Marta Bílková

Uniform Interpolation and Propositional Quantifiers in Modal Logics

Abstract. We investigate uniform interpolants in propositional modal logics from the proof-theoretical point of view.

Our approach is adopted from Pitts' proof of uniform interpolation in intuitionistic propositional logic [15]. The method is based on a simulation of certain quantifiers ranging over propositional variables and uses a terminating sequent calculus for which structural rules are admissible.

We shall present such a proof of the uniform interpolation theorem for normal modal logics K and T. It provides an explicit algorithm constructing the interpolants.

Keywords: modal logic, sequent calculus, interpolation, propositional quantifiers

1. Introduction

The uniform interpolation property for a propositional logic is a strengthening of the Craig interpolation property. It states that for every formula A and any choice of propositional variables $\bar{q}$, there is a *post-interpolant* $I_{post}(A, \bar{q})$ such that (i) $A \to I_{post}(A, \bar{q})$ is provable and (ii) whenever $A \to B$ is provable for a formula B whose variables shared with A are among $\bar{q}$, one has $I_{post}(A, \bar{q}) \to B$. Similarly, for every formula B and any choice of propositional variables $\bar{r}$ there is a *pre-interpolant* $I_{pre}(B, \bar{r})$ such that (i) $I_{pre}(B, \bar{r}) \to B$ is provable and (ii) whenever $A \to B$ is provable for a formula A whose variables shared with A are among $\bar{r}$, one has $A \to I_{pre}(B, \bar{r})$.

Uniform interpolants are unique up to the provable equivalence. Concerning Craig interpolation this means that every implication has the minimal and the maximal interpolants w.r.t. the provability ordering.

The situation is easy when dealing with logics satisfying *local tabularity* [5], which means that there is only *finitely* many nonequivalent formulas for each finite number of propositional variables. If a logic satisfies both local tabularity and Craig interpolation then the conjunction of all formulas $I(\bar{q})$ implied by $A(\bar{p}, \bar{q})$ is the post-interpolant of A, and the disjunction of all formulas $J(\bar{r})$ implying $B(\bar{r}, \bar{s})$ is the pre-interpolant of B. This simple

Presented by **Heinrich Wansing**; *Received* February 15, 2005

Studia Logica (2007) 85: 1–31
DOI: 10.1007/s11225-007-9021-5

argument works e.g. in the case of classical propositional logic or modal logic **S5**, while it is not the case of modal logics **K**, **T**, **K4**, **S4**.

The phenomenon of the existence of uniform interpolants can also be viewed as the possibility of a simulation (or equivalently an elimination) of certain propositional quantifiers. If we can simulate propositional quantification satisfying usual reasonable properties (given e.g. by usual quantifier axioms and rules) then the simulations of $\exists\bar{p}A$ and $\forall\bar{r}B$ are the post-interpolant of $A(\bar{p},\bar{q})$ and the pre-interpolant of $B(\bar{q},\bar{r})$ respectively.

The main point is that even if the logic does not satisfy local tabularity we can still keep the information "to be the uniform interpolant" *finite* and thus represented by a single formula (a conjunction in the case of the existential quantifier or a disjunction in the case of the universal quantifier).

A semantic proof of uniform interpolation based on a simulation of propositional quantifiers was given by Visser in [19] for modal logics **K**, Gödel-Löb's logic of provability **GL** and Grzegorczyk's logic **S4Grz**. Visser's semantic proof uses a model theoretical argument based on bisimulations on Kripke models. The proof yields a semantical characterization of the simulated so-called *bisimulation* quantifiers and also a complexity bound of uniform interpolants in terms of $\Box$-depth is obtained. However, the proof does not provide us with a construction of the interpolants. A similar semantic argument should also work for modal logic **T** but it is not given in Visser's paper. For **GL**, uniform interpolation was first proved by Shavrukov in [17].

So far no proof theoretical proof which would provide us with a construction of uniform interpolants has been given for modal logics.

We concentrate on a proof-theoretical method which was introduced by Pitts in [15] where he proved that intuitionistic propositional logic satisfies uniform interpolation, which had not been known before. In this case, a semantic argument using bisimulations on Kripke models was given later by Ghilardi and Zawadowski in [10] and independently by Visser in [19]. Pitts' argument uses a simulation of propositional quantifiers in the framework of a sequent proof system. The main point of keeping the information "to be the uniform interpolant" finite and thus represented by a single formula is in a use of a terminating sequent proof system, i.e., a proof system in which any backward proof-search terminates.

The main advantage of the proof-theoretical method is that it provides an explicit effective (and also easily implementable) construction of uniform interpolants. In this paper we shall apply this method to modal logics **K** and **T**. We find the case of logic **T** interesting since, although the proof is analogous to that for **K**, it makes use of a sequent calculus that includes a loop-preventing mechanism to enforce its termination.

Ghilardi and Zawadowski in [9] proved that uniform interpolation fails for modal logic **S4**. It follows immediately that it fails for modal logic **K4** as well. Since this easy observation is not mentioned in literature and in our further work we are interested in logics extending **K4** we include it in this paper.

The paper is organized as follows:

- In Section 3 we briefly discuss the failure of uniform interpolation in modal logic **K4**.
- In Section 4 we define sequent calculi Gm_K for modal logic **K** and Gm_T for modal logic **T** in a standard way (at least from our point of view).
  - In Subsection 4.1 we briefly explain how proof search in modal logic looks and discuss the termination problem. Then we define a terminating calculus Gm_T^+ for the logic **T**.
  - In Subsection 4.2 we show that structural rules are admissible in our calculi and we prove that Gm_T^+ is equivalent to Gm_T. Later only the cut admissibility is needed.
- In Section 5 we prove the main theorem which shows how to construct the formula which serves as the uniform pre-interpolant in the logic **K** (or, equivalently, as the formula simulating universal propositional quantification).
  - In Subsection 5.1 we discuss propositional quantification in modal logics and we introduce a calculus for the second order logic $\mathbf{K}^2$ (i.e. Gm_K extended by propositional quantification). We prove that the formula constructed in the previous theorem simulates propositional universal quantifier in Gm_K.
  - In Subsection 5.2 we conclude that the logic **K** satisfies the uniform interpolation property which is an immediate corollary of the previous simulation of propositional quantification in Gm_K. We show that we have indeed constructed the interpolants proving the main theorem.
- In Section 6 we prove an analogue of the main theorem also for the logic **T**.

2. Preliminaries

We consider propositional modal logics and quantified propositional modal logics. We follow literature in referring to quantified propositional modal logics as to second order propositional modal logics.

The letters $A, B, \ldots$ range over formulas, the letters $p, q, \ldots$ range over propositional variables, Greek letters $\Gamma, \Delta, \ldots$ range over finite multisets of formulas. We write Γ, Δ for the multiset union of Γ and Δ.

We use the following propositional second order modal language and definition of formulas:

$$A := p|\Box A|A \wedge B|\neg A|\forall pA.$$

Logical connectives $\vee, \rightarrow, \leftrightarrow$, and the constants $\top, \bot$ are defined as usual, and $\exists pA \equiv_{df} \neg\forall p\neg A$, $\Diamond A \equiv_{df} \neg\Box\neg A$.

$\Box\Gamma$ denotes the multiset $\{\Box A|A \in \Gamma\}$. $\Gamma^{\Box}$ denotes the multiset $\{A|\Box A \in \Gamma\}$. Writing $A(\bar{p}, \bar{q})$ we mean that *all* propositional variables of A are among $\bar{p}, \bar{q}$. $Var(\Gamma)$ stands for the set of all variables free in the multiset Γ.

Quantifiers bind propositional variables; we adopt the usual definition of the scope, free, and bounded variables.

The *weight* $w(A)$ of a formula A is defined as follows:

- $w(p) = 1$
- $w(B \circ C) = w(B) + w(C) + 1$
- $w(\neg B) = w(\Box B) = w(B) + 1$

The weight $w(\Gamma)$ of a multiset Γ is the sum of the weights of the formula occurrences from Γ.

We consider the minimal propositional normal modal logic **K** with its Hilbert style formalization H_K (we treat axioms as schemata):

- (classical) propositional tautologies
- **K**: $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- Rules: Modus Ponens and the Necessitation rule: $A/\Box A$.

The calculus H_K is complete w.r.t. the class of all Kripke frames.

The calculus H_T for modal logic **T** results from adding the reflexivity scheme **T**: $\Box A \rightarrow A$ to H_K and is complete w.r.t. the class of reflexive Kripke frames.

The calculus H_{K4} is the system that results from adding the transitivity scheme **4**: $\Box A \rightarrow \Box\Box A$ to H_K. H_{K4} is complete w.r.t. the class of transitive Kripke frames.

Extending H_{K4} by **T** yields the calculus H_{S4} complete w.r.t. the class of reflexive and transitive Kripke frames.

We use $\vdash_{H_L}$ for provability in the calculus H_L.

More on modal logics and their semantics can be found in [1], [5].

3. Logic K4

Since our aim of further work is to investigate uniform interpolation in provability logics **GL** and **S4Grz** which extend modal logic **K4** let us briefly discuss the failure of uniform interpolation in **K4**.

It is known that modal logic **S4** does not have the uniform interpolation property. A counterexample was provided by Ghilardi and Zawadowski in [9].

Using the following translation from **S4** to **K4** and the fact that **K4** is a subsystem of **S4** we conclude that **K4** does not have the uniform interpolation either. Although it is an easy observation, we include it here since as far as we know it is not mentioned in the literature.

DEFINITION 3.1. Translation $A^\star$ of a modal formula A:

- $p^\star = p$
- $(A \circ B)^\star = A^\star \circ B^\star$
- $(\Box A)^\star = \Box A^\star \wedge A^\star$, i.e., $(\Box A)^\star = \boxdot A^\star$

LEMMA 3.2. *[2]*

$$\vdash_{H_{S4}} A \quad \textit{iff} \quad \vdash_{H_{K4}} A^\star$$

$$\vdash_{H_{S4}} A \leftrightarrow A^\star$$

LEMMA 3.3. *[9] There is a modal formula $B(p_1, p_2, q)$ which does not have a uniform post-interpolant $I_{post}(B, q)$ in **S4**, i.e., there is no formula $I_{post}(B, q)$ satisfying*

- $\vdash_{H_{S4}} B \to I_{post}(B, q)$
- *for all $C(q, \bar{r})$ such that $\vdash_{H_{S4}} B \to C$, $\vdash_{H_{S4}} I_{post}(B, q) \to C$*

The counterexample provided in [9] is :

$$B \equiv p_1 \wedge \Box(p_1 \to \Diamond p_2) \wedge \Box(p_2 \to \Diamond p_1) \wedge \Box(p_1 \to q) \wedge \Box(p_2 \to \neg q)$$

There is no formula simulating $\exists p_1 \exists p_2 B$. It follows that B cannot have a uniform post-interpolant. See also [19].

