A characterization of lifting generics for Sacks-like forcings

RADEK HONZIK

Charles University, Department of Logic, Celetná 20, Praha 1, 116 42, Czech Republic
radek.honzik@ff.cuni.cz

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Abstract This article gives a uniform characterization of generics which are used in the context of liftings of elementary embeddings for the generalized Sacks forcing, and variations thereof. It applies to majority of arguments using such forcings providing that the trees are required to be *singular-splitting*, i.e. no splitting at regular levels. The use of singular-splitting trees eliminates the necessity to choose a generic branch from the "tuning fork" object, defined in [5].

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1 Introduction

The purpose of this note is to give a simple uniform characterization of the generic filters which are used in the context of lifting arguments for Sacks-like forcings (both product-style and iteration-style). Examples of such forcings are given in [5] (this is the paper where the technique first appeared), [4], [6], [2], and [3]. The characterization in this paper applies to all these papers, providing that the trees are required to have *singular-splitting* (see below).

An important exception is [4], where the number of liftings is of crucial importance (the characterization describes a uniform way of lifting, and is thus not suitable for situations where we need to tweak the liftings in a special way).

We should also state that the results obtained using the Sacks-like forcings are completely new because they allow for the lifting of iteration-style forcings. Before this technique was discovered in [5], the only way to achieve the failure of GCH at a measurable from the assumption of a κ^{++}-strong cardinal (or $H(\kappa^{++})$-strong cardinal, depending on the notation) was the "surgery argument" due to H. Woodin (see [1] for some details). Woodin's argument is however very fragile and applies only in very special circumstances (such as a product-style Cohen forcing).

2 Singular splitting κ-Sacks forcing

For simplicity, we focus here only on the single κ-Sacks forcing. Other application are straightforward generalizations.

In [8], A. Kanamori generalizes the Sacks forcing at ω for a regular cardinal κ. Although κ can be any regular cardinal, we will deal here only with the case when κ is a (strongly) inaccessible cardinal. There are two new non-trivial requirements which need to be added so that the forcing is κ-closed. If $p \subseteq 2^{<\kappa}$ is a κ-tree with unbounded splitting, then we also demand:

(2.1)

(i) If $s_0 \subseteq s_1 \subseteq \ldots$ is a sequence of nodes in p indexed by $\alpha < \lambda$ for some limit $\lambda < \kappa$, then $\bigcup_{\alpha<\lambda} s_\alpha$ is a node in p;

(ii) If s is a node in p, length(s) is a limit ordinal $< \kappa$, and splitting nodes $t \subseteq s$ are unbounded in s, then s splits ("continuous splitting").

In general, we say that p is a κ-perfect tree for an inaccessible κ, if $p \subseteq 2^{<\kappa}$ is a tree of height κ, for each $s \in p$ there is $t \supseteq s$ in p which splits, and p satisfies the conditions (i) and (ii) in (2.1). We write $p \leq q$ to denote that p is a stronger condition, where $p \leq q \leftrightarrow p \subseteq q$. The forcing notion consisting of all perfect κ-trees together with the inclusion relation will be denoted Sacks$'(\kappa)$, and called the generalized Sacks forcing.

We define some notation which we find useful. Let us write for $p \in$ Sacks$'(\kappa)$, $\mathrm{Split}_\alpha(p) = \{s \in p \,|\, s \text{ is the } \alpha\text{-th splitting node in } p\}$. We write $p \leq_\alpha q$ if $p \leq q$ and $\mathrm{Split}_\alpha(p) = \mathrm{Split}_\alpha(q)$.

Recall that $\langle p_i \,|\, i < \kappa \rangle$ is a fusion sequence if $p_{i+1} \leq_i p_i$ for each $i < \kappa$ (and intersections are taken at limits). We say that the forcing satisfies κ-fusion if every fusion sequence has the greatest lower bound. One can show (see [8]) that Sacks$'(\kappa)$ is κ-closed[1] and satisfies κ-fusion (and hence preserves cardinals up κ^+). Under GCH, the forcing is κ^{++}-cc, and so preserves all cardinals.

In order to obtain a nice definition of lifting (see below in Lemma 4.3), we wish to restrict the splitting levels in p at regular levels $\lambda < \kappa$.

