\mathbb{P})$ in the interval $(\kappa, j(\kappa))$ can be constructed in $L[E][G_{\kappa+1}]$ using the $(\kappa^{+3})^M$-closure of this forcing applied to all relevant dense open sets in $M[G_\kappa * g^\kappa]$.

By an argument similar to Theorem 3.8 one can show that $M[G_\kappa * g^\kappa * F]$ is closed under κ-sequences in $L[E][G_{\kappa+1}]$: $M[G_\kappa * g^\kappa * F]$ contains the same subsets of κ as $L[E][G_{\kappa+1}]$, and a fusion-based argument as in Theorem 3.8 ensures that this is enough for $M[G_\kappa * g^\kappa * F]$ to contain all κ-sequences of ordinals.

We will show how to lift to g^κ. First, we shall formulate the natural concept of the κ-closure of $j[g^\kappa]$ and show that it is a filter. We shall later show that this closure is in fact a generic filter. The generic on the M-side is thus in a very elegant way generated by $j[g^\kappa]$.

Working in $V[G_{\kappa+1}]$, set $h = \mathrm{Cl}_\kappa(j[g^\kappa])$ to consist of all conditions $p \in (P^{j(\kappa)})^{M[G_\kappa * g^\kappa * F]}$ (the top forcing in $j(\mathbb{P})$) such that there is a decreasing sequence of length κ of conditions in $j[g^\kappa]$, such that p is greater or equal to the greatest lower bound of this sequence.

Fact 4.2 *$h = \mathrm{Cl}_\kappa(j[g^\kappa])$ is a filter which contains $j[g^\kappa]$.*

Proof. Use the fact that $M[G_\kappa * g^\kappa * F]$ is closed under κ-sequences in $V[G_{\kappa+1}]$ and that the forcing $(P^{j(\kappa)})^{M[G_\kappa * g^\kappa * F]}$ is $j(\kappa)$-closed, so in particular κ^+-closed. Given two elements p, q in h, construct by an induction of length κ a sequence of conditions which is eventually below all conditions in the sequences determining p, q. The limit of this sequence is below p, q and lies in h. □

It follows that in order to lift j, it suffices to show that h meets every dense open set in $M[G_\kappa * g^\kappa * F]$.

Let $j(f)(d) = D$ be a dense open set for some f whose range contains only dense open sets of P^κ in $L[E][G_\kappa]$.

The first part of the proof, meeting of $f(i)$'s, is practically identical to the argument in the proof of Theorem 3.8. We just concentrate on the differences.

First part: Stages 0,1.

We say that q reduces f with respect to the sequence $\langle p_i \mid i < \kappa\rangle$ and the sequence $\langle F_i \mid i < \kappa\rangle$, if q is the greatest lower bound of this sequence and

(4.13)

(*) For every $k < \kappa$ there is $\bar{k} \leq \kappa$ and $F_k \subseteq \mathrm{supp}(q)$ of size $< \kappa$ such that for every $r \leq q$ if for each $\xi^2 \in F_k \cap \mathscr{F}$, $r \restriction \xi^2$ forces that $r(\xi^2)$ has a stem $\dot{x}$ of length at least $\bar{k}$, and moreover $r \restriction \xi^2$ determines $\dot{x}$ up to $\bar{k}$, then r meets $f(k)$.

Lemma 4.3 *For every p in P^κ there is $q \leq p$ as above in (4.13). It follows that some such q is in g^κ.*

Proof. Choose a continuous sequence of submodels $M_k^\kappa[G_\kappa]$ for $k < \kappa$ of $L_\theta[E][G_\kappa]$ as in Fact 2.8, with the full elementarity as is ensured by Fact 2.9, with the parameter $R = G_\kappa$ (here we identify G_κ with a subset of κ), such that $M_0^\kappa[G_\kappa]$ contains p, P^κ, f and other relevant parameters. Then proceed as in (3.9) to fix $i \in S_{-1}^\kappa$, $i = M_i^\kappa[G_\kappa] \cap \kappa^+$, which does not belong to any S_β^κ for $\beta \in \kappa^{++} \cap M_i^\kappa[G_\kappa]$ and $\beta \neq -1$. Fix this $M[G_\kappa] = M_i^\kappa[G_\kappa]$. Let $\langle i_k \,|\, k < \kappa\rangle$ be a sequence cofinal in this i. Choose this sequence to belong to the least $L_\mu[E][G_\kappa]$, in which the transitive collapse $\bar{M}[G_\kappa]$ of $M[G_\kappa]$ has size κ. Let $\langle D_k \,|\, k < \kappa\rangle$ be a listing of all dense open sets of P^κ which lie in $M[G_\kappa]$.

Using the model $M[G_\kappa]$ and the sequence $\langle i_k \,|\, k < \kappa\rangle$ fixed above, construct $\langle p_k \,|\, k < \kappa\rangle$, where the successor step from p_k to p_{k+1} looks as follows:

Stage 0. Determination. Proceed as in Stage 0, Determination on page 21.

Stage 1. Proceed as in Stage 1 on page 23 with the following modification:

- In Stage 1, when constructing p_{k+1}, hit D_k and $f(k)$ (note that because f is in $M[G_\kappa]$, the sequence of $f(k)$'s is actually included in D_k's, but for easy indexing we prefer to hit $f(k)$ at stage $k+1$).

Note that by choosing i in S_{-1}^κ we need not worry that by adding i on top of the condition in the stationarity-killing forcing we might compromise the coding (recall that the stationary set S_{-1}^κ is set aside and not used for coding). □

Fix $q \in g^\kappa$ satisfying (4.13) above.

Second part. Stages A, 0, 1. By elementarity, there are $\bar{d} < j(\kappa)$ and $T_d \subseteq \mathrm{supp}(j(q))$ of size $< j(\kappa)$ such that if $t \leq j(q)$ and for each $\xi^2 \in T_d \cap \mathscr{F}$, $t \restriction \xi^2$ forces that $t(\xi^2)$ has a stem $\dot{x}$ of length at least $\bar{d}$, and moreover $t \restriction \xi^2$ determines $\dot{x}$ up to $\bar{d}$, then t meets $j(f)(d)$. T_d can be taken to be the d-th element of the sequence $j(\langle F_k \,|\, k < \kappa\rangle)$, where $\langle F_k \,|\, k < \kappa\rangle$ is the sequence of supports in the construction of q. Let $C_{\bar{d}}$ be a club in κ so that $j(C_{\bar{d}})(\kappa+1) \geq \bar{d}$, where we write $C_{\bar{d}}(\xi)$ for $\xi < \kappa$ to denote the ξ-th element of $C_{\bar{d}}$ (for existence of such a club, see [10]).

Let $r' \leq q$ be given. We construct $r \leq r'$, in preparation to find t as above, as the fusion limit of $r' = r_0 \geq_{0,E_0 \cap \mathscr{F}} r_1 \geq_{1,E_1 \cap \mathscr{F}} \dots$ using the following construction.