COROLLARY 3.4. *There is a modal formula which does not have a uniform post-interpolant in **K4**.*

PROOF. Consider the **S4** counterexample $B(p_1, p_2, q)$. Consider for the contradiction that **K4** does have the uniform interpolation property. This means that for $B^\star$, there is a formula $I_{post}(B^\star, q)$ such that $\vdash_{H_{K4}} B^\star \rightarrow I_{post}(B^\star, q)$ and for all $C(p_1, p_2, \overline{r})$ we have that $\vdash_{H_{K4}} B^\star \rightarrow C$ implies $\vdash_{H_{K4}} I_{post}(B^\star, q) \rightarrow C$. Then we have the same for all $C^\star$ of the form of a translation of a formula C. Moreover by the fact that $\vdash_{H_{K4}} A$ implies $\vdash_{H_{S4}} A$ and that $\vdash_{H_{S4}} A \leftrightarrow A^\star$ we obtain $\vdash_{H_{S4}} B \rightarrow I_{post}(B^\star, q)$.

Using the property of the translation $\vdash_{H_{S4}} A$ iff $\vdash_{H_{K4}} A^\star$, and again the fact that $\vdash_{H_{K4}} A$ implies $\vdash_{H_{S4}} A$, and that $\vdash_{H_{S4}} A \leftrightarrow A^\star$, yields the following: for all C, $\vdash_{H_{S4}} B \rightarrow C$ implies $\vdash_{H_{S4}} I_{post}(B^\star, q) \rightarrow C$. But then we have obtained the uniform interpolant for B in **S4** which is the desired contradiction. ■

4. Sequent calculi

Our proofs are based on sequent calculi Gm_K for modal logic **K** and Gm_T^+ for modal logic **T** which have certain properties: both calculi that we shall use are *terminating*, which means that we can define a well founded quasi-ordering on sequents such that each pair of a premiss and a conclusion of a rule lies in this relation. In other words, this means that every backward proof-search in the calculus terminates. This is unproblematic in the case of modal logic **K** where a naturaly defined sequent calculus suffices. As of logic **T**, a simple loop-checker has to be included in the definition of a sequent calculus to enforce termination.

First we introduce the sequent calculus Gm_K for modal logic **K** and Gm_T for modal logic **T** in a natural way. Then we discuss termination and define the sequent calculus Gm_T^+ for modal logic **T** including a loop preventing mechanism. Next we prove their structural properties - admissibility of weakening, contraction, and the cut rules; and we show that Gm_T^+ is indeed equivalent to Gm_T.

For more on sequent calculi for modal logics see e.g. [20], [18], [7].

DEFINITION 4.1. Sequent calculus Gm_K:

$$\Gamma, p \Rightarrow p, \Delta$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \wedge\text{-l} \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ \vee\text{-r}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\ \neg\text{-r} \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}\ \neg\text{-l}$$

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ \wedge\text{-r} \qquad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \vee\text{-l}$$

$$\frac{\Gamma \Rightarrow\ A}{\Box\Gamma, \Pi \Rightarrow \Box A, \Sigma}\ \Box_K$$

Π contains only propositional variables and Σ contains only propositional variables and boxed formulas in the $\Box_K$ rule.

Sequent calculus Gm_T results from adding the following modal rule to Gm_K:

$$\frac{\Gamma, \Box A, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \Box_T$$

Antecedents, *succedents* and *principal formulas* are defined as usual. We have chosen a formalization which is usual when dealing with proof search related problems, in particular we use multisets of formulas to treat sequents. Therefore we consider antecedents and succedents to be finite multisets of fomulas.

In the $\Box_K$-rule, $\Box A$ and all formulas from $\Box\Gamma$ are principal. The *height* of a proof is just its height as a tree.

The *weight* of a sequent is the sum of the weight of its antecedent and the weight of its succedent. Observe that this function decreases in each backward application of a rule of Gm_K, while this is not the case of the $\Box_T$ rule where it increases.

In what follows, we suggest reader to read rules and proof figures bottom up.

4.1. Termination

Let us briefly explain how a proof search in modal logics **K** and **T** works.

We start with a sequent $\Gamma \Rightarrow \Delta$. Applying rules of the calculus backwards we create a tree whose nodes are labelled by sequents. Applying a rule, we create a predecessor node(s) of the current node labelled by the conclusion of the applied rule and label the new node(s) by the premiss(es) of the rule. We proceed using the invertible rules until we reach a sequent in which all formulas are either atomic or boxed, say $\Box\Gamma, \Pi \Rightarrow \Box\Delta, \Lambda$. Let us call it a *critical sequent.* If it is not an initial sequent and $\Box\Delta$ is nonempty we apply the $\Box_K$-rule (we make a modal jump) and create a predecessor node(s) labelled by sequents $\Gamma \Rightarrow B$, for all $B \in \Delta$. We continue until there is no rule to be applied.

Leaves of the tree are labelled by sequents, on which no rule can be applied - they are either initial sequents or unprovable sequents. We mark the leaves as follows - the initial sequents as positive and the others as negative. We continue marking the sequents in the tree as follows: a critical sequent is marked as positive if *at least one* of its predecessors has been marked as positive. Any other sequent is marked as positive if *all* its predecessors have been marked as positive.

If the bottom sequent has been marked as positive, it is provable and by deleting all negative sequents we obtain its proof.

A proof search terminates if the corresponding tree is finite. In other words, it terminates if there is a function defined on sequents which decreases in every backward application of a rule.

Any backward proof search in the calculus Gm_K obviously terminates: we consider the weight of a sequent to be the function and observe that for each rule, the weight function decreases in every backward application of the rule.

This is not the case in the calculus Gm_T due to the $\Box_T$ rule in which a contraction is hidden and therefore the weight function can increase in a backward application of the $\Box_T$ rule (the rule can always be applied backwards to a critical sequent). Moreover, no other function does the job - the calculus is not terminating. A counterexample is e.g. a proof search for sequent $p \Rightarrow \Diamond(p \wedge q)$ which creates a loop.

This defect can be easily avoided by a simple loop-preventing mechanism: once we handle $\Box A$ going backward the $\Box_T$ rule, we mark it. To do it we add the third multiset Σ to each sequent to store formulas of the form $\Box A$ already handled. We empty this multiset whenever we go backward through the $\Box_K$ rule since in this case the boxed content of the antecedent properly changes. This results in the following calculus similar to the calculus used in [13],[6] and [7] (in [13], it can be recognized in the decision procedure; in [7] and [6], the one-sided form of the calculus is used).

We suggest reader to read the figures bottom up and understand the third multiset as formulas which have been marked.

Definition 4.2. Sequent calculus Gm_T^+:

$$\Sigma|\Gamma, p \Rightarrow p, \Delta$$

$$\frac{\Sigma|\Gamma, A, B \Rightarrow \Delta}{\Sigma|\Gamma, A \wedge B \Rightarrow \Delta}\ \wedge\text{-l} \qquad \frac{\Sigma|\Gamma \Rightarrow A, B, \Delta}{\Sigma|\Gamma \Rightarrow A \vee B, \Delta}\ \vee\text{-r}$$

$$\frac{\Sigma|\Gamma, A \Rightarrow \Delta}{\Sigma|\Gamma \Rightarrow \neg A, \Delta}\ \neg\text{-r} \qquad \frac{\Sigma|\Gamma \Rightarrow A, \Delta}{\Sigma|\Gamma, \neg A \Rightarrow \Delta}\ \neg\text{-l}$$

$$\frac{\Sigma|\Gamma \Rightarrow A, \Delta \qquad \Sigma|\Gamma \Rightarrow B, \Delta}{\Sigma|\Gamma \Rightarrow A \wedge B, \Delta}\ \wedge\text{-r} \qquad \frac{\Sigma|\Gamma, A \Rightarrow \Delta \qquad \Sigma|\Gamma, B \Rightarrow \Delta}{\Sigma|\Gamma, A \vee B \Rightarrow \Delta}\ \vee\text{-l}$$

$$\frac{\emptyset|\Gamma \Rightarrow\ A}{\Box\Gamma|\Pi \Rightarrow \Box A, \Delta}\ \Box_K^+ \qquad \frac{\Box A, \Sigma|\Gamma, A \Rightarrow \Delta}{\Sigma|\Gamma, \Box A \Rightarrow \Delta}\ \Box_T^+$$

In the $\Box_K^+$ rule, Π contains only propositional variables and Δ contains only propositional variables and boxed formulas. Σ contains only boxed formulas.

Now let us see that this calculus is terminating.

Lemma 4.3. *Backward proof search in Gm_T^+ always terminates.*

Proof. We define $b(\Sigma, \Pi, \Lambda)$ to be the number of boxed subformulas in Σ, Π, Λ counted as a *set*.

With each sequent $\Sigma|\Pi \Rightarrow \Lambda$ occurring during a proof search we associate an ordered pair of natural numbers $\langle b(\Sigma, \Pi, \Lambda), w(\Pi, \Lambda)\rangle$. We consider the pairs lexicographically ordered. In every backward application of a rule this measure decreases in terms of the lexicographical ordering.

For all rules except the $\Box_K^+$ rule w decreases while b remains the same. For classical rules this is obvious since they do not change the set of boxed subformulas. For the $\Box_T$ rule observe that $b(\Box A, A) = b(\Box A)$.

For the $\Box_K^+$ rule b decreases. It follows from the fact that $b(\Box\Gamma) > b(\Gamma)$ for a finite multiset of formulas Γ. To see this, let us $sf(\Gamma)$ denote the set of subformulas of a multiset Γ. Moreover, let $\preceq$ denote the well quasi-ordering on formulas defined $A \preceq B$ iff $w(A) \leq w(B)$, and let $\prec$ denote the corresponding strict ordering. Observe that, for $A \in sf(B)$, it holds that $A \preceq B$. There are two possibilities:

Either there is $\Box B \in sf(\Box\Gamma)$ such that $\Box B \notin sf(\Gamma)$ and we are done (in this case $\Box B \in \Box\Gamma$).