Definition 2.1 *We say that p is a* singular-splitting-Sacks tree *if p is a tree as above, but the condition (ii) is modified as follows: the tree p only splits if* lenght(s) *for $s \in p$ has countable cofinality.*

[1] A forcing is κ-closed if all decreasing sequences of length less than κ have a lower bound.

The collection of all singular-splitting Sacks trees will be denoted as Sacks(κ).

Lemma 2.2 *Singular-splitting-Sacks forcing* Sacks(κ) *is still κ-closed and satisfies κ-fusion.*

Proof. This is a routine argument using the fact that the set of all ordinals below κ which are not regular cardinals (and so where splitting is allowed) is a stationary set. Since κ-perfect trees have continuous splitting (see (ii) in (2.1)) and are closed under increasing sequences of length $< \kappa$ (see (i) in (2.1), the argument follows. $\square$

3 A lifting primer

We will assume GCH throughout. For a review of lifting arguments and related large cardinal concepts, please see [FRIEDMAN, in this journal].

In what follows, j will denote a non-trivial elementary extender ultrapower embedding with a critical point κ as detailed below, unless stated otherwise.

We say (for the purposes of this paper) that an elementary embedding $j : V \to M$ with $\lambda < j(\kappa)$ is an *extender ultrapower embedding* if

$$M = \{j(f)(\alpha) \mid f : \kappa \to V, \alpha < \lambda\}, \tag{3.2}$$

for some cardinal $\lambda > \kappa$ with cofinality at least κ^+ (this implies that M is closed under κ-sequences in V). In what follows not much is lost if you substitute κ^{++} for λ. We say that κ is λ-strong, if moreover $H(\lambda) \subseteq M$.

The basic context is as follows: Suppose we want to force some property φ (such as failure of GCH) at a regular cardinal κ (which is a standard task), but *in addition* preserve some degree of largeness of κ. In order to achieve this goal, we will typically need to do more than in the standard setting when the preservation of largeness of κ is not an issue.

Common to all such argument (with the important exception of Prikry-type iterations, see [7]) is the way how the largeness of κ is verified in a $\mathbb{P}$-generic extension $V[G]$: we will define $\mathbb{P}$ in such a way so that an embedding $j : V \to M$ existing in V lifts to an embedding $j^* : V[G] \to M^*$ existing in $V[G]$. This technique hinges on the following simple fact:

Fact 3.1 (Lifting lemma, Silver) *Let $j : V \to M$ be elementary and $\mathbb{P}$ a forcing notion. If G is $\mathbb{P}$-generic and H is $j(\mathbb{P})$-generic over M and $j[G] \subseteq H$, then j lifts to an elementary $j^* : V[G] \to M[H]$ such that $j^* \restriction V = j$ and $j^*(G) = H$. If moreover j was an extender ultrapower embedding, so is j^*.*

Proof. Define $j^*(\dot{x}^G) = (j(\dot{x}))^H$ for every $\dot{x}$ in V. To show that j^* is well-defined, elementary, and $j^*(G) = H$ is a standard argument using the Forcing Theorem and the fact that $j[G] \subseteq H$. As regards the ultrapower representation, for $f : \kappa \to V$ which takes its range in $\mathbb{P}$-names, define $f^* : \kappa \to V[G]$ by $f^*(\alpha) = (f(\alpha))^G$. Then $M[H] = \{j^*(f^*)(\alpha) \mid f : \kappa \to V, \alpha < \lambda\}$. □

Let X be a subset of a forcing notion $\mathbb{P}$. We write $\mathscr{G}(X)$ for the upper closure of X, i.e.

$$\mathscr{G}(X) = \{q \in \mathbb{P} \mid (\exists p \in X)\ p \leq q\}.$$

If $\mathbb{P}$ does not add new κ-sequences, the lifting argument can be carried out in a uniform way using the following fact:

Fact 3.2 *Let $j : V \to M$ be an extender ultrapower embedding for some $\lambda > \kappa$ as in (3.2) and $\mathbb{P}$ a κ^+-distributive forcing notion in V and let G be $\mathbb{P}$-generic. Then $\mathscr{G}(j[G])$ is $j(\mathbb{P})$-generic over M. By Lifting lemma, j lifts to $V[G] \to M[\mathscr{G}(j[G])]$.*