First fix a submodel $M[G_\kappa]$, an ordinal i and a sequence $\langle i_k \,|\, k < \kappa\rangle$ as in the proof of Lemma 4.3, so that $M[G_\kappa]$ contains as parameters P^κ, r', q, $\langle F_k \,|\, k < \kappa\rangle$ (where $\langle F_k \,|\, k < \kappa\rangle$ is the sequence of supports used to construct the condition q above), and other relevant objects. Let $\langle D_k \,|\, k < \kappa\rangle$ be an enumeration of all dense open sets of P^κ in $M[G_\kappa]$.

We will need to do more than in Lemma 4.3 when we define the successor step r_{k+1}. Also, there is a special condition on the selection of supports E_k's:

- For each k, choose E_k so that $F_k \subseteq E_k$ and $E_k \cap (\mathrm{supp}(q) \setminus F_k) = \emptyset$, where $\langle F_k \mid k < \kappa \rangle$ is the sequence of supports used to construct the condition q above.

The construction of r_{k+1} from r_k is subdivided into three stages A, 0, and 1, obtaining conditions $r_k \geq_{k,E_k \cap \mathscr{F}} r^A_{k+1} \geq_{k,E_k \cap \mathscr{F}} r^*_{k+1} \geq_{k,E_k \cap \mathscr{F}} r_{k+1}$:

Stage A. Recall the notion of a κ-tree p not splitting between $\mathrm{Split}_k(p)$ and β for $k, \beta < \kappa$, defined before Lemma 3.6.

Construct r^A_{k+1} by thinning out (using just maximal names) each tree at the coordinates ξ^2 in $(E_k \cup F_{C_{\bar{d}}(k+1)}) \cap \mathscr{F}$ so that:

– For each $\xi^2 \in E_k \cap \mathscr{F}$, $r^A_{k+1} \restriction \xi^2$ forces that $r^A_{k+1}(\xi^2)$ does not split between $\mathrm{Split}_k(r_k(\xi^2))$ and $C_{\bar{d}}(k+1)$.
– For each $\xi^2 \in F_{C_{\bar{d}}(k+1)} \setminus F_k$, $r^A_{k+1} \restriction \xi^2$ forces that $r^A_{k+1}(\xi^2)$ has its stem of length at least $C_{\bar{d}}(k+1)$.

Stage 0. As in Stage 0 on page 21, construct the determined version of r^A_{k+1}, to be denoted as r^*_{k+1}, so that:

(4.14)

- If ξ^2 is in $(E_k \cup F_{C_{\bar{d}}(k+1)}) \cap \mathscr{F}$ and σ is suitable for r^*_{k+1} on the domain $E_k \cap \mathscr{F}$ and $s = r^*_{k+1} | \sigma$, then $s \restriction \xi^2$ determines (where $\mathrm{Length}_\alpha(p)$ for a tree p denotes all elements of p of length α):
  – $\mathrm{Split}_k(r^A_{k+1}(\xi^2))$ and $\mathrm{Length}_{C_{\bar{d}}(k+1)}(r^A_{k+1}(\xi^2))$ for $\xi^2 \in E_k \cap \mathscr{F}$;
  – $\mathrm{Length}_{C_{\bar{d}}(k+1)}(r^A_{k+1}(\xi^2))$ for $\xi^2 \in (F_{C_{\bar{d}}(k+1)} \setminus F_k) \cap \mathscr{F}$.
- There is some $\bar{k} < \alpha$ such that whenever σ is suitable for r^*_{k+1} on the domain $E_k \cap \mathscr{F}$ then σ is a function from $E_k \cap \mathscr{F}$ to $2^{\bar{k}+1}$.

Notice here that unlike in Stage 0 on page 21, we also need to determine a part of the trees at coordinates ξ^2 in $(F_{C_{\bar{d}}(k+1)} \setminus F_k) \cap \mathscr{F}$: the induction process as in Stage 0 on page 21 thus proceeds on $(E_k \cup F_{C_{\bar{d}}(k+1)}) \cap \mathscr{F}$, but only the coordinates in $E_k \cap \mathscr{F}$, where the k-th splitting nodes must be preserved, will be added to the domain of suitable σ's.

Stage 1. As in Stage 1 on page 23, extend r^*_{k+1} to r_{k+1} in order to hit the dense open set D_k.

Note that by extending $r_{k+1} \leq_{k,E_k \cap \mathscr{F}} r^*_{k+1}$, we will not change the determined parts of trees $r^*_{k+1}(\xi^2)$ in (4.14) for the coordinates $\xi^2 \in (E_k \cup F_{C_{\bar{d}}(k+1)}) \cap \mathscr{F}$.

Choose some $r \leq q$ constructed above which is in g.

Lemma 4.4 *Let g^* denote the function which to each $j(\xi^2)$ in $j[\kappa^{++}] \cap \mathscr{F}$ assigns the value $g^\kappa(\xi^2)$, where we write $g^\kappa(\xi^2)$ for the function $\kappa \to \kappa$ which interprets the name for the generic for the perfect-tree coding at ξ^2 in $L[E][G_\kappa][g^\kappa_\xi * H^\kappa_\xi * Y^\kappa_\xi]$. Then the condition $r^* = j(r)|(g^* \upharpoonright j[\mathrm{supp}(r)])$ meets $j(f)(d)$.*

Proof. Let us write $\langle E^*_k \,|\, k < j(\kappa)\rangle$ for the j-image of the sequence of supports $\langle E_k \,|\, k < \kappa\rangle$ used in the construction of r; it follows that $\bigcup_{k<j(\kappa)} E^*_k = \mathrm{supp}(j(r))$. Also, $E^*_\kappa = j[\mathrm{supp}(r)] = j[\bigcup_{k<\kappa} E_k]$. Analogously we write $\langle F^*_k \,|\, k < j(\kappa)\rangle$ for the j-image of the sequence of supports $\langle F_k \,|\, k < \kappa\rangle$ used in the construction of q. It follows that $\bigcup_{k<j(\kappa)} F^*_k = \mathrm{supp}(j(q))$. It also holds $F^*_\kappa = j[\mathrm{supp}(q)] = j[\bigcup_{k<\kappa} F_k]$. By elementarity, $F^*_\kappa \subseteq E^*_\kappa$ and $E^*_\kappa \cap (\mathrm{supp}(j(q)) \setminus F^*_\kappa) = \emptyset$.

We will show that $r^* \leq j(q)$ satisfies:

(†) For each ξ^2 in $F^*_{j(C_{\bar{d}})(\kappa+1)} \cap \mathscr{F}$, $r^* \upharpoonright \xi^2$ forces that $r^*(\xi^2)$ has a stem $\dot{x}$ of length at least $j(C_{\bar{d}})(\kappa+1)$, and moreover $r^* \upharpoonright \xi^2$ determines $\dot{x}$ up to to $j(C_{\bar{d}})(\kappa+1)$.

Since by our choice of $C_{\bar{d}}$ it holds that $d \leq \bar{d} \leq j(C_{\bar{d}})(\kappa+1)$, it follows by (*)(4.13) and elementarity that r^* meets $j(f)(d)$.