Or, for all $\Box B \in sf(\Box\Gamma)$, it holds $\Box B \in sf(\Gamma)$. Then each $\Box B \in \Box\Gamma$ is a subformula of a formula from Γ. Consider any formula from $\Box\Gamma$ and denote it $\Box B_1$. Then $\Box B_1$ is a subformula of a formula from Γ, say B_2. Obviously $B_1 \prec B_2$ since $\Box B_1 \preceq B_2$. Since $\Box B_2 \in \Box\Gamma$, it is a subformula of some $B_3 \in \Gamma$ such that $B_1 \prec B_2 \prec B_3$. We continue this way and create a sequence of B_i from Γ where each $\Box B_i$ is a subformula of B_{i+1} and for any $j < i$, $B_j \prec B_i$. Since Γ is finite, the sequence is also finite. Consider its last element B_n. Since the $\prec$ ordering is well founded, there is no such formula in Γ, a subformula of which is $\Box B_n$ - a contradiction.

So there is $\Box B \in sf(\Box\Gamma)$ such that $\Box B \notin sf(\Gamma)$ and hence $b(\Box\Gamma) > b(\Gamma)$.

For a termination argument see also [6] or [7], where another (however closely related) function is considered which depends on the weight of the sequent for which the proof search is considered. See also Remark 4.7. Here we can do without referring to the input sequent using the lexicographical ordering. Referring to the input sequent becomes necessary (even in our lexicographical setting) when dealing with modal logics that requires more complicated loop checking mechanisms, as e.g. **GL** or **S4Grz**. ■

4.2. Structural rules

Structural rules, i.e., the weakening rules, the contraction rules, and the cut rule are not listed among our rules in definitions of the calculi, but they are admissible in our systems.

Admissibility of a rule, elimination of a rule, and closure under a rule are three slightly different notions from the point of view of structural proof theory. For a discussion on this topic see [14]. What follows are proofs of a rule-admissibility established through induction on derivations.

We shall prove admissibility of structural rules for the calculus Gm_T^+. For the calculi Gm_K and Gm_T, admissibility of structural rules can be proved similarly but since it is an immediate consequence of their admissibility in Gm_T^+, we omit it.

For the cut-elimination in modal logics based on multisets see e.g. [18], where a slightly different symmetric definition of sequent calculi is used (treating both $\Box$ and $\Diamond$ modalities as primitive).

In what follows, the horizontal lines in proof figures stand for instances of rules of Gm_T^+ as well as for instances of admissible rules (see the appropriate labels).

Definition 4.4. We call a rule *admissible* if whenever, for an instance of the rule, all premisses are provable, there is a proof of its conclusion.

We call a rule *height-preserving admissible* if whenever, for an instance of the rule, all premisses are provable and the sum of the heights of their proofs is n, there is a proof of height $\leq n$ of the conclusion.

We call a rule *height-preserving invertible* if whenever, for an instance of the rule, the conclusion of a rule has a proof of height n, each premiss has a proof of height $\leq n$.

Note that all rules except the $\Box_K$-rule and the $\Box_K^+$-rule are *height-preserving invertible*. This can be easily shown by induction on the height of the proof of the conclusion.

LEMMA 4.5. *The weakening rules are admissible in Gm_T^+.*

PROOF. The weakening rules are:

$$\frac{\Sigma|\Gamma \Rightarrow \Delta}{\Sigma|\Gamma, A \Rightarrow \Delta}\text{ weak-l} \qquad \frac{\Sigma|\Gamma \Rightarrow \Delta}{\Sigma|\Gamma \Rightarrow \Delta, A}\text{ weak-r} \qquad \frac{\Sigma|\Gamma \Rightarrow \Delta}{\Sigma, \Box A|\Gamma \Rightarrow \Delta}\text{ weak-l+}$$

The proof is by induction on the weight of the weakening formula and, for each weight, on the height of the proof of the premiss. The induction runs simultaneously for all the weakening rules. Note that in the weak-l+ rule, the weakening formula is always of the form $\Box A$.

For an atomic weakening formula the proof is obvious - note that weakening is built into initial sequents as well as in the $\Box_K^+$-rule.

For non atomic and not boxed formula we use the height-preserving invertibility of the appropriate rule, weaken by a formula(s) of lower weight, and then apply the appropriate rule.

Let us consider the weakening formula of the form $\Box A$. If the last inference is a classical inference or a $\Box_T^+$ inference, we just use the i.h., weaken one step above, and use the appropriate rule again. Let the last inference be a $\Box_K^+$ inference. The case of weak-r is then obvious since it is built-in the $\Box_K^+$ rule. weak-l+ and weak-l are captured as follows using the i.h.:

$$\frac{\dfrac{\emptyset|\Sigma^\Box \Rightarrow B}{\emptyset|\Sigma^\Box, A \Rightarrow B}\text{ weak-l}}{\Sigma, \Box A|\Gamma \Rightarrow \Box B, \Delta}\;\Box_K^+ \qquad \frac{\dfrac{\dfrac{\dfrac{\emptyset|\Sigma^\Box \Rightarrow B}{\emptyset|\Sigma^\Box, A \Rightarrow B}\text{ weak-l}}{\Sigma, \Box A|\Gamma \Rightarrow \Box B, \Delta}\;\Box_K^+}{\Sigma, \Box A|A, \Gamma \Rightarrow \Box B, \Delta}\text{ weak-l}}{\Sigma|\Box A, \Gamma \Rightarrow \Box B, \Delta}\;\Box_T^+$$

The latter is the only non height-preserving step in the proof. It is easy to see that this problem does not occur when dealing with Gm_T or Gm_K where the the height-preserving admissibility of weakening rules can easily be obtained. However, the height-preserving admissibility of weakening rules is not necessary in what follows. ■

LEMMA 4.6. *The contraction rules are height-preserving admissible in* Gm_T^+.

PROOF. The contraction rules are:

$$\frac{\Sigma|\Gamma, A, A \Rightarrow \Delta}{\Sigma|\Gamma, A \Rightarrow \Delta}\text{contr-l} \qquad \frac{\Sigma|\Gamma \Rightarrow \Delta, A, A}{\Sigma|\Gamma \Rightarrow \Delta, A}\text{contr-r}$$

$$\frac{\Sigma, \Box A, \Box A|\Gamma \Rightarrow \Delta}{\Sigma, \Box A|\Gamma \Rightarrow \Delta}\text{contr-l+}$$

The proof is by induction on the weight of the contraction formula and, for each weight, on the height of the proof of the premiss. The induction runs simultaneously for all the contraction rules. We use the height preserving invertibility of rules. Note that in the contr-l+ rule the contraction formula is always of the form $\Box A$.

For A atomic, if the premiss is an initial sequent, the conclusion is an initial sequent as well. If not, A is not principal and we use the i.h. and apply a contraction one step above or, in the case of $\Box_K^+$ rule, we apply the rule so that the conclusion is weakened by only one occurrence of A.

For A not atomic and not boxed we use the height preserving invertibility of the appropriate rule and by the i.h. we apply contraction on formula(s) of lower weight and then the rule again. The third multiset does not make any difference here and all works precisely as in the classical logic.

All the steps are obviously height preserving.

Now suppose the contraction formula to be of the form $\Box B$. We distinguish three cases:
(i) The contraction formula is the principal formula of a $\Box_K^+$ inference in the antecedent. Then we permute the proof as follows using the i.h.:

$$\dfrac{\dfrac{\emptyset|B, B, \Gamma \Rightarrow C}{\Box B, \Box B, \Box\Gamma|\Pi \Rightarrow \Box C, \Sigma}\Box_K^+}{\Box B, \Box\Gamma|\Pi \Rightarrow \Box C, \Sigma}\text{contr-l+} \quad \Longrightarrow \quad \dfrac{\dfrac{\emptyset|B, B, \Gamma \Rightarrow C}{\emptyset|B, \Gamma \Rightarrow C}\text{contr-l}}{\Box B, \Box\Gamma|\Pi \Rightarrow \Box C, \Sigma}\Box_K$$

The permutation is obviously height preserving.
(ii) The contraction formula is the principal formula of a $\Box_T^+$ inference in the antecedent. Then we permute the proof as follows using the i.h. and the height preserving invertibility of the $\Box_T^+$ rule:

$$\dfrac{\dfrac{\Sigma, \Box B | \Box B, B, \Gamma \Rightarrow \Delta}{\Sigma | \Box B, \Box B, \Gamma \Rightarrow \Delta}\,\Box_T^+}{\Sigma | \Box B, \Gamma \Rightarrow \Delta}\,\text{contr-l} \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\Sigma, \Box B | \Box B, B, \Gamma \Rightarrow \Delta}{\Sigma, \Box B, \Box B | B, B, \Gamma \Rightarrow \Delta}\,\text{invert.}}{\Sigma, \Box B | B, \Gamma \Rightarrow \Delta}\,\text{contr-l,l+}}{\Sigma | \Box B, \Gamma \Rightarrow \Delta}\,\Box_T^+$$

The permutation is height preserving since the steps contr-l, contr-l+, and invert. do not change the height of the proof.
(iii) The contraction formula is the principal formula in the succedent and we want to have admissible the following contraction:

$$\dfrac{\dfrac{\emptyset | \Gamma \Rightarrow B}{\Box\Gamma | \Pi \Rightarrow \Box B, \Box B, \Sigma}\,\Box_K^+}{\Box\Gamma, \Pi \Rightarrow \Box B, \Sigma}\,\text{contr-r}$$

Then we use the $\Box_K^+$ rule so that the conclusion is not weakened by the other occurrence of $\Box B$. This step is obviously height preserving.
(iv) The contraction formula is not the principal formula. If the last step is a $\Box_K^+$ inference, $\Box B$ is in Δ. Then we use the $\Box_K^+$ rule so that the conclusion is weakened by only one occurrence of the contraction formula. If the last step is another inference, we use contraction one step above on the proof of lower height. If it is an initial sequent, the conclusion of the desired contraction is an initial sequent as well. Again, all the steps are height preserving. ■

Remark 4.7. **Removing duplicate formulas**.

As long as we have the height-preserving admissibility of the contraction rules, we can always remove duplicate formulas during a backward proof search. It is important for the space complexity. Consider the $\Box_T^+$ rule is applied backwards. It can be split into two cases: either the principal formula $\Box A$ is already in the third multiset Σ, and then we do not add it there, or it is not, and the inference stays as it is and we add $\Box A$ to Σ. This corresponds to treating the third multiset as a set. Try for example to search for a proof of $\emptyset | \Box\Box\Box\Box\Box p \Rightarrow \Box\Box\Box\Box p$ in both versions of the calculus. If we allow duplicate formulas in Σ, the increase of the weight of the sequent can be exponential. For more on this topic see Heuerding [6], the calculus $KT^{S,2}$. We do not change Gm_T^+ this way to prove uniform interpolation. However, our proof can be easily reformulated in this manner.