Proof. Each dense open set $D \in M$ can be represented as $j(f)(\alpha)$ for some $\alpha < \lambda$ and some f which has its range included in dense open sets in $\mathbb{P}$. On the V-side, the intersection $\bar{D} = \bigcap_{\alpha<\kappa} f(\alpha)$ is a dense set in $\mathbb{P}$ and hence there is some $p \in \bar{D} \cap G$. By elementarity, $j(p)$ is in $j(f)(\alpha) = D$. □

We wish to formulate an "as good as possible" variation of Fact 3.2 for forcings $\mathbb{P}$ which add new subsets of κ. Typically, $\mathbb{P}$ would be κ-closed and satisfy some sort of κ-fusion. The paradigmatic example is the forcing Sacks(κ) defined above.

4 Characterization

In this section, we assume for simplicity that our forcing $\mathbb{P}$ is a complete Boolean algebra. In particular all subsets of the domain have supremum and infimum. Also recall that the ordering on $\mathbb{P}$ is then always separative.

Definition 4.1 *Let $\mathbb{P}$ be a α^+-closed forcing notion for some regular cardinal α. For a non-empty $X \subseteq \mathbb{P}$, we denote by $\mathscr{G}_\alpha(X)$ the collection of all conditions p in $\mathbb{P}$ such that there exists a $\leq$-decreasing α-sequence $\langle p_i \mid i < \alpha\rangle$ of elements in X satisfying $\bigwedge_{i<\alpha} p_i \leq p$. We say that $\mathscr{G}_\alpha(X)$ is α-generated by X.*

Example. Let G be $\mathbb{P}$-generic for an α^+-closed forcing $\mathbb{P}$. Then (in $V[G]$) $\mathscr{G}_\alpha(G) = G$.

Proof. Note first that since $\mathbb{P}$ is α^+-closed, it does not add new α-sequences and hence the meaning of $\mathscr{G}_\alpha(G)$ is unambiguous for V and $V[G]$.

We will show by induction that if $\langle p_i \,|\, i < \bar{\alpha}\rangle$ for some limit $\bar{\alpha} \leq \alpha$ is a decreasing sequence of conditions in G, then $\bigwedge_{i<\bar{\alpha}} p_i$ is in G. To this end, notice that $D = \{p \,|\, p \leq \bigwedge_{i<\bar{\alpha}} p_i\} \cup \{p \,|\, (\exists j < \bar{\alpha}) p \perp p_j\}$ is (using separativity) dense, which implies the claim. □

Let us assume that κ is a critical point of an embedding $j : V \to M$. Let $\mathbb{P} = (\mathbb{P}_\alpha, \dot{Q}_\alpha)_{\alpha \leq \kappa}$ be a forcing iteration on inaccessibles $\alpha \leq \kappa$ (typically with Easton support), where each $\dot{Q}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be α-closed. Let $G * g$ be $\mathbb{P}_\kappa * \dot{Q}_\kappa$ generic, and let us write $Q = (\dot{Q}_\kappa)^G$. Assume we have lifted $j : V \to M$ in $V[G * g]$ to

$$j^* : V[G] \to M[G * g * H], \tag{4.3}$$

where H is the "middle generic" for the forcing $j(\mathbb{P})$ in the interval $(\kappa, j(\kappa))$. Assume further that

$$V[G * g] \cap {}^\kappa M[G * g * H] \subseteq M[G * g * H]. \tag{4.4}$$

Note that this is a typical situation in forcing arguments preserving largeness of cardinals, and one which is relatively easy to get; this applies for instance when Q is a κ^+-cc forcing, or Q satisfies κ-fusion (κ-fusion basically ensures the same thing as κ^+-cc in this context: namely that a set of size at most κ in the generic extension is included in a set of size at most κ in the ground model).