In order to verify (†) consider the following points:

- $j(C_{\bar{d}})(\kappa) = \kappa$, and so $F^*_{j(C_{\bar{d}})(\kappa+1)} \cap \mathscr{F} \subseteq [E^*_\kappa \cup (F^*_{j(C_{\bar{d}})(\kappa+1)} \setminus F^*_\kappa)] \cap \mathscr{F}$, where E^*_κ and $(F^*_{j(C_{\bar{d}})(\kappa+1)} \setminus F^*_\kappa)$ are disjoint;
- $g^* \upharpoonright E^*_\kappa$ is suitable for $j(r)$ on the domain E^*_κ because r is in g^κ, and κ is a limit ordinal of uncountable cofinality (and hence the suitability just refers to the elements in $\mathrm{Split}_\kappa(j(r)(\xi^2))$ for ξ^2 in E^*_κ).
- By construction of r, the j-version of (4.14) holds for $j(r)|(g^* \upharpoonright E^*_\kappa)$ (note that $j(r)|(g^* \upharpoonright E^*_\kappa)$ is below the restriction of $r^*_{\kappa+1}$ to $g^* \upharpoonright E^*_\kappa$, where $r^*_{\kappa+1}$ is the $\kappa+1$-th stage of the j-version of the construction in (4.14); in particular $\kappa = \bar{\kappa}$ in the j-version of (4.14)). It follows:
  - If ξ^2 is in $(E^*_\kappa \cup F^*_{j(C_{\bar{d}})(\kappa+1)}) \cap \mathscr{F}$ and $s = j(r)|(g^* \upharpoonright E^*_\kappa)$, then $s \upharpoonright \xi^2$ determines:
  - $\mathrm{Split}_\kappa(j(r)(\xi^2))$ and $\mathrm{Length}_{j(C_{\bar{d}})(\kappa+1)}(j(r)(\xi^2))$ for $\xi^2 \in E^*_\kappa \cap \mathscr{F}$. In this case the tree $j(r)(\xi^2)$ does not split between κ and $j(C_{\bar{d}})(\kappa+1)$ and so $g^*(\xi^2)$ determines not only the branch of length κ, but the unique branch of length at least $j(C_{\bar{d}})(\kappa+1)$. (Note that, crucially, the cof ω-splitting ensures that the tree does not split at κ, which is a regular cardinal).

– $\text{Length}_{j(C_{\vec{d}})(\kappa+1)}(j(r)(\xi^2))$ for $\xi^2 \in (F^*_{j(C_{\vec{d}})(\kappa+1)} \setminus F^*_\kappa) \cap \mathscr{F}$.
In this case, the tree $j(r)(\xi^2)$ has the stem of length at least $j(C_{\vec{d}})(\kappa+1)$.

We have shown that r^* indeed meets $j(f)(d)$ as desired. □

Since r^* is in h, the κ-closure of $j[g^\kappa]$ (because one can build a decreasing sequence of conditions of length κ below r with the length of the stems on the fusion coordinates in $\text{supp}(r)$ being cofinal in κ, and the greatest lower bound of the j-image of this sequence witnesses that r^* is in h), we can conclude that h meets D. Since D was arbitrary, h is a generic filter, and so we can lift j in $L[E][G_{\kappa+1}]$ to an embedding from $L[E][G_{\kappa+1}]$ to $M[G_\kappa * g^\kappa * F * h]$. It follows that κ remains measurable in $L[E][G_{\kappa+1}]$, which finishes the proof of Theorem 4.1.

5 A definable failure of SCH

In this section, we start with the generic extension $L[E][G_{\kappa+1}]$ constructed in Theorem 4.1, and proceed to "definably" collapse κ to $\aleph_\omega$, thus obtaining a definable failure of SCH at the first limit cardinal.

Theorem 5.1 *(GCH) Starting from a κ^{++}-strong cardinal κ, it is consistent that GCH fails at $\aleph_\omega$, $2^{\aleph_n} < \aleph_\omega$ for every $n < \omega$, and there is a lightface definable wellorder of $H(\aleph_{\omega+1})$.*

The proof will be given in the following subsections.

5.1 Definition of the definable collapse forcing

Work for the moment in $L[E]$. Assume $\mu < \nu \leq \kappa$ are Mahlo cardinals, or assume that $\mu = \aleph_0$ and ν is Mahlo. (Mahloness is not really necessary; in fact inaccessible limits of inaccessibles would suffice for the arguments to follow.)

Let I be the set of all inaccessible cardinals in the interval $[\mu, \nu)$. Then in $L[E]$ there exists a sequence of mutually stationary sets with the following properties.

Lemma 5.2 *Work in $L[E]$. There exists a definable sequence $\langle S^\mu_i \mid i \in I\rangle$ such that:*

(i) S^μ_i is a subset of $i^{+4} \cap \text{Cof}(\mu^{++})$ for each $i \in I$.

(ii) The sequence $\langle S_i^\mu \mid i \in I\rangle$ is mutually stationary with closure: For any regular cardinal $\theta > \mu^{++}$ and parameter x in $H(\theta)$, there is a μ^{++}-closed (i.e. closed under μ^+-sequences) elementary submodel N of $H(\theta)$ of size μ^{++} containing x such that $\sup(N \cap i^{+4})$ belongs to S_i^μ for all $i \in I \cap N$.

(iii) Property (ii) also holds if we replace a single S_i^μ by its complement in $i^{+4} \cap \mathrm{Cof}(\mu^{++})$.

Proof. Ad (i). For any ordinal γ which is not a cardinal, let $\langle \beta(\gamma), n(\gamma)\rangle$ be the lexicographically least pair $\langle \beta, n\rangle$ so that there is a Σ_n over $\langle L_\beta[E], \in, E \restriction \beta, E_\beta\rangle$ definable injection from γ to a smaller ordinal. We define S_i^μ to consist of all γ's in $i^{+4} \cap \mathrm{Cof}(\mu^{++})$ greater than i^{+3} such that $n(\gamma)$ is even.

Ad (ii). Suppose it fails. Let the pair θ, x be the least counterexample. Also let $L^E_{\theta_0}$, $L^E_{\theta_1} \ldots$ enumerate the transitive Σ_2-elementary submodels of L^E and set $\theta^* = \theta_{\mu^{++}}$. For any ordinal δ let $N^+(\delta)$ be the least Σ_3-elementary submodel of $H(\theta^*)$ containing δ as a subset and let C be the club of δ's less than μ^{+4} such that $N^+(\delta) \cap \mu^{+4} = \delta$. Let δ^* be the μ^{++}-th element of C; then the transitive collapse of $N^+(\delta^*)$ is sound with Σ_3-projectum δ^* and therefore by condensation (see Fact 2.6) is an initial segment of $L[E]$. Therefore $\langle \beta(\delta^*), n(\delta^*)\rangle$ equals $\langle \beta, 4\rangle$, for some β. Now $N^+(\delta^*)$ is μ^{++}-closed as $\mathrm{cf}(\delta^*) = \mu^{++}$ and therefore $N = H(\theta) \cap N^+(\delta^*)$ is a μ^{++}-closed elementary submodel of $H(\theta)$ containing x such that $N \cap \mu^{+4}$ belongs to S^μ_μ. If i belongs to $I \cap N$, then $\delta_i^* = \sup(N \cap i^{+4})$ is the same as $N^+(\delta_i^*) \cap i^{+4}$ as the latter is the union of the Σ_3-Skolem hull of γ in $L^E_{\theta_i}$ for γ in $N \cap i^{+4}$, $i < \mu^{++}$, and each of these hulls is an element of $N^+(\delta^*)$. By soundness of the transitive collapse of $N^+(\delta_i^*)$ and condensation, δ_i^* is an element of S_i^μ. But then N contradicts the fact that θ, x were chosen to be a counterexample to (ii).