If we consider a proof-search for a sequent $\emptyset | \Pi \Rightarrow \Lambda$ and put $c = w(\Pi, \Lambda)$, an analogous function to that in [6] would be $f(\Sigma | \Gamma; \Delta) = c^2 \cdot b(\Sigma, \Gamma, \Delta) + w(\Gamma, \Delta)$. It decreases in each backward application of a rule of the variant of Gm_T^+ where we do not duplicate formulas in the third multiset Σ. Then possible increase of $w(\Gamma, \Delta)$ in a backward application of the $\Box_K^+$ rule is balanced by c^2. If we do not remove duplicate formulas, the constant c^2 has to be replaced by an exponential function of c. [1]

LEMMA 4.8. *The following cut rules are admissible in Gm_T^+.*

$$\frac{\emptyset|\Gamma \Rightarrow \Delta, A \qquad \emptyset|A, \Pi \Rightarrow \Lambda}{\emptyset|\Gamma, \Pi \Rightarrow \Delta, \Lambda}\text{cut} \qquad \frac{\Sigma|\Gamma \Rightarrow \Delta, \Box A \qquad \Theta, \Box A|\Pi \Rightarrow \Sigma}{\Sigma, \Theta|\Gamma, \Pi \Rightarrow \Delta, \Lambda}\text{cut+}$$

The above cut rule cannot be replaced by the expected form of cut:

$$\frac{\Sigma|\Gamma \Rightarrow \Delta, A \qquad \Theta|A, \Pi \Rightarrow \Lambda}{\Sigma, \Theta|\Gamma, \Pi \Rightarrow \Delta, \Lambda}\text{cut'}\ ,$$

since it is not admissible in Gm_T^+. The counterexample is the following use of cut':

$$\frac{\dfrac{\emptyset|p \Rightarrow p}{\Box p|\emptyset \Rightarrow \Box p}\Box_K^+ \qquad \emptyset|\Box p \Rightarrow p}{\Box p|\emptyset \Rightarrow p}\text{cut'}$$

which results sequent $\Box p|\emptyset \Rightarrow p$ unprovable in Gm_T^+.

However, the cut rule above suffices to go through the proof of Theorem 6.1 and it corresponds to the system Gm_T in view of Lemma 4.9. The cut+ rule is needed to prove admissibility of the cut rule and it will not be used in the proof of Theorem 6.1. What we are care about here are only sequents with the third multiset empty since they match usual sequents of the system Gm_T and therefore they have a clear meaning (see Lemma 4.12, 4.9).

PROOF. The proof of cut-admissibility is by induction on the weight of the cut formula and, for each weight, on the sum of the heights of the proofs of the premisses. The main step is the following: Given cut-free proofs of the premisses we have to show that there is a proof of the conclusion using

[1] In Heuerding [6] (where one-sided version of the calculus is used treating both $\Box, \Diamond$ as primitive), $b(\Gamma)$ is replaced by the number of boxes in Γ. There is a gap since the function can increase in a backward application of the $(\Diamond, new)$ rule of his calculus $KT^{S,2}$. An example is a proof search for $\Diamond\Box p$ where $f(\emptyset|\Diamond\Box p) < f(\Diamond\Box p|\Box p)$ since then the number of boxes in the sequent increases.

only cuts where either the cut formula is of lower weight or the cut formula is of the same weight but the sum of the heights of the proofs of the premisses is lower.

We proceed simultaneously for both the cut rules. Note that in the cut+ rule, the cut formula is always of the form $\Box A$.

If the cut formula is an atom and principal in one premiss (which is then an initial sequent) then we can replace the cut inference by weakening inferences. If the cut formula is principal in both premisses, the conclusion is an initial sequent. If it is principal in neither premiss, we can apply the cut rule one step above so that the sum of the heights of the proofs of its premisses is lower, then apply the original rule and finally some contractions (if one premiss is an initial sequent, the conclusion is an initial sequent as well).

Let us consider a non atomic and not boxed cut formula. If it is not the principal formula in one premiss we can apply the cut rule one step above so that the sum of the heights of the proofs of its premisses is lower, then apply the original rule and finally some contractions. If the cut formula is principal in both premisses we proceed the same way as in the case of classical sequent calculus. For missed details (reduction steps treating classical connectives) see the proof for calculus G3cp in [18] or [14]. We deal with the cut rule where the third multiset is empty and therefore it does not make any change here.

Let the cut formula be of the form $\Box B$. Again, if it is not principal in one premiss we can apply the cut rule one step above so that the sum of the heights of the proofs of its premisses is lower, then apply the original rule and finally some contractions. So let the cut formula be principal in both premisses. Then there are two cases to distinguish:

(i) The cut formula is the principal formula of a $\Box^+_K$ inference in both premisses (i.e. the following instance of the cut+ rule):

$$\dfrac{\dfrac{\emptyset|\Gamma \Rightarrow B}{\Box\Gamma|\Gamma' \Rightarrow \Box B, \Delta}\,\Box^+_K \qquad \dfrac{\emptyset|\Pi, B \Rightarrow C}{\Box B, \Box\Pi|\Pi' \Rightarrow \Box C, \Lambda}\,\Box^+_K}{\Box\Gamma, \Box\Pi|\Gamma', \Pi' \Rightarrow \Delta, \Box C, \Lambda}\,\text{cut+}$$

Here we apply the i.h. and use the following cut inference with the cut formula of lower weight and the $\Box^+_K$ rule to permute the proof as follows:

$$\dfrac{\dfrac{\emptyset|\Gamma \Rightarrow B \qquad \emptyset|\Pi, B \Rightarrow C}{\emptyset|\Gamma, \Pi \Rightarrow C}\,\text{cut}}{\Box\Gamma, \Box\Pi|\Gamma', \Pi' \Rightarrow \Delta, \Box C, \Lambda}\,\Box^+_K$$

(ii) The cut formula is the principal formula of a $\Box_K^+$ inference in one premiss while it is the principal formula of a $\Box_T^+$ inference in the other. The only possibility how this situation can occur is the following instance of the cut rule:

$$\dfrac{\dfrac{\emptyset|\emptyset \Rightarrow B}{\emptyset|\Gamma \Rightarrow \Box B, \Delta}\,\Box_K^+ \qquad \dfrac{\Box B|\Pi, B \Rightarrow \Lambda}{\emptyset|\Box B, \Pi \Rightarrow \Lambda}\,\Box_T^+}{\emptyset|\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\text{cut}$$

In this case we use, by the i.h., one cut+ inference with a lower sum of the heights of its premisses and one cut inference with the cut formula of a lower weight to permute the proof as follows:

$$\dfrac{\emptyset|\emptyset \Rightarrow B \qquad \dfrac{\dfrac{\emptyset|\emptyset \Rightarrow B}{\emptyset|\Gamma \Rightarrow \Box B, \Delta}\,\Box_K^+ \qquad \Box B|\Pi, B \Rightarrow \Lambda}{\emptyset|\Gamma, B, \Pi \Rightarrow \Delta, \Lambda}\,\text{cut+}}{\emptyset|\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\text{cut}$$

■

LEMMA 4.9. *Gm_T^+ is equivalent to Gm_T:*

$$\vdash_{Gm_T} \Gamma \Rightarrow \Delta \ \textit{iff} \vdash_{Gm_T^+} \emptyset|\Gamma \Rightarrow \Delta$$

PROOF. The right-left implication follows immediately since deleting the "|" symbol from all sequents in a Gm_T^+ proof of $\emptyset|\Gamma \Rightarrow \Delta$ yields a Gm_T proof of $\Gamma \Rightarrow \Delta$.

The left-right implication is proved by induction on the height of the proof $\vdash_{Gm_T} \Gamma \Rightarrow \Delta$ using admissibility of structural rules (weakening and contraction suffice here).

The steps for initial sequents and classical rules are obvious since they do not change the third multiset. So let us consider the box rules.

The $\Box_K$ rule is captured in Gm_T^+ as follows

$$\dfrac{\dfrac{\dfrac{\emptyset|\Gamma \Rightarrow A}{\Box\Gamma|\Pi \Rightarrow \Box A, \Delta}\,\Box_K^+}{\Box\Gamma|\Gamma, \Pi \Rightarrow \Box A, \Delta}\,\text{admiss. weak.}}{\emptyset|\Box\Gamma, \Pi \Rightarrow \Box A, \Delta}\,\Box_T^+ \text{ inferences}$$

The $\Box_T$ rule is captured as follows:

$$\dfrac{\dfrac{\dfrac{\emptyset|\Gamma, \Box A, A \Rightarrow \Delta}{\Box A|\Gamma, A, A \Rightarrow \Delta}\text{invert. of } \Box_T^+}{\Box A|\Gamma, A \Rightarrow \Delta}\text{admiss. contr.}}{\emptyset|\Gamma, \Box A \Rightarrow \Delta}\Box_T^+$$

■

As an immediate consequence of Lemma 4.6 , 4.5, and 4.9 we obtain:

COROLLARY 4.10. *The weakening and the contraction rules are admissible in Gm_T and Gm_K.*

The height preserving admissibility of the weakening and the contraction rules in Gm_T and Gm_K can also be obtained using a similar proof as for Gm_T^+.

As an immediate consequence of Lemma 4.8 and 4.9 we obtain the following admissibility of the usual cut rule in Gm_T and Gm_K:

COROLLARY 4.11. *The cut rule*

$$\dfrac{\Gamma \Rightarrow \Delta, A \qquad A, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\, cut$$

is admissible in Gm_T and Gm_K.