Lemma 4.2 (Filter lemma) *Assume the situation in (4.3) above. Then $\mathscr{G}_\kappa(j^*[g])$ is a filter on $j^*(Q)$ which contains $j^*[g]$.*

In particular, if $\mathscr{G}_\kappa(j^*[g])$ happens to hit all dense open sets in $j^*(Q)$ in $M[G * g * H]$, then j^* lifts (in $V[G * g]$) to $j^{**} : V[G * g] \to M[G * g * H * \mathscr{G}_\kappa(j^*[g])]$.

Proof. Since Q is κ-closed, $j^*(Q)$ is κ^+-closed in $M[G * g * H]$ and by (4.4) also in $V[G * g]$. It suffices to show that if $\langle p_i \,|\, i < \kappa\rangle$ and $\langle q_i \,|\, i < \kappa\rangle$ are decreasing sequences of elements in $j^*[g]$, then we can find a decreasing sequence of elements in $j^*[g]$ $\langle r_i \,|\, i < \kappa\rangle$ such that $\bigwedge_{i<\kappa} r_i \leq \bigwedge_{i<\kappa} p_i$ and $\bigwedge_{i<\kappa} r_i \leq \bigwedge_{i<\kappa} q_i$. We define the sequence $\langle r_i \,|\, i < \kappa\rangle$ by induction. Set $r_0 = 1_{j^*(Q)}$. If r_i is constructed, choose r_{i+1} in $j^*[g]$ below p_{i+1}, q_{i+1}, r_i. This is possible because $p_{i+1} = j^*(\bar{p}), q_{i+1} = j^*(\bar{q})$, and $r_i = j^*(\bar{r})$ for some $\bar{p}, \bar{q}, \bar{r}$ in g, and if we choose any $s \leq \bar{p}, \bar{q}, \bar{r}$ in g, then clearly $r_{i+1} = j^*(s)$ is as required. If $k < \kappa$ is a limit ordinal, set $\tilde{r}_k = \bigwedge_{i<k} r_i$. By the construction at the successor step, it holds that $\tilde{r}_k$ is below $\bigwedge_{i<k} p_i$ and $\bigwedge_{i<k} q_i$. It also holds that $\tilde{r}_k$ is in $j^*[g]$: let $\langle \bar{r}_i \,|\, i < k\rangle$ be the decreasing sequence of elements in

g such that $j^*(\bar{r}_i) = r_i$ for each $i < k$. The set $\{p \in Q \,|\, p \leq \bigwedge_{i<k} \bar{r}_i\} \cup \{p \in Q \,|\, (\exists i < k)\ p \perp \bar{r}_i\}$ is dense (we take Q to be a separative order), and any element in g must intersect this set in $\{p \in Q \,|\, p \leq \bigwedge_{i<k} \bar{r}_i\}$. It follows that $\bigwedge_{i<k} \bar{r}_i$ is in g. By elementarity, $j^*(\bigwedge_{i<k} \bar{r}_i) = \tilde{r}_k$ is in $j^*[g]$. As in the successor step, choose r_k to be any condition in $j^*[g]$ below $p_k, q_k, \tilde{r}_k$.

The sequence $\langle r_i \,|\, i < \kappa \rangle$ consists of elements in $j^*[g]$ and by the construction it is obvious that $\bigwedge_{i<\kappa} r_i$ is a lower bound of $\{p_i \,|\, i < \kappa\}$ and $\{q_i \,|\, i < \kappa\}$, that is $\bigwedge_{i<\kappa} r_i \leq \bigwedge_{i<\kappa} p_i$ and $\bigwedge_{i<\kappa} r_i \leq \bigwedge_{i<\kappa} q_i$ as required. □

The filter $\mathscr{G}_\kappa(j^*[g])$ is the least possible:

Lemma 4.3 (Characterization lemma) *Assume the situation in (4.3) above. Let h be any $M[G * g * H]$-generic filter for $j^*(Q)$ which exists in $V[G * g]$ and contains $j^*[g]$. Then $\mathscr{G}_\kappa(j^*[g]) \subseteq h$.*

It follows that if $\mathscr{G}_\kappa(j^*[g])$ is generic, then it is the unique generic in $V[G*g]$ which contains $j^*[g]$.