Ad (iii). Fix i. Suppose that the property (iii) fails and let the pair θ, x be the least counterexample. Define $N^+(\delta)$ for each δ as well as δ^* exactly as in the previous paragraph above, and let j be the least inaccessible greater than i. As above, define $\delta_j^* = \sup(N^+(\delta^*) \cap j^{+4})$. Now we set $N_0 = N^+(\delta_j^*)$ and take N_1 to be the μ^{++}-th Σ_4 elementary submodel of N_0 containing i^{+3} as a subset. More precisely, for $\delta < i^{+4}$ define $N_0(\delta)$ to be the least Σ_4 elementary submodel of N_0 containing δ as a subset, and let C_1 be the club of $\delta < i^{+4}$ such that $N_0(\delta) \cap i^{+4} = \delta$; set $N_1 = N_0(\delta_i^*)$, where δ_i^* is the μ^{++}-the element of C_1. Then by the same argument as in the previous paragraph, $N_1 \cap i^{+4} = \delta_i^*$ is an element of $\mathrm{Cof}(\mu^{++})$ not in S_i^μ (as $n(\delta_i^*)$ equals 5). Finally, set N_2 to be the μ^{++}-th Σ_5 elementary submodel of N_1 containing μ^{+3} as a subset. Then $N = N_2 \cap H(\theta)$ is the desired elementary submodel of $H(\theta)$ which shows that the pair θ, x was not a counterexample to property (iii) after all. □

We now show that these mutually stationary sets survived the first forcing in Theorem 4.1.

Lemma 5.3 *Let $\langle S_i^\mu \mid i \in I\rangle$ be as in Lemma 5.2. Then properties (i)–(iii) still hold in $L[E][G_{\kappa+1}]$.*

Proof. Suppose that p is a condition in $\mathbb{P} = \mathbb{P}_{\kappa+1}$, θ is a regular cardinal greater than κ^{++} and p forces $\dot{x}$ to be an element of $\dot{H}(\theta)$ (a name for $H(\theta)$ in the generic extension). Note that it suffices to deal with θ larger than κ^{++} because the general case then follows. For property (ii), we must find $q \le p$ and $\dot{N}$ so that q forces $\dot{N}$ to be a μ^{++}-closed elementary submodel of $\dot{H}(\theta)$ containing $\dot{x}$ and $\dot{N} \cap i^{+4}$ belongs to S_i^μ for each inaccessible $i \in \dot{N}$ which is at least μ. In the ground model, let M be a μ^{++}-closed elementary submodel of $H(\theta)$ containing the name $\dot{x}$ and the forcing $\mathbb{P}$ such that $\sup(M \cap i^{+4})$ belongs to S_i^μ for each inaccessible $i \in M$ which is at least μ. If G is any generic for $\mathbb{P}$, then $M[G_{\mu+1}]$ is elementary in $H(\theta)[G_{\mu+1}]$ simply because the forcing $\mathbb{P}_{\mu+1}$ is contained in M. Moreover $M[G_{\mu+1}]$ is μ^{++}-closed in $H(\theta)[G_{\mu+1}]$. In the tail forcing $\mathbb{P}(>\mu)$ in $L[E][G_{\mu+1}]$, let $q(>\mu)$ be a condition which meets all dense sets on $\mathbb{P}(>\mu)$ which belong to $M[G_{\mu+1}]$ and let $q = p_{\mu+1}{}^\frown q(>\mu)$. Then if G is any $\mathbb{P}$-generic containing q as an element we have that $M[G]$ is elementary and μ^{++}-closed in $H(\theta)[G]$. Moreover $M[G]$ contains $\dot{x}^G$ and $M[G] \cap \mathrm{ORD} = M \cap \mathrm{ORD}$. Hence $M[G]$ witnesses property (ii) for θ and $\dot{x}$, and q is the desired extension of p with $\dot{N}$ a name for $M[G]$.

Verification of property (iii) proceeds in the same way. □

We can now define the forcing $\mathrm{DefCol}(\mu^{+3}, <\nu)$ which will definably collapse the cardinals in the interval $[\mu^{+4}, \nu)$.

Definition 5.4 *Work in $L[E][G_{\kappa+1}]$. Given $c \subseteq \mu^{+3}$ cofinal in μ^{+3}, we denote by $\mathrm{Stat}^\mu(c)$ the forcing defined below which will code c while collapsing the first μ^{+3}-many inaccessible cardinals in I greater than μ. $\mathrm{Stat}^\mu(c)$ will be the basic building block of the forcing $\mathrm{DefCol}(\mu^{+3}, <\nu)$, which will be defined later.*

Let $\langle i_\xi \mid \xi < \mu^{+3}\rangle$ be the increasing enumeration of the first μ^{+3} inaccessible cardinals $\ge \mu$. Let I_c denote the first μ^{+3}-many inaccessibles whose indices belong to c in the sense that $i_\xi \in I_c \leftrightarrow \xi \in c$. The forcing $\mathrm{Stat}^\mu(c)$ shall kill the costationarity of S_i^μ in $\mathrm{Cof}(\mu^{++})$ for each $i \in I_c$, while preserving the costationarity of S_i^μ in $\mathrm{Cof}(\mu^{++})$ for $i \notin I_c$ (by the expression "S_i^μ is costationary in $\mathrm{Cof}(\mu^{++})$" we mean that $(\mathrm{Cof}(\mu^{++}) \cap i^{+4}) \setminus S_i^\mu$ is stationary).

More precisely, a condition p in $\mathrm{Stat}^\mu(c)$ is a subset p of $\bigcup\{(i^{+3}, i^{+4}) \mid i \in I_c\}$ of size at most μ^{++} such that for each i in I_c, $p \cap (i^{+3}, i^{+4})$ is closed and $p \cap (\mathrm{Cof}(\mu^{++}) \cap (i^{+3}, i^{+4}))$ is contained in S_i^μ. A condition p extends a condition q if $p \cap (i^{+3}, i^{+4})$ end-extends $q \cap (i^{+3}, i^{+4})$ for each $i \in I_c$. We call $\{i \in I_c \mid p \cap (i^{+3}, i^{+4}) \neq \emptyset\}$ the support of p.