LEMMA 4.12. *Gm_K and Gm_T are equivalent to the corresponding Hilbert style definitions H_K and H_T:*

$$\vdash_{Gm_K} \Gamma \Rightarrow \Delta \text{ iff } \vdash_{H_K} \bigwedge \Gamma \rightarrow \bigvee \Delta$$
$$\vdash_{Gm_T} \Gamma \Rightarrow \Delta \text{ iff } \vdash_{H_T} \bigwedge \Gamma \rightarrow \bigvee \Delta$$

PROOF. Easy induction on the height (the length) of the proof of $\Gamma \Rightarrow \Delta$ ($\bigwedge \Gamma \rightarrow \bigvee \Delta$ resp.) using admissibility of structural rules. ■

5. Logic K

Our main technical result is the following theorem. Its proof provides us with an explicit algorithm which for a sequent $\Gamma \Rightarrow \Delta$ constructs a formula $A_p(\Gamma; \Delta)$ to simulate universal quantification over p. The formula $\forall p B(p, \bar{q})$ (or equivalently the pre-interpolant $I_{pre}(B, \bar{q})$) is to be simulated by $A_p(\emptyset; B)$. To do the job, the formula $A_p(\Gamma; \Delta)$ has to satisfy the following:

THEOREM 5.1. *Let Γ, Δ be finite multisets of formulas. For every propositional variable p there exists a formula $A_p(\Gamma; \Delta)$ such that:*

- *(i)*
$$Var(A_p(\Gamma;\Delta)) \subseteq Var(\Gamma,\Delta)\backslash\{p\}$$
- *(ii)*
$$\vdash_{Gm_K} \Gamma, A_p(\Gamma;\Delta) \Rightarrow \Delta$$
- *(iii) moreover let* Π, Σ *be multisets of formulas not containing* p *and* $\vdash_{Gm_K} \Pi, \Gamma \Rightarrow \Lambda, \Delta$. *Then*
$$\vdash_{Gm_K} \Pi \Rightarrow A_p(\Gamma;\Delta), \Lambda.$$

We define a formula $A_p(\Gamma;\Delta)$ inductively on the weight of the multiset Γ,Δ as described in the following table. In the line 8, q and r are any propositional variables other than p, and Φ and Ψ are multisets containing only propositional variables and it is not the case that $p \in \Phi \cap \Psi$ (if it is, the line 1 is used instead). Moreover we require that at least one of the multisets $\Gamma', \Delta', \Phi, \Psi$ is nonempty in the line 8, so that $\emptyset;\emptyset$ does not match the line (to prevent looping).

The procedure uses the lines 1 - 7 (in the case of 1 it ends up with $\top$) until it reaches a critical sequent which does not match the line 1 - then the line 8 (the modal jump) is used. The procedure is nondeterministic in the sense that the sequent can match more then one of the lines 1 - 7, however, it is easy to see that the result does not depend (up to logical equivalence) on a particular order in which lines 1 - 7 are used (this corresponds to the fact that these lines treat invertible rules).

$A_p(\Gamma;\Delta)$ where $\Gamma;\Delta$ does not match any line of the table is defined to equal $\bot$. (In particular, $A_p(\emptyset;\emptyset) \equiv \bot$.)

	$\Gamma;\Delta$ matches	$A_p(\Gamma;\Delta)$ equals
1	$\Gamma', p; \Delta', p$	$\top$
2	$\Gamma', C_1 \wedge C_2; \Delta$	$A_p(\Gamma', C_1, C_2; \Delta)$
3	$\Gamma', \neg C; \Delta$	$A_p(\Gamma'; C, \Delta)$
4	$\Gamma; C_1 \vee C_2, \Delta'$	$A_p(\Gamma; C_1, C_2, \Delta')$
5	$\Gamma; \neg C, \Delta'$	$A_p(\Gamma, C; \Delta')$
6	$\Gamma', C_1 \vee C_2; \Delta$	$A_p(\Gamma', C_1; \Delta) \wedge A_p(\Gamma', C_2; \Delta)$
7	$\Gamma; C_1 \wedge C_2, \Delta'$	$A_p(\Gamma; C_1, \Delta') \wedge A_p(\Gamma; C_2, \Delta')$
8	$\Phi, \Box\Gamma'; \Box\Delta', \Psi$	$\bigvee_{q\in\Psi} q \vee \bigvee_{r\in\Phi} \neg r \vee \bigvee_{B\in\Delta'} \Box A_p(\Gamma'; B) \vee \Diamond A_p(\Gamma'; \emptyset)$

Consider for example $\mathcal{A}_p(\Box(p \wedge q); \Box p)$. It matches the line 8 and thus we obtain $\Box A_p(p \wedge q; p) \vee \Diamond A_p(p \wedge q; \emptyset)$. This yields $\Box A_p(p, q; p) \vee \Diamond A_p(p, q; \emptyset)$ using the line 2, and then, using lines 8 and 1, $\Box\top \vee \Diamond\neg q$. We have obtained $A_p(\Box(p \wedge q); \Box p) \equiv \Diamond\neg q \vee \Box\top$, which is provably equivalent to $\top$.

Proof. The definition of $A_p(\Gamma; \Delta)$ runs inductively on the weight of Γ, Δ. Note that recursively called arguments of A_p are strictly less in terms of the weight function than the corresponding match of $\Gamma; \Delta$ in the second column at each line of the table. Thus our definition always terminates.

(i) follows easily by induction on Γ, Δ inspecting the table just because we never add p during the definition of the formula $A_p(\Gamma; \Delta)$.

(ii) We proceed by induction on the weight of Γ, Δ. We prove $\Gamma, A_p(\Gamma; \Delta) \Rightarrow \Delta$ for each line of the table (i.e., for each match of $\Gamma; \Delta$). For the lines 1-7, (ii) follows from the induction hypothesis by easy proofs in Gm_K.

For the line 8 we have the following:

- for each $B \in \Delta'$, we have $\Gamma', A_p(\Gamma'; B) \Rightarrow B$ by the i.h., which gives $\Box\Gamma', \Phi, \Box A_p(\Gamma'; B) \Rightarrow \Box B, \Box\Delta'', \Psi$ by a $\Box_K$ inference.
- by the i.h. we also have $\Gamma', A_p(\Gamma'; \emptyset) \Rightarrow \emptyset$, which gives, using negation rules and the $\Box_K$ rule, $\Box\Gamma', \Phi, \Diamond A_p(\Gamma'; B) \Rightarrow \Box\Delta', \Psi$.
- for each $r \in \Phi$ obviously $\Phi, \neg r, \Box\Gamma' \Rightarrow \Box\Delta', \Psi$.
- for each $q \in \Psi$ obviously $\Phi, q, \Box\Gamma' \Rightarrow \Box\Delta', \Psi$.

Together this yields, using $\vee$-l inferences,

$$\Phi, \Box\Gamma', \bigvee_{q\in\Psi} q \vee \bigvee_{r\in\Phi} \neg r \vee \bigvee_{B\in\Delta'} \Box A_p(\Gamma'; B) \vee \Diamond A_p(\Gamma'; \emptyset) \Rightarrow \Box\Delta', \Psi,$$

that is, by the line 8, $\Phi, \Box\Gamma', A_p(\Phi, \Box\Gamma'; \Box\Delta', \Psi) \Rightarrow \Box\Delta', \Psi$.

(iii) We proced by induction on the height n of a proof of $\Pi, \Gamma \Rightarrow \Lambda, \Delta$.

<u>$(n = 0)$</u>

Then $\Pi, \Gamma \Rightarrow \Lambda, \Delta$ is an axiom, say $\Sigma, r \Rightarrow r, \Theta$. We distinguish two cases - either $r \equiv p$ or not:

- $r \equiv p$: then $p \in \Gamma \cap \Delta$, which means that $A_p(\Gamma; \Delta) \equiv \top$ and since obviously $\Pi \Rightarrow \top, \Lambda$, we obtain (iii).
- $r \neq p$: there are four cases:
  - $r \in \Pi \cap \Lambda$, then (iii) is an axiom.
  - $r \in \Pi \cap \Delta$ then the line 7 gives, by invertibility of the $\vee$-l rule, $r \Rightarrow A_p(\Gamma; r, \Delta')$.
  - $r \in \Gamma \cap \Lambda$ then the line 8 gives, by invertibility of the $\vee$-l rule, $\neg r \Rightarrow A_p(\Gamma'; r, \Delta)$.

– $r \in \Gamma \cap \Delta$ then the line 8 gives, by invertibility of the $\vee$-l rule, $r \vee \neg r \Rightarrow A_p(\Gamma; \Delta)$, and so, by cut admissibility, $\emptyset \Rightarrow A_p(\Gamma; \Delta)$.

In all the three cases above admissibility of the weakening rule yields what is required.

$(n > 0)$

We consider the last inference of the proof.

- $\wedge$-l (the case of $\vee$-r is dual using the line 4 of the table): If the principal formula $A \wedge B \in \Pi$, we just use the i.h. and apply the rule again. So suppose $A \wedge B \in \Gamma$. Then by the line 2 of the table we have $A_p(\Gamma', A, B; \Delta) \equiv A_p(\Gamma', A \wedge B; \Delta)$ which together with the i.h. $\Pi \Rightarrow A_p(\Gamma', A, B; \Delta), \Lambda$ yields (iii).

- $\wedge$-r (the case of $\vee$-l is dual using the line 6 of the table) Again suppose the principal formula $A \wedge B \in \Delta$. Then the i.h. gives by a $\wedge$-r inference $\Pi \Rightarrow (A_p(\Gamma; A, \Delta') \wedge A_p(\Gamma; B, \Delta')), \Lambda$ which together with the line 7: $A_p(\Gamma; A, \Delta') \wedge A_p(\Gamma; B, \Delta') \equiv A_p(\Gamma; A \wedge B, \Delta')$ yields (iii).

- $\neg$-r (again the case of $\neg$-l is dual using the line 3 of the table) First suppose the principal formula $\neg A \in \Lambda$. Then A doesn't contain p and by the i.h. we have $\Pi, A \Rightarrow A_p(\Gamma; \Delta), \Lambda$, which gives (iii) by a $\neg$-r inference.
  Now suppose the principal formula $\neg A \in \Delta$. The induction hypothesis yields $\Pi \Rightarrow A_p(\Gamma, A; \Delta), \Lambda$, while the line 5 of the table says $A_p(\Gamma, A; \Delta) \equiv A_p(\Gamma; \neg A, \Delta)$. This together yields (iii).