Proof. Let us denote $M^* = M[G * g * H * h]$. Since h is a generic filter for a $j^*(\kappa)$-closed forcing in $M[G * g * H]$, by the above Example (just below Definition 4.1), $(\mathscr{G}_{j^*(\kappa)}(h))^{M^*}$ (let us denote it as h^*) is included in h. Since M^* contains as elements all κ-sequences of its elements which are available in $V[G * g]$, $\mathscr{G}_\kappa(j^*[g])$ (which is the same whether taken in $V[G * g]$ or in M^*) is included in h^*, and so in h. □

5 Application

Let us assume that Q in (4.3) is just the forcing Sacks(κ) defined in Definition 2.1. In practice, at least the product of κ^{++}-many of these forcings is relevant, but for illustration the single Sacks(κ) will need to suffice.

By an observation first stated in [5], the intersection of j^*-images of all the clubs in κ which exist in $V[G]$ is very thin: it contains just one element, namely κ (this requires the extender ultrapower representation of $M[G * g * H]$):

(5.5) $$\bigcap\{j^*(C) \,|\, C \subseteq \kappa \text{ a club, } C \in V[G]\} = \{\kappa\}.$$

We argue that $\mathscr{G}_\kappa(j^*[g])$ is a generic filter as follows. Given D a dense open set in $j^*(Q)$, we can represent it as $j^*(f)(\alpha)$ for some $\alpha < \lambda$ and some f which has its range included in dense open sets in Q. Using the κ-fusion in Q we can find $p \in g$ so that:

(*) For each $\xi < \kappa$ there is $\bar{\xi} < \kappa$ such that whenever $q \leq p$ and the stem of q has length at least $\bar{\xi}$, then q meets the dense open set $f(\xi)$.

Using (5.5), find $C \subseteq \kappa$ in $V[G]$ so that $j^*(C)(\kappa+1)$, the $\kappa+1$-th element of $j^*(C)$, is greater or equal to $\bar{\alpha}$, where $\bar{\alpha}$ is obtained from α using the elementarity of j^* and (*). Back in $V[G]$, find by fusion $q \leq p$, $q \in g$ so that:

(**) q does not split between $C(\xi)$ and $C(\xi+1)$ for every $\xi < \kappa$.

In $M[G*g*H]$, the tree $j^*(q)$ does not split in the interval $(\kappa, \alpha]$. Crucially, $j^*(q)$ cannot split at κ either, because the definition of singular-splitting trees was specifically designed to avoid splitting at regulars (and κ is regular in $M[G*g*H]$). That is $j^*(q)$ does not split in the interval $[\kappa, \bar{\alpha}]$.

However, $j^*(q)$ obviously does split below κ because $j^*(q)$ restricted to V_κ of $M[G*g*H]$ is just q. Here is the place where the κ-closure of $j^*[g]$ comes in: for each $\xi < \kappa$ there is by density $p \in g$ such that the stem of p has length at least ξ. It follows that there is $r \in \mathscr{G}_\kappa(j^*[g])$ which has its stem of length at least κ. Since $\mathscr{G}_\kappa(j^*[g])$ is always a filter by Lemma 4.2, there is a condition $\bar{r}$ in $\mathscr{G}_\kappa(j^*[g])$ below r and $j^*(q)$. It follows that $\bar{r}$ has the stem of length at least $\bar{\alpha}$, and by (*) (and elementarity) this means that $\bar{r} \in j^*(f)(\alpha) = D$.

It follows that $\mathscr{G}_\kappa(j^*[g])$ is a filter which meets every dense open set, and so is a generic filter. By Lemma 4.3, it is also the unique generic to which Q lifts in $V[G*g]$.

Remark 5.1 This characterization helps to treat uniformly certain technical issues occurring in liftings of Sacks-like forcings (which are not apparent here, but occur in more complex settings). For instance in [5], a separate argument was needed to show that the object defined is also a filter (note that it is not difficult to hit all dense open sets; it is difficult to hit them compatibly, i.e. with a filter).

Remark 5.2 Note that the κ-closure $\mathscr{G}_\kappa(j^*[g])$ can only give non-trivial information for forcings Q where the conditions $q \in Q$ are of size at least κ. For instance if Q is just the single κ-Cohen forcing, then the κ-closure $\mathscr{G}_\kappa(j^*[g])$ is equal to g.

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