Lemma 5.5 *Work in $L[E][G_{\kappa+1}]$, and let $c \subseteq \mu^{+3}$ be an arbitrary set cofinal in μ^{+3}. Let γ be the supremum of the first μ^{+3}-many inaccessible cardinals in I. Then:*

(i) $\mathrm{Stat}^\mu(c)$ *collapses cardinals in the interval $[\mu^{+4}, \gamma]$.*
(ii) $\mathrm{Stat}^\mu(c)$ *is μ^{+3}-distributive. Hence it preserves all cardinals up to μ^{+3}.*
(iii) The size of $\mathrm{Stat}^\mu(c)$ *is $\gamma^{\mu^{++}} = \gamma$, and so the forcing preserves cardinals $\geq \gamma^+$. In particular, γ^+ becomes the new μ^{+4}.*
(iv) The desired stationary sets are preserved: more precisely, in the generic extension by $\mathrm{Stat}^\mu(c)$*, the set c is defined by the costationarity of S^μ_i in* $\mathrm{Cof}(\mu^{++})$ *for $i < \gamma$: $\xi \in c$ iff $S^\mu_{i_\xi}$ is costationary in* $\mathrm{Cof}(\mu^{++})$.

Proof. Ad (i). For $\xi \in c$, the generic adds a club C_ξ through i^{+4}_ξ or order type μ^{+3}. It follows that the generic adds a surjection from μ^{+3} onto i^{+4}_ξ because it is possible to partition $i^{+4}_\xi \cap \mathrm{Cof}(\omega)$ into i^{+4}_ξ-many disjoint, cofinal subsets and by a density argument, C_ξ hits each of these pieces.

Ad (ii). Let $\langle D_k \mid k < \mu^{++}\rangle$ be a sequence of dense open sets and let p be a condition. Fix a μ^{++}-closed elementary submodel $M \prec H(\gamma^+)$ of size μ^{++} which contains $\langle D_k \mid k < \mu^{++}\rangle$ (and other relevant parameters) and such that $\sup(M \cap i^{+4})$ is in S^μ_i for each $i \in M \cap I_c$. This is possible by Lemma 5.3(ii). Let us denote as δ_i the supremum $\sup(M \cap i^{+4})$, and fix a sequence $\langle m^i_k \mid k < \mu^{++}\rangle$ cofinal in δ_i, for each $i \in I_c \cap M$. Now build in M a decreasing chain of conditions $p = p_0 \geq p_1 \geq \dots$ of length μ^{++} such that p_{k+1} is in D_k for each $k < \mu^{++}$, and p_{k+1} extends past m^i_k in i^{+4} for each i in the support of p_k. By the choice of M, it follows that the limit of this sequence exists (the limit is the union of the conditions p_k's with the ordinals δ_i added on top) and lies in the intersection $\bigcap\{D_k \mid k < \mu^{++}\}$.

Ad (iii). Obvious.

Ad (iv). Fix an inaccessible $j = i_\xi$ such that $\xi \notin c$ (and so $j \notin I_c$). We want to show that that S^μ_j remains costationary in $\mathrm{Cof}(\mu^{++})$ in the generic extension by $\mathrm{Stat}^\mu(c)$. Let a condition p force that $\dot{C}$ is a club in j^{+4}. We wish to show that there is a condition $q \leq p$ which forces that $\dot{C}$ meets $(j^{+4} \cap \mathrm{Cof}(\mu^{++})) \setminus S^\mu_j$. Fix a μ^{++}-closed elementary submodel $M \prec H(\gamma^+)$ of size μ^{++} which contains $j, p, \dot{C}$ (and other relevant parameters), and such that $\sup(M \cap i^{+4})$ is in S^μ_i for each $i \in M \cap I_c$, and $\sup(M \cap j^{+4})$ is in the complement of S^μ_j in $\mathrm{Cof}(\mu^{++})$. This is possible by Lemma 5.3(iii). As in the previous paragraph, fix sequences $\langle m^i_k \mid k < \mu^{++}\rangle$ cofinal in δ_i for $i \in I_c \cap M$. Also fix a sequence $\langle n_k \mid k < \mu^{++}\rangle$ cofinal in $\sup(M \cap j^{+4})$. Now build in M a decreasing chain of conditions $p = p_0 \geq p_1 \geq \dots$ of length μ^{++}

such that p_{k+1} forces that the k-th element of $\dot{C}$ is greater than n_k, and p_{k+1} extends past m^i_k in i^{+4} for each i in the support of p_k. By the choice of M, it follows that the limit of this sequence exists (the limit is the union of the conditions p_k's with the ordinals δ_i added on top) and forces $\sup(M \cap j^{+4})$ to be in $\dot{C}$. □

We introduce now one more parameter into the definition of $\mathrm{Stat}^\mu(c)$.

Definition 5.6 *Work in $L[E][G_{\kappa+1}]$. We denote by $\mathrm{Stat}^\mu(c,k)$ for $k < \nu$ the forcing $\mathrm{Stat}^\mu(c)$ above with the modification that instead of the first μ^{+3}-many inaccessibles in I we use the k-th segment of μ^{+3}-many inaccessibles: $c \subseteq [\mu^{+3}k, \mu^{+3}k + \mu^{+3})$ and we kill the costationarity of $S^\mu_{i_\xi}$ in $\mathrm{Cof}(\mu^{++})$ for $\xi \in c$.*

In particular $\mathrm{Stat}^\mu(c) = \mathrm{Stat}^\mu(c, 0)$.

Definition 5.7 *Work in $L[E][G_{\kappa+1}]$. We denote by $\mathrm{DefCol}(\mu^{+3}, < \nu)$ the forcing detailed below which will definably collapse all cardinals in the interval $[\mu^{+4}, \nu)$.*

$\mathrm{DefCol}(\mu^{+3}, < \nu)$ is an iteration $\langle (P_i, \dot{Q}_i) \,|\, i \in I\rangle$ of length ν with support of size $\leq \mu^{++}$. To initiate the construction, set $c_0 = \mu^{+3}$ and let $\dot{Q}_0$ be the forcing $\mathrm{Stat}^\mu(c_0) = \mathrm{Stat}^\mu(c_0, 0)$. Let f_0 be $\mathrm{Stat}^\mu(c_0)$-generic. By Lemma 5.5, the generic f_0 can be viewed as a subset of μ^{+3} because the cardinals in the interval $[\mu^{+4}, \gamma]$ were collapsed (where γ is the supremum of the first μ^{+3}-many inaccessibles in I). In the next stage, we force with $\mathrm{Stat}^\mu(f_0, 1)$. In general we define by induction:

$$\dot{Q}_i = \begin{cases} \mathrm{Stat}^\mu(\mu^{+3}, i) & \text{for } i \text{ limit or } i = 0 \\ \mathrm{Stat}^\mu(f_{i-1}, i) & \text{for } i \text{ a successor,} \end{cases}$$

where $F_i = F_{i-1} * f_{i-1}$ is the generic for the i-th stage $P_i = P_{i-1} * \dot{Q}_{i-1}$ of the iteration, where we view f_{i-1} as an (unbounded) subset of μ^{+3} coding the collapsing function. Also, to make sense of the definition, the blocks of inaccessibles are computed in $L[E]$. (Note that $L[E]$ is definable in the generic extension $L[E][G_{\kappa+1}][F_{i-1}]$, and so the i-th segment of inaccessibles above μ is definable here as well.)