- $\Box_K$: consider the principal formula $\Box A \in \Lambda$ first, i.e. A doesn't contain p. Then the proof ends with:

$$\frac{\Pi', \Gamma' \Rightarrow A}{\Box\Pi', \Box\Gamma', \Pi'', \Gamma'' \Rightarrow \Box A, \Lambda', \Delta}\ \Box_K$$

  where $\Box\Pi', \Pi''$ is Π; $\Box\Gamma', \Gamma''$ is Γ; and $\Box A, \Lambda'$ is Λ.
  Then the induction hypothesis gives $\Pi' \Rightarrow A_p(\Gamma'; \emptyset), A$ and by a $\neg$-l inference we obtain $\Pi', \neg A_p(\Gamma'; \emptyset) \Rightarrow A$. Now, by a $\Box_K$ and a negation inference, we obtain

$$\Box\Pi', \Pi'' \Rightarrow \Diamond A_p(\Gamma'; \emptyset), \Box A, \Lambda'.$$

By the line 8 of the table and invertibility of the $\vee$-l rule we have

$$\Diamond A_p(\Gamma'; \emptyset) \Rightarrow A_p(\Box\Gamma', \Gamma''; \Delta).$$

The two sequents above yield (iii) by cut admissibility.

- $\Box_K$, consider the principal formula $\Box A \in \Delta$. Then the proof ends with:

$$\frac{\Pi', \Gamma' \Rightarrow A}{\Box\Pi', \Box\Gamma', \Pi'', \Gamma'' \Rightarrow \Box A, \Delta', \Lambda}\ \Box_K$$

where $\Box\Pi', \Pi''$ is Π; $\Box\Gamma', \Gamma''$ is Γ; and $\Box A, \Delta'$ is Δ.
Now the induction hypothesis gives $\Pi' \Rightarrow A_p(\Gamma'; A)$ and by a $\Box_K$ inference we obtain

$$\Box\Pi', \Pi'' \Rightarrow \Box A_p(\Gamma'; A), \Lambda.$$

The line 8 of the table and invertibility of the $\vee$-l rule yields

$$\Box A_p(\Gamma'; A) \Rightarrow A_p(\Box\Gamma', \Gamma''; \Box A, \Delta').$$

We obtain (iii) again by cut admissibility. ■

5.1. Propositional quantifiers

Propositional quantifiers are usually introduced via their semantical meaning. In the framework of Kripke semantics they are defined as ranging over propositions, i.e., sets of possible worlds. This definition is used in Fine [8], see also Bull [3] and Kremer [11]. The second order modal systems over logics **K**, **T**, **K4**, **S4** obtained this way are recursively isomorphic to full second order classical logic. This was proved independently by Fine and Kripke shortly after Fine's paper [8] was published, as Kremer remarked in [11]. Also Kremer's strategy from [12] can be extended to prove the same result, as he claims in [11]. In particular it means that these systems are undecidable while their propositional counterparts are decidable.

Another way of defining quantified propositional logic is by extending a proof system for the propositional logic by new axioms and analogues of usual quantifier rules. This approach was applied e.g. in Bull's paper [3], or in [15] in the case of intuitionistic logic. Bull in [3] proved completeness of such second order calculi over S4 and S5 w.r.t. Kripke semantics. This sort of proof is analogous to standard completeness proofs in first order predicate modal logics. It can also be given for second order $\mathbf{K}^2$ and $\mathbf{T}^2$ considered here but it is outside the scope of this paper. The difference is that Bull doesn't allow quantifiers to range over *all* subsets of possible worlds but only over those given by validating some formula. In this case we quantify over *substitutions*. These two possible semantical definitions are different and do not seem to yield systems of the same complexity.

We adopt the syntactical approach and define quantified propositional modal logic $\mathbf{K}^2$ as follows. Consider the following sequent calculus Gm_{K^2}:

DEFINITION 5.2. Sequent calculus Gm_{K^2} results from extending Gm_K by weakening rules, contraction rules and the cut rule,
initial sequents of the form

$$\forall p \Box A \Rightarrow \Box \forall p A,$$

and two quantifier rules:

$$\frac{\Gamma, A[p/B] \Rightarrow \Delta}{\Gamma, \forall p A \Rightarrow \Delta} \forall\text{-l} \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \forall p A, \Delta} \forall\text{-r}, p \text{ not free in } \Gamma, \Delta$$

The added axiom represents the propositional version of the *Barcan formula.* Note that its converse is easily provable in the calculus using the quantifier rules.

The desirability of the Barcan formula is usually discussed in first order predicate modal logics where it relates to the question whether there is a constant domain in all possible worlds or not. Since it is certainly the case here, because the domain of propositional quantification (i.e. the set of propositional variables) is the same at each world, we include this scheme to our calculus.

The calculus as defined here does not have nice structural properties but is transparent and suffices to capture the semantical meaning of $\mathbf{K}^2$ quantifiers in the sense of Bull's paper. If we want to do without cut (if it is at all possible), we should include the Barcan formula another way.

COROLLARY 5.3. *Let C be a modal formula and let Γ, Δ be multisets of formulas not containing p. There is a formula $A_p(C)$ not containing p such that:*
(i) $\vdash_{Gm_K} \Gamma \Rightarrow C, \Delta$ *implies* $\vdash_{Gm_K} \Gamma \Rightarrow A_p(C), \Delta$
(ii) $\vdash_{Gm_K} \Gamma \Rightarrow A_p(C), \Delta$ *implies for all B,* $\vdash_{Gm_K} \Gamma \Rightarrow C[p/B], \Delta$*.*

PROOF. We define $A_p(\Delta) = A_p(\emptyset; \Delta)$. The first part follows immediately from 5.1 (iii).

By 5.1 (ii) we have $A_p(C) \Rightarrow C$. As $A_p(C)$ does not contain p, we obtain $A_p(C) \Rightarrow C[p/B]$ by substitution, which yields the second part. ∎

To simulate propositional quantifiers of Gm_{K^2} in Gm_K we define the following translation A^* of a second order modal formula A:

- $p^* := p$
- $(C \circ B)^* := C^* \circ B^*$
- $(\neg C)^* := \neg C^*$
- $(\Box B)^* := \Box(B^*)$
- $(\forall p C)^* := A_p(C^*)$

Observe that for a quantifier-free formula B, $B^* = B$ holds.

Now let us see that our Theorem 5.1 yields the desired simulation of propositional quantifiers. To obtain it, we moreover need our construction of A_p to commute with substitution:

COROLLARY 5.4. $\vdash_{Gm_K} A_p(C[q/B]) \Rightarrow (A_p(C))[q/B]$ *and* $\vdash_{Gm_K} (A_p(C))[q/B] \Rightarrow A_p(C[q/B])$, *where* B *doesn't contain* p.

PROOF. The first direction uses the following congruence property of modal logic **K**: $C[q/A] \leftrightarrow C[q/B]$ whenever $A \leftrightarrow B$.

By 5.1 (ii) we have that $A_p(C[q/B]) \Rightarrow C[q/B]$. Now by the congruence property we get $(q \leftrightarrow B), A_p(C[q/B]) \Rightarrow C$, and since the antecedent doesn't contain p also $(q \leftrightarrow B), A_p(C[q/B]) \Rightarrow A_p(C)$. Substituting $[q/B]$ it results $A_p(C[q/B]) \Rightarrow A_p(C)[q/B]$.

The other direction: by 5.1 (ii) we have $A_p(C) \Rightarrow C$. By substitution we get $(A_p(C))[q/B] \Rightarrow C[q/B]$ and since the antecedent doesn't contain p, we also get by 5.1 (iii) $(A_p(C))[q/B] \Rightarrow A_p(C[q/B])$. ■

Now we are ready to prove:

COROLLARY 5.5. *If* $\vdash_{Gm_{K^2}} \Gamma \Rightarrow \Delta$ *then* $\vdash_{Gm_K} \Gamma^* \Rightarrow \Delta^*$.

PROOF. By induction on the proof of $\Gamma \Rightarrow \Delta$ in Gm_{K^2} using Corollary 5.3 and Corollary 5.4.

As for the added initial sequent $\forall p \Box B \Rightarrow \Box \forall p B$, note that $A_p(\Box B)$ yields $\Box A_p(B)$ and thus $\vdash_{Gm_K} A_p(\Box B) \Rightarrow \Box A_p(B)$ can be easily proved from the line 6 of the table in 5.1. ■

The other direction cannot be obtained. An example of a schema valid on our simulated quantifiers in **K** and not valid on propositional quantifiers in $\mathbf{K}^2$ is the $\forall$ quantifier commuting with the $\Diamond$ modality:

$$(\Diamond \forall \bar{p} A)^* \leftrightarrow (\forall \bar{p} \Diamond A)^*,$$

which can be easily proved from the line 8 of the table in 5.1. The right-left implication can be seen not to hold using Kripke semantics in the sense of Bull's paper, i.e., quantifying over substitutions.

5.2. Uniform interpolation

LEMMA 5.6. $\mathbf{K}^2$ *satisfies the uniform interpolation property: For any formula* $A(\bar{p},\bar{q})$ *and variables* $\bar{q}$ *there is a formula* $I_{post}(A,\bar{q})$ *such that*

- $\vdash_{Gm_{K^2}} A \Rightarrow I_{post}(A,\bar{q})$
- *for any formula* $B(\bar{q},\bar{r})$*, where* $\bar{r},\bar{p}$ *are disjoint sets of variables, if* $\vdash_{Gm_{K^2}} A(\bar{p},\bar{q}) \Rightarrow B(\bar{q},\bar{r})$ *then* $\vdash_{Gm_{K^2}} I_{post}(A,\bar{q}) \Rightarrow B(\bar{q},\bar{r})$.

For any formula $B(\bar{q},\bar{r})$ *and variables* $\bar{q}$ *there is a formula* $I_{pre}(B,\bar{q})$ *such that*

- $\vdash_{Gm_{K^2}} I_{pre}(B,\bar{q}) \Rightarrow B$
- *for any formula* $A(\bar{p},\bar{q})$*, where* $\bar{r},\bar{p}$ *are disjoint sets of variables, if* $\vdash_{Gm_{K^2}} A(\bar{p},\bar{q}) \Rightarrow B(\bar{q},\bar{r})$ *then* $\vdash_{Gm_{K^2}} A(\bar{p},\bar{q}) \Rightarrow I_{pre}(B,\bar{q})$.

PROOF. It is easy to see that $\exists\bar{p}A(\bar{p},\bar{q})$ and $\forall\bar{r}B(\bar{q},\bar{r})$ are the interpolants $I_{post}(A,\bar{q})$ and $I_{pre}(B,\bar{q})$ respectively.

Obviously $\forall\bar{r}B(\bar{q},\bar{r}) \Rightarrow B$. Let $A(\bar{p},\bar{q}) \Rightarrow B(\bar{q},\bar{r})$ be provable. Since B does not contain p free, we can use the $\forall$-r rule to conclude $A(\bar{p},\bar{q}) \Rightarrow \forall\bar{r}B(\bar{q},\bar{r})$.