Lemma 5.8 *Work in $L[E][G_{\kappa+1}]$. The forcing $\mathrm{DefCol}(\mu^{+3}, < \nu)$ satisfies the following properties:*

(i) It is μ^{+3}-distributive.
(ii) It has ν-cc.
(iii) It collapses the cardinals in the interval $[\mu^{+4}, \nu)$.

(iv) If F is $\mathrm{DefCol}(\mu^{+3}, <\nu)$*-generic, then F is definable in $H(\kappa^+)$ of $L[E][G_{\kappa+1}][F]$.*

Proof. Ad (i). This is basically just like Lemma 5.5(ii), modified for the iteration in the obvious way.

Ad (ii). This is true by the usual Δ-lemma argument because ν is inaccessible, and the supports are bounded in ν.

Ad (iii). Obvious by the construction of the forcing.

Ad (iv). In $H(\kappa^+)$ of $L[E][G_{\kappa+1}][F]$, the restriction $E \restriction \nu^+$ is certainly definable (and E_{ν^+} is empty). It follows that the generic F can be decoded by looking at the costationarity in $\mathrm{Cof}(\mu^{++})$ of the stationary sets S^μ_i for $i \in I$. The verification that the iteration preserves costationarity in $\mathrm{Cof}(\mu^{++})$ of S^μ_i's, whose costationarity is not explicitly killed by the forcing, is just like the argument in Lemma 5.5(iv), modified for the iteration in the obvious way. □

5.2 Definition of the Prikry-type forcing

In order to prove Theorem 5.1, we will use a Prikry-type forcing with definable collapses. To ensure that it satisfies the Prikry property (while having a nice chain condition), we will use the idea of the *guiding generic* which originated in [19] and [6]. The construction of the guiding generic will make crucial use of the chain condition of our definable collapse forcing.

Let $j^* : L[E][G_{\kappa+1}] \to M^*$ be the lifting of the embedding $j : L[E] \to M$, where j was fixed at the beginning of theorem 4.1. We will abuse notation a little and will denote the lifted embedding again by j, so that $j : L[E][G_{\kappa+1}] \to M^*$. The lifted embedding j is still an extender ultrapower embedding: $M^* = \{j(f^*)(\alpha) \mid f^* : \kappa \to L[E][G_{\kappa+1}], \alpha < \kappa^{++}\}$, where every f^* is derived from some f in $L[E]$ with $f^*(\alpha) = (f(\alpha))^{G_{\kappa+1}}$, for $\alpha < \kappa$. See [2] for more details. (In fact this extender ultrapower embedding reduces to a simple measure ultrapower embedding; see Fact 5.10 below.)

We now show that we have in $L[E][G_{\kappa+1}]$ an appropriate guiding generic which will be used to define our forcing.

Lemma 5.9 *Consider the forcing* $\mathrm{DefCol}(\kappa^{+3}, < j(\kappa))$ *defined in M^*. Then there exists in $L[E][G_{\kappa+1}]$ a* $\mathrm{DefCol}(\kappa^{+3}, < j(\kappa))$*-generic over M^*. Let us fix such a generic and denote it as G^{guide}.*

Proof. By Lemma 5.8, $\mathrm{DefCol}(\kappa^{+3}, < j(\kappa))$ is κ^{+3}-distributive in M^* and has $j(\kappa)$-cc. The $j(\kappa)$-cc ensures that all the maximal antichains in M^* can be represented as $j(f^*)(\alpha)$, where $f : \kappa \to H(\kappa^+)$, $f \in L[E]$, has its range included in $\mathbb{P}_\kappa$-names (recall that $\mathbb{P}_\kappa$ is the forcing $\mathbb{P}$ used in Theorem 4.1 without the top forcing at κ, i.e. $\mathbb{P} = \mathbb{P}_\kappa * \dot{P}^\kappa$), and f^* is defined from f by $f^*(\alpha) = (f(\alpha))^{G_\kappa}$ for every $\alpha < \kappa$. More precisely, due to the bounded support of conditions in $\mathrm{DefCol}(\kappa^{+3}, < j(\kappa))$, the forcing $\mathrm{DefCol}(\kappa^{+3}, < j(\kappa))$ can be viewed as a subset of $H(j(\kappa))$ of M^*. If A is a maximal antichain in $\mathrm{DefCol}(\kappa^{+3}, < j(\kappa))$ belonging to M^*, then by the $j(\kappa)$-cc of this forcing, A is smaller than $j(\kappa)$ and so is present in $H(j(\kappa))$ of M^*. It follows by the $j(\kappa)$-closure of $j(\mathbb{P})(j(\kappa))$ (the top forcing at $j(\kappa)$) that A must have been added by the forcing $j(\mathbb{P})_{j(\kappa)}$. It follows that there is a $j(\mathbb{P})_{j(\kappa)}$-name $\dot{A}$ for A which lies in $H(j(\kappa))$ of the original M. Back on the $L[E]$-side, this translates into the above claim that there is a function $f : \kappa \to H(\kappa)$ in $L[E]$ such that $j(f^*)(\alpha) = A$ for some $\alpha < \kappa^{++}$.

Since $L[E]$ satisfies GCH, there are only κ^+-many such functions f^*. Using the κ^{+3}-distributivity of the forcing, and also the κ^+-closure, it is possible to build a sequence of length κ^+ of conditions in $\mathrm{DefCol}(\kappa^{+3}, < j(\kappa))$ which will hit all maximal antichains existing in M^*. □

Now we can define the forcing $\mathbb{P}^{\mathrm{DefCol}}$, combining the definable collapses with Prikry-style conditions, which will be used to prove Theorem 5.1. The forcing is a variation of the by now standard forcing with Levy collapses, used in this form in [12]. The new aspect of our forcing is to collapse κ to $\aleph_\omega$ in such a way that the collapsing generic is definable in the appropriate sense in the extension.

In order to define the forcing, we will use another representation of our model M^*.

Fact 5.10 *The embedding $j : L[E][G_{\kappa+1}] \to M^*$ is also a measure ultrapower embedding: if a measure U is defined in $L[E][G_{\kappa+1}]$ by $X \in U \leftrightarrow \kappa \in j(X)$ then $M^* = \mathrm{Ult}(L[E][G_{\kappa+1}], U)$ (the ultrapower of $L[E][G_{\kappa+1}]$ via U), and the canonical ultrapower embedding is identical to j.*

Proof. Factor j as $i \circ k$, where $i : L[E][G_{\kappa+1}] \to N = \mathrm{Ult}(L[E][G_{\kappa+1}], U)$ is the canonical ultrapower embedding, and $k([f]_U) = j(f)(\kappa)$. Since $\mathscr{P}(\kappa)$ of $L[E][G_{\kappa+1}]$ is included in N and has size κ^{++} (which implies that κ^{++} in N is the same as in $L[E]$), k is the identity below κ^{++}. We now show that $\mathrm{rng}(k) = M^*$ which implies that k is the identity function, and so $j = i$ and $N = M^*$. Every element of M^* is of the form $j(f)(\alpha)$, for some $\alpha < \kappa^{++}$ and some function f in $L[E][G_{\kappa+1}]$ with domain κ. Since k fixes the ordinals $< \kappa^{++}$, it follows that $k(\alpha) = \alpha$, and so $j(f)(\alpha) = (k \circ i)(f)(\alpha) = (k \circ i)(f)(k(\alpha)) = k(i(f)(\alpha))$, which proves the fact because $i(f)(\alpha)$ is in N. □

Definition 5.11 *Work in $L[E][G_{\kappa+1}]$. We define $\mathbb{P}^{\text{DefCol}}$ similarly as in [12]. A condition p in $\mathbb{P}^{\text{DefCol}}$ is a finite sequence of the form $(q_0, \kappa_1, q_1, \ldots, \kappa_n, q_n, I)$, where:*

(i) $\kappa_1 < \cdots < \kappa_n < \kappa$ *are Mahlo cardinals.*
(ii) q_0 *is a condition in* $\text{DefCol}(\omega_3, < \kappa_1)$*, for* $i \in \{1, \ldots, n-1\}$*,* q_i *is a condition in* $\text{DefCol}(\kappa_i^{+3}, < \kappa_{i+1})$*, and finally* q_n *is a condition in* $\text{DefCol}(\kappa_n^{+3}, < \kappa)$.
(iii) I *is a function defined on an element of* U *(the normal measure derived from* j*) such that* $I(\alpha) \in \text{DefCol}(\alpha^{+3}, < \kappa)$ *for every* $\alpha \in \text{dom}(I)$*, and* $[I]_U \in G^{\text{guide}}$.

The extension relation is defined as follows.

Definition 5.12 *A condition $p = (q_0, \kappa_1, q_1, \ldots, \kappa_m, q_m, I)$ extends a condition $\bar{p} = (\bar{q}_0, \bar{\kappa}_1, \bar{q}_1, \ldots, \bar{\kappa}_n, \bar{q}_n, \bar{I})$ in $\mathbb{P}^{\text{DefCol}}$ iff:*

(i) $n \leq m$,
(ii) For $i \in \{0, \ldots, n\}$, $q_i \leq \bar{q}_i$ *and for* $i \in \{1, \ldots, n\}$, $\kappa_i = \bar{\kappa}_i$,
(iii) For $i \in \{n+1, \ldots, m\}$, $\kappa_i \in \text{dom}(I)$*, and* $q_i \leq I(\kappa_i)$.
(iv) $\text{dom}(I) \subseteq \text{dom}(\bar{I})$ *and* $I(\alpha)$ *extends* $\bar{I}(\alpha)$ *for every* $\alpha \in \text{dom}(I)$.

Definition 5.13 *A condition p directly extends $\bar{p}$, which we write as $p \leq^* \bar{p}$, if $p \leq \bar{p}$ and $m = n$.*

If $p = (q_0, \kappa_1, q_1, \ldots, \kappa_m, q_m, I)$ is a condition, then we call the sequence $(q_0, \kappa_1, q_1, \ldots, \kappa_m, q_m)$ the lower part of p and denote it as $\text{lp}(p)$. The I is called the upper part, and denoted as $\text{up}(p)$. Furthermore, let $\kappa(p)$ denote the set $\{\aleph_0, \kappa_1, \ldots, \kappa_m\}$. If κ_i lies in $\kappa(p)$, we write $p(\kappa_i)$ to denote the condition q_i (and $p(\aleph_0)$ denotes q_0). This convention is also extended to lower parts of conditions: if r is equal to $\text{lp}(p)$ for some p, then we write $\kappa(r)$ to denote the cardinals $\kappa(p)$.

The forcing $\mathbb{P}^{\text{DefCol}}$ satisfies the same basic properties as the forcing in [12].

Lemma 5.14 *Work in $L[E][G_{\kappa+1}]$ and let H be $\mathbb{P}^{\text{DefCol}}$-generic.*

(i) The forcing $\mathbb{P}^{\text{DefCol}}$ *has the* κ^+*-cc.*
(ii) (The Prikry property). If σ *is a sentence in the forcing language and* p *is a condition, then there is a direct extension* $q \leq^* p$ *deciding* σ.
(iii) The forcing $\mathbb{P}^{\text{DefCol}}$ *forces* κ *to be* $\aleph_\omega$*, while preserving all cardinals* $\tau, \tau^+, \tau^{++}, \tau^{+++}$ *such that* τ *occurs in* $\kappa(p)$ *for some* $p \in H$.

Proof. Ad (i). Any two conditions with the same lower part are compatible because the upper parts are compatible due to the membership of their U-classes in the guiding generic G^{guide}.

Ad (ii). The argument is virtually identical to the argument in [12], using the fact that $\mathrm{DefCol}(\kappa^{+3}, < j(\kappa))$ is κ^{+3}-distributive in M^* and also κ^{++}-closed in M^* (although only κ^+-closure is relevant for the argument).

Ad (iii). The argument for (iii) is again analogous to [12], in particular $\mathbb{P}^{\mathrm{DefCol}}$ factors naturally below a condition p and $\kappa_i \in \kappa(p)$ into $\mathbb{P}^{\mathrm{DefCol}}_{\leq\kappa_i} \times \mathbb{P}^{\mathrm{DefCol}}_{>\kappa_i}$. The forcing $\mathbb{P}^{\mathrm{DefCol}}_{\leq\kappa_i}$ contains finitely many DefCol-collapses, and the tail forcing $\mathbb{P}^{\mathrm{DefCol}}_{>\kappa_i}$ is κ_i^{+4}-closed under the direct extension relation $\leq^*$. This in combination with the Prikry property implies that all cardinals $\tau, \tau^+, \tau^{++}, \tau^{+++}$ such that τ occurs in $\kappa(p)$ for some $p \in H$, are preserved. (In particular, for each such $\tau > \aleph_0$, $2^\tau = \tau^{++}$, and so GCH fails cofinally often below $\aleph_\omega$.) $\square$

Now, we turn to the question of definability of the generic. We first show that, analogously to other Prikry-type forcings, the generic H is definable from its "lower part".

Let H_0 denote the *lower part* of H, i.e. H_0 contains the lower part of all conditions which lie in H:

$$H_0 = \{\mathrm{lp}(p) \mid p \in H\}.$$

Lemma 5.15 *The generic filter H is definable from the lower part H_0, in particular $L[E][G_{\kappa+1}][H] = L[E][G_{\kappa+1}][H_0]$.*

Proof. Set $T = \{\tau \mid \exists p \in H \ \tau \in \kappa(p)\}$. Note that T is definable just from H_0. Given a sequence r in H_0, we say that I, where I is as in Definition 5.11, and in particular $[I]_U \in G^{\mathrm{guide}}$, does not conflict with H_0 outside r if $(T \setminus \kappa(r)) \subseteq \mathrm{dom}(I)$ and for each λ in $T \setminus \kappa(r)$ there is a condition $s \in H_0$ such that $\lambda \in \kappa(s)$ and $I(\lambda) \geq s(\lambda)$.