The other case is dual. ■

COROLLARY 5.7. ***K*** *satisfies the uniform interpolation property.*

PROOF. The result follows immediately from Corollary 5.5 and Lemma 5.6.

To see that we have in fact constructed the interpolants proving Theorem 5.1, observe that our construction of A_p works as well for more then one propositional variable p. We can construct $A_{\bar{p}}$ using the procedure for all $\bar{p}$ simultaneously.

Let us have $A(\bar{p},\bar{q})$. Theorem 5.1 yields the formula $\neg A_{\bar{p}}(A;\emptyset)$ (constructed only from A and containing only the variables $\bar{q}$) such that from (ii) it follows:

$$A \Rightarrow \neg A_{\bar{p}}(A;\emptyset).$$

Let $A(\bar{p},\bar{q}) \Rightarrow B(\bar{q},\bar{r})$ be provable. From (iii) we get:

$$\neg A_{\bar{p}}(A;\emptyset) \Rightarrow B.$$

Let us have $B(\bar{q}, \bar{r})$. Theorem 5.1 yields the formula $A_{\bar{r}}(B)$ (constructed only from B and containing only the variables $\bar{q}$) such that it follows from (ii):

$$A_{\bar{r}}(B) \Rightarrow B.$$

Let $A(\bar{p}, \bar{q}) \Rightarrow B(\bar{q}, \bar{r})$ be provable. From (iii) we get:

$$A \Rightarrow A_{\bar{r}}(B).$$

■

6. Logic T

The following analogue of Theorem 5.1 holds for the calculus Gm_T^+:

THEOREM 6.1. *Let Σ, Γ, Δ be finite multisets of formulas. For every propositional variable p there exists a formula $A_p(\Sigma|\Gamma;\Delta)$ such that:*

- *(i)*
$$Var(A_p(\Sigma|\Gamma;\Delta)) \subseteq Var(\Sigma, \Gamma, \Delta)\backslash\{p\}$$
- *(ii)*
$$\vdash_{Gm_T^+} \Sigma|\Gamma, A_p(\Sigma|\Gamma;\Delta) \Rightarrow \Delta$$
- *(iii) moreover let Π, Λ, Θ be multisets of formulas not containing p and $\vdash_{Gm_T^+} \Theta, \Sigma|\Pi, \Gamma \Rightarrow \Lambda, \Delta$. Then*
$$\vdash_{Gm_T^+} \emptyset|\Theta, \Pi \Rightarrow A_p(\Sigma|\Gamma;\Delta), \Lambda.$$

We define a formula $A_p(\Sigma|\Gamma;\Delta)$ inductively as in 5.1, changing the table as follows (again, q and r are any propositional variables other then p, multisets Φ and Ψ in the line 9 contain only propositional variables, $p \notin \Phi\cap\Psi$, and at least one of the four multisets in the line 9 is required to be nonempty):

	$\Sigma\|\Gamma;\Delta$ matches	$A_p(\Sigma\|\Gamma;\Delta)$ equals
1	$\Sigma\|\Gamma', p; \Delta', p$	$\top$
2	$\Sigma\|\Gamma', C_1 \wedge C_2; \Delta$	$A_p(\Sigma\|\Gamma', C_1, C_2; \Delta)$
3	$\Sigma\|\Gamma', \neg C; \Delta$	$A_p(\Sigma\|\Gamma'; C, \Delta)$
4	$\Sigma\|\Gamma; C_1 \vee C_2, \Delta'$	$A_p(\Sigma\|\Gamma; C_1, C_2, \Delta')$
5	$\Sigma\|\Gamma; \neg C, \Delta'$	$A_p(\Sigma\|\Gamma, C; \Delta')$
6	$\Sigma\|\Gamma', \Box B; \Delta$	$A_p(\Sigma, \Box B\|\Gamma', B; \Delta)$
7	$\Sigma\|\Gamma', C_1 \vee C_2; \Delta$	$A_p(\Sigma\|\Gamma', C_1; \Delta) \wedge A_p(\Sigma\|\Gamma', C_2; \Delta)$
8	$\Sigma\|\Gamma; C_1 \wedge C_2, \Delta'$	$A_p(\Sigma\|\Gamma; C_1, \Delta') \wedge A_p(\Sigma\|\Gamma; C_2, \Delta')$
9	$\Box\Gamma'\|\Phi; \Box\Delta', \Psi$	$\bigvee_{q\in\Psi} q \vee \bigvee_{r\in\Phi} \neg r \vee \bigvee_{B\in\Delta'} \Box A_p(\emptyset\|\Gamma'; B) \vee \Diamond A_p(\emptyset\|\Gamma'; \emptyset)$

Proof. The procedure runs precisely as that from Theorem 5.1. This time the recursively called arguments of A_p are strictly less then the corresponding match of $\Sigma|\Gamma;\Delta$ in the second column at each line of the table in terms of the function used in 4.3 to prove termination of Gm^+_T.
(i) holds since we never add p during a run of the procedure constructing the formula A_p.

(ii) Similarly as in Theorem 5.1 (ii), we proceed by induction on the complexity of $\Sigma|\Gamma;\Delta$ given by the termination function. We prove the following: $\vdash_{Gm^+_T} \Sigma, \Gamma, A_p(\Sigma|\Gamma;\Delta) \Rightarrow \Delta$ for each match $\Sigma|\Gamma;\Delta$ occurring during a run of the procedure, i.e., for each line of the table.

For lines 1-5,7,8 (ii) follows by easy Gm^+_T proofs as in Theorem 5.1, the third multiset does not cause any problems here.

The step for the line 6 treating the $\Box^+_T$ rule:

By the induction hypothesis, $\vdash_{Gm^+_T} \Sigma, \Box B|\Gamma', B, A_p(\Sigma, \Box B|\Gamma', B;\Delta) \Rightarrow \Delta$. By the line 6 of the table, $A_p(\Sigma|\Gamma', \Box B;\Delta) \equiv A_p(\Sigma, \Box B|\Gamma', B;\Delta)$. Now we obtain $\vdash_{Gm^+_T} \Sigma|\Gamma', \Box B, A_p(\Sigma|\Gamma', \Box B;\Delta) \Rightarrow \Delta$ by a $\Box^+_T$ inference.

For the line 9 treating the $\Box^+_K$ rule we have, similarly as in 5.1, the following:

- for each $B \in \Delta'$, we have $\vdash_{Gm^+_T} \emptyset|\Gamma', A_p(\emptyset|\Gamma';B) \Rightarrow B$ by the i.h., which gives $\vdash_{Gm^+_T} \Box\Gamma', \Box A_p(\emptyset|\Gamma';B)|\Phi \Rightarrow \Box\Delta', \Psi$ by a $\Box^+_K$ inference. Now, using admissibility of weakening we obtain

  $$\vdash_{Gm^+_T} \Box\Gamma', \Box A_p(\emptyset|\Gamma';B)|A_p(\emptyset|\Gamma';B), \Phi \Rightarrow \Box\Delta', \Psi$$

  by a $\Box^+_K$ inference and hence, by a $\Box^+_T$ inference,

  $$\vdash_{Gm^+_T} \Box\Gamma'|\Box A_p(\emptyset|\Gamma';B), \Phi \Rightarrow \Box\Delta', \Psi.$$

- by the i.h. we also have $\vdash_{Gm^+_T} \emptyset|\Gamma', A_p(\emptyset|\Gamma';\emptyset) \Rightarrow \emptyset$, which gives, using negation rules and the $\Box^+_K$ rule,

  $$\vdash_{Gm^+_T} \Box\Gamma'|\Diamond A_p(\emptyset|\Gamma';\emptyset), \Phi \Rightarrow \Box\Delta', \Psi.$$

- for each $r \in \Phi$ obviously $\vdash_{Gm^+_T} \Box\Gamma'|\Phi, \neg r \Rightarrow \Box\Delta', \Psi$.
- for each $q \in \Psi$ obviously $\vdash_{Gm^+_T} \Box\Gamma'|\Phi, q \Rightarrow \Box\Delta', \Psi$.

Together this yields, using $\vee$-l inferences,

$$\vdash_{Gm^+_T} \Box\Gamma'|\Phi, \bigvee_{q\in\Psi} q \vee \bigvee_{r\in\Phi} \neg r \vee \bigvee_{B\in\Delta'} \Box A_p(\emptyset|\Gamma';B) \vee \Diamond A_p(\emptyset|\Gamma';\emptyset) \Rightarrow \Box\Delta', \Psi,$$

that is, by the line 9, $\vdash_{Gm_T^+} \Box\Gamma'|\Phi, A_p(\Box\Gamma'|\Phi; \Box\Delta', \Psi) \Rightarrow \Box\Delta', \Psi$.

(iii) We proceed by induction on the height of the proof of the sequent $\Theta, \Sigma|\Pi, \Gamma \Rightarrow \Lambda, \Delta$ in Gm_T^+. All the steps for initial sequent and classical rules are similar as in 5.1, the third multiset has no influence here. So let us consider the two modal rules.

The last inference of the proof of $\Theta, \Sigma|\Pi, \Gamma \Rightarrow \Lambda, \Delta$ is a $\Box_K^+$ inference.