Set

(5.15)
$H^* = \{p \in \mathbb{P}^{\mathrm{DefCol}} \mid \mathrm{lp}(p) \in H_0 \wedge$ and $\mathrm{up}(p)$ does not conflict with H_0 outside $\kappa(p)\}$.

It is routine to verify that $H = H^*$. $\square$

Lemma 5.16 *H_0 is definable in $H(\aleph_{\omega+1})$ of $L[E][G_{\kappa+1}][H] = L[E][G_{\kappa+1}][H_0]$.*

Proof. The $H(\aleph_{\omega+1})$ of $L[E][G_{\kappa+1}][H]$ includes $L_{\kappa^+}[E]$, where the sequence $E \restriction \kappa^+$ is definable. Since $\mathbb{P}^{\mathrm{DefCol}}$ factors naturally into $\mathbb{P}^{\mathrm{DefCol}}_{\leq\kappa_i} \times \mathbb{P}^{\mathrm{DefCol}}_{>\kappa_i}$ below each condition p such that $\kappa_i \in \kappa(p)$ (where the second part is κ^{+4}-closed under the relation $\leq^*$), H_0 is definable from the ω-many blocks (given by elements $\tau \in \kappa(p)$ for some $p \in H$) of mutually stationary sets as in Lemma 5.8 (iv). $\square$

5.3 The definable wellorder

Let W denote $L[E][G_{\kappa+1}]$. We need to show that all elements of $H(\aleph_{\omega+1})^{W[H]}$ can be wellordered definably in $H(\aleph_{\omega+1})^{W[H]}$. The basic idea is that every element x of $H(\aleph_{\omega+1})^{W[H]}$ is in $H(\aleph_{\omega+1})^{W[H]}$ constructed from a pair $(a(x), H_0)$, where $a(x)$ lies in $H(\kappa^+)^W$. After verifying that the wellordering of $H(\kappa^+)^W$ (constructed in Theorem 4.1) is still definable in $H(\aleph_{\omega+1})^{W[H]}$, we are finished because H_0 is itself definable in $H(\aleph_{\omega+1})^{W[H]}$ by Lemma 5.16.

Lemma 5.17 *$H(\kappa^+)^W$ is definable in $H(\aleph_{\omega+1})^{W[H]}$, and the ordering $<_\kappa$ defined at (W**) on page 13 still wellorders $H(\kappa^+)^W$ in $H(\aleph_{\omega+1})^{W[H]}$.*

Proof. Note that in the definition of the wellordering $<_\kappa$ in (W**) on page 13, we can equivalently consider just suitable models of the form $\langle L_\alpha[E][z], \in, E \cap L_\alpha[E]\rangle$ for $\alpha < \kappa^+$ and z a subset of κ. We show that the righthand side of the equivalence in (W**) holds in $H(\kappa^+)^{W[H]}$ if and only if it holds in $H(\kappa^+)^W$, which proves the claim. Clearly, if the right-hand side holds in W it will still hold in $W[H]$. Conversely, if the right-hand side holds in $W[H]$ then by reflection, in $W[H]$ there really is a κ-block of canonical $L[E]$-stationary sets which were killed according to $x * y$; but as the forcing that added H is κ^+-cc and therefore does not kill stationarity, this stationary kill already occurred in W. But then $x <_\kappa y$ holds (by the equivalent definition (W*)). □

We now show that all subsets of κ are constructible from $H(\kappa^+)^W$ together with the set H_0. We present this result in the form of a general lemma.

Lemma 5.18 *Suppose that G is P-generic over V, where P is a forcing notion, P is κ^+-cc and $V[G] = V[A]$, where A is a subset of κ. Then any subset of κ in $V[G]$ belongs to $H(\kappa^+)^V[A]$.*

Proof. Suppose that σ_0 is a P-name for A and σ is an arbitrary nice P-name for a subset of κ. Then by the κ^+-cc, σ has size at most κ. Choose some large $H(\theta)$ such that $H(\theta)[A] = H(\theta)^{V[G]}$. In V, let M be an elementary submodel of $H(\theta)$ of size κ containing P, σ_0, σ as elements and κ as a subset. Then $M[G]$ contains A and σ^G as elements. Now $G \cap M$ hits all maximal antichains in M, as by the κ^+-cc, any such maximal antichain is a subset of M. Thus $M[G]$ is elementary in $H(\theta)[G] = H(\theta)^{V[G]}$. Let

$$\pi : \bar{M}[\bar{G}] \to M[G] \tag{5.16}$$

be the inverse of the transitive collapse of $M[G]$. Then π is an elementary embedding to $H(\theta)[G]$ and A, σ^G belong to $\bar{M}[\bar{G}]$. Moreover by elementarity, $\bar{M}[\bar{G}] = \bar{M}[A]$. So we have shown that σ^G, an arbitrary subset of κ in $V[G]$, belongs to $\bar{M}[A]$ and therefore to $H(\kappa^+)[A]$, as desired. □

Based on the above Lemmas, we can define in $H(\aleph_{\omega+1})^{W[H]}$ a wellordering $\prec$ of subsets of $\aleph_\omega$ as follows:

Given x a subset of $\aleph_\omega^{W[H]} = \kappa$, find the $<_\kappa$-least subset $a(x)$ of κ in W such that x is in $L[a(x), H_0]$ and the $\alpha(x)$ such that x is the $\alpha(x)$-th subset of κ in that model. Then $x \prec y$ iff $a(x) <_\kappa a(y)$ or $[a(x) = a(y)$ and $\alpha(x) < \alpha(y)]$.

This finishes the proof of Theorem 5.1.

6 Final comments

We close the paper with some open problems.

(1) The iteration in this paper cannot be used to make 2^κ bigger than κ^{++}. This is analogous to difficulties in obtaining $2^{\aleph_0} > \aleph_2$ with countable support iteration. Another type of forcing seems to be required to obtain $2^\kappa > \kappa^{++}$ in Theorem 4.1. However, there are no obvious candidates because the fusion property was essential for the preservation of measurability (forcings with $< \kappa$ support are extremely hard to lift, and not much is known beyond the surgery argument of W. H. Woodin).

(2) In Theorem 5.1, GCH fails cofinally often below $\aleph_\omega$. The reason is that the new $\aleph_\omega$ was first a measurable cardinal violating GCH before it was collapsed. If one wishes to modify Theorem 5.1 so that $\aleph_\omega$ is the first cardinal which fails to satisfy GCH (thus showing that it is possible to *definably* violate GCH first at $\aleph_\omega$), the most obvious way to proceed would be to combine the extender-based Prikry forcing with collapses (see [13], Section 4) together with some form of coding similar to the one in this paper. Technical details involved might require new ideas in the proof, though.

(3) An essential assumption of the proof was the existence of a "nice" initial model for the large cardinal in question (the extender model $L[E]$). An interesting problem is whether one can extend Theorem 4.1 for instance to a supercompact cardinal κ, i.e. to have that GCH fails at κ, κ is supercompact and there is a definable wellordering of $H(\kappa^+)$. There are techniques to define a wellordering for large cardinals without appropriate inner models, see for instance [1], but the resulting models always satisfy GCH.

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