- Consider the principal formula $\Box A \in \Delta$. Then the proof ends with:

$$\frac{\emptyset|\Theta^\Box, \Sigma^\Box \Rightarrow A}{\Theta, \Sigma|\Gamma, \Pi \Rightarrow \Box A, \Delta', \Lambda}\ \Box_K^+$$

 where $\Box A, \Delta'$ is Δ.
 Then by the induction hypothesis $\vdash_{Gm_T^+} \emptyset|\Theta^\Box \Rightarrow A_p(\emptyset|\Sigma^\Box; A)$ and by a $\Box_K^+$ inference

$$\vdash_{Gm_T^+} \Theta|\Pi \Rightarrow \Box A_p(\emptyset|\Sigma^\Box; A), \Lambda.$$

 By weakening inferences

$$\vdash_{Gm_T^+} \Theta|\Theta^\Box, \Pi \Rightarrow \Box A_p(\emptyset|\Sigma^\Box; A), \Lambda.$$

 By $\Box_T^+$ inferences we obtain

$$\vdash_{Gm_T^+} \emptyset|\Theta, \Pi \Rightarrow \Box A_p(\emptyset|\Sigma^\Box; A), \Lambda.$$

 By the line 9 of the table and invertibility of the $\vee$-l rule we have

$$\vdash_{Gm_T^+} \emptyset|\Box A_p(\emptyset|\Sigma^\Box; A) \Rightarrow A_p(\Sigma|\Gamma; \Box A, \Delta').$$

 The two sequents above yield (iii) by admissibility of the cut rule in Gm_T^+.
- Consider the principal formula $\Box A \in \Lambda$, so, A doesn't contain p. Then the proof ends with:

$$\frac{\emptyset|\Theta^\Box, \Sigma^\Box \Rightarrow A}{\Theta, \Sigma|\Gamma, \Pi \Rightarrow \Delta, \Box A, \Lambda'}\ \Box_K^+$$

 where $\Box A, \Lambda'$ is Λ.
 Then by the induction hypothesis $\vdash_{Gm_T^+} \emptyset|\Theta^\Box \Rightarrow A_p(\emptyset|\Sigma^\Box; \emptyset), A$ and by a $\neg$-l inference and a $\Box_K^+$ inference $\vdash_{Gm_T^+} \Theta, \Box\neg A_p(\emptyset|\Sigma^\Box; \emptyset)|\Pi \Rightarrow \Box A, \Lambda'$.
 Since weakening is admissible in Gm_T^+, we obtain

$$\vdash_{Gm_T^+} \Theta, \Box\neg A_p(\emptyset|\Sigma^\Box; \emptyset)|\neg A_p(\emptyset|\Sigma^\Box; \emptyset), \Pi \Rightarrow \Box A, \Lambda'$$

and now $\Box_T^+$ inferences and a $\neg$-l inference yield

$$\vdash_{Gm_T^+} \Theta|\Pi \Rightarrow \Diamond A_p(\emptyset|\Sigma^\Box;\emptyset), \Box A, \Lambda'.$$

By weakening inferences

$$\vdash_{Gm_T^+} \Theta|\Theta^\Box, \Pi \Rightarrow \Diamond A_p(\emptyset|\Sigma^\Box;\emptyset), \Box A, \Lambda'.$$

By $\Box_T^+$ inferences

$$\vdash_{Gm_T^+} \emptyset|\Theta, \Pi \Rightarrow \Diamond A_p(\emptyset|\Sigma^\Box;\emptyset), \Box A, \Lambda'.$$

By the line 9 of the table and invertibility of the $\vee$-l rule we have

$$\vdash_{Gm_T^+} \emptyset|\Diamond A_p(\emptyset|\Sigma^\Box;\emptyset) \Rightarrow A_p(\Sigma|\Gamma;\Box A, \Delta').$$

The two sequents above yield (iii) by admissibility of the cut rule in Gm_T^+.

The last inference of the proof of $\Theta, \Sigma|\Pi, \Gamma \Rightarrow \Lambda, \Delta$ in Gm_T^+ is a $\Box_T^+$ inference.

- Consider the principal formula $\Box A \in \Pi$ so A doesn't contain p. Then the proof ends with:

$$\frac{\Theta, \Sigma, \Box A|A, \Pi', \Gamma \Rightarrow \Lambda, \Delta}{\Theta, \Sigma|\Box A, \Pi', \Gamma \Rightarrow \Lambda, \Delta}\ \Box_T^+$$

The induction hypothesis yields

$$\emptyset|\Theta, \Box A, A, \Pi' \Rightarrow A_p(\Sigma, \Gamma; \Delta), \Lambda.$$

By invertibility of the $\Box_T^+$ rule (applied on $\Box A$) and by a contraction inference (applied on A) we obtain

$$\Box A|\Theta, A, \Pi' \Rightarrow A_p(\Sigma, \Gamma; \Delta), \Lambda.$$

Now a $\Box_T^+$ inference yields (iii).

- Consider the principal formula $\Box A \in \Gamma$. Then the proof ends with:

$$\frac{\Theta, \Sigma, \Box A|A, \Pi, \Gamma' \Rightarrow \Lambda, \Delta}{\Theta, \Sigma|\Box A, \Pi, \Gamma' \Rightarrow \Lambda, \Delta}\ \Box_T^+$$

Then by the induction hypothesis

$$\vdash_{Gm_T^+} \emptyset|\Theta, \Pi \Rightarrow A_p(\Sigma, \Box A|A, \Gamma'; \Delta), \Lambda.$$

By the line 6 of the table

$$A_p(\Sigma, \Box A|A, \Gamma'; \Delta) \equiv A_p(\Sigma|\Box A, \Gamma'; \Delta).$$

This immediately yields (iii). ■

Corollary 6.2. *Let Γ, Δ be finite multisets of formulas. For every propositional variable p there exists a formula $A_p(\Gamma; \Delta)$ such that:*

- *(i)*
$$Var(A_p(\Gamma; \Delta)) \subseteq Var(\Gamma, \Delta) \backslash \{p\}$$
- *(ii)*
$$\vdash_{Gm_T} \Gamma, A_p(\Gamma; \Delta) \Rightarrow \Delta$$
- *(iii) moreover let Π, Λ be multisets of formulas not containing p and $\vdash_{Gm_T} \Pi, \Gamma \Rightarrow \Lambda, \Delta$ then*
$$\vdash_{Gm_T} \Pi \Rightarrow A_p(\Gamma; \Delta), \Lambda$$

Proof. We define $A_p(\Gamma; \Delta) \equiv A_p(\emptyset|\Gamma; \Delta)$. The corollary now follows from Theorem 6.1 and Lemma 4.9. ■

Analogues of Corollaries 5.3, 5.4, 5.5 and 5.7 hold also for modal logic T.

7. Conclusion and further research

We have given purely syntactic proofs of uniform interpolation for modal logics **K** and **T**. The latter is interesting since it makes use of a sequent calculus including a loop-preventing mechanism. Our proofs are closely related to decision procedures [13] and proof-search in modal logics.

Our work is intended as a basic step to be continued by giving a similar proof-theoretical argument for modal logics **GL** and **S4Grz** having arithmetical interpretations. In the case of **GL**, a motivation can be given by observing that uniform interpolation entails fixed point theorem. Already ordinary interpolation does the job: a fixed point of a formula is an interpolant of a sequent expressing uniqueness of the fixed point, see [16] and [2]. However, this method is not useful for implementations. Sambin and Valentini in [16] presented another construction of explicit fixed points which is implementable. Our approach would then provide an alternative implementable solution to that given by them. Let us consider a formula $B(p, \bar{q})$ with p modalized in B (i.e., any occurrence of p is in the scope of a $\Box$). The fixed point of B then would be the simulation of $\exists p(\Box(p \leftrightarrow B) \wedge B)$ or, equivalently, of $\forall p(\Box(p \leftrightarrow B) \rightarrow B)$.

Another interesting class of logics for which it may be useful to investigate this sort of uniform interpolation proofs are intuitionistic modal logics.

Acknowledgements. This work was partially supported by Grant no. 401/03/H047 of the Grant Agency of the Czech Republic and by Grant no. A1019401 of the Grant Agency of the Academy of Sciences of the Czech Republic. Part of this work was done while the author was supported by Huygens scholarship at Utrecht University.

This paper has profited from useful comments and suggestions of two anonymous referees.

References

[1] Blackburn, P., M. de Rijke, and Y. Venema, *Modal Logic*, Cambridge University Press, 2001.

[2] Boolos, G., *The Logic of Provability*, Cambridge University Press, New York and Cambridge, 1993.

[3] Bull, R. A., 'On modal logic with propositional quantifiers', *The Journal of Symbolic Logic* 34:257–263, 1969.

[4] Buss, S. R. (ed.), *Handbook of Proof Theory*, Elsevier, Amsterdam, 1998.

[5] Chagrov, A., and M. Zakharyaschev, *Modal Logic*, Oxford University Press, 1998.

[6] Heuerding, A., *Sequent Calculi for Proof Search in Some Modal Logics*, Thesis, University of Bern, 1998.

[7] Heuerding, A., M. Seyfried, and H. Zimmermann, 'Efficient loop-check for backward proof search in some non-classical propositional logics', in P. Miglioli, U. Moscato, D. Mundici, and M. Ornaghi (eds.), 5th Workshop on Theorem Proving with Analytic Tableaux and Related Methods, LNCS 1071, Springer, 1996, pp. 210–225.

[8] Fine, K., 'Propositional quantifiers in modal logic', *Theoria* 36:336–346, 1970.

[9] Ghilardi, S., and M. Zawadowski, 'Undefinability of Propositional Quantifiers in the Modal System S4', *Studia Logica* 55:259–271, 1995.

[10] Ghilardi, S., and M. Zawadowski, 'A sheaf representation and duality for finitely presented Heyting algebras', *The Journal of Symbolic Logic* 60:911–939, 1995.

[11] Kremer, P., 'On the complexity of propositional quantification in intuitionistic logic', *The Journal of Symbolic Logic* 62:529–544, 1997.

[12] Kremer, P., 'Quantifying over propositions in relevance logics: Non-axiomatizibility of ∀ and ∃', *The Journal of Symbolic Logic* 58:334–349, 1993.

[13] LADNER, R., 'The computational complexity of provability in systems of modal propositional logic', *SIAM Journal of Computation* Vol. 6, No. 3:467–480, 1977.

[14] NEGRI, S., and J. VON PLATO, *Structural Proof Theory*, Cambridge University Press, 2001.

[15] PITTS, A., 'On an interpretation of second order quantification in first order intuitionistic propositional logic', *The Journal of Symbolic Logic* 57:33–52, 1992.

[16] SAMBIN, G., and S. VALENTINI, 'The modal logic of provability. The sequential approach', *Journal of Philosophical Logic* 11: 311-342, 1982.

[17] SHAVRUKOV, V. Yu., *Subalgebras of diagonalizable algebras of theories containing arithmetic*, Dissertationes Mathematicae CCCXXIII, Polska Akademia Nauk, Mathematical Institute, Warszawa, 1993.

[18] TROELSTRA, A. S., and H. SCHWICHTENBERG, *Basic Proof Theory*, Cambridge University Press, 1996.

[19] VISSER, A., Bisimulations, 'Model Descriptions and Propositional Quantifiers', *Logic Group Preprint Series* No. 161, Utrecht, 1996.

[20] WANSING, H., 'Sequent systems for modal logics', in D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, Vol. 8, Kluwer Academic Publishers, 2002, pp. 61–145.

MARTA BÍLKOVÁ
Department of Logic
Charles University
Celetná 20, 116 42 Praha 1
Czech Republic
marta.bilkova@ff.cuni.cz