Grigorieff forcing and the tree property

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Abstract. In this paper we use Grigorieff forcing to obtain the tree property at the second successor of a regular uncountable cardinal κ. We also show that Silver forcing can be used to obtain the tree property at $\aleph_2$.

Keywords: Grigorieff forcing, Silver forcing and the tree property.

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Contents

1 Introduction

Let μ be an infinite cardinal. We say that a tree T of height μ^+ is a μ^+-tree if its levels have size less than μ^+. A μ^+-tree T is *Aronszajn* if it has no cofinal branches; T is a *special Aronszajn tree* if there is a function f from T to μ which is injective on chains in T, i.e. if x, y in T are comparable, then $f(x) \neq f(y)$. We say that μ^+ has the *tree property* if there are no

μ^+-Aronszajn trees. In 1930's, Nachman Aronszajn proved in ZFC that there is a special Aronszajn tree at ω_1. Therefore ω_1 does not have the tree property. In 1949, Ernst Specker [Spe49] generalized Aronszajn's original result by proving that if $\mu^{<\mu} = \mu$ then there exists a special Aronszajn tree at μ^+.[1] Hence to obtain the tree property at κ^{++}, we need to violate GCH at κ.

In 1972, William Mitchell (using ideas of Silver) proved in [Mit72] that the tree property at κ^{++}, where κ is regular, is consistent under the assumption of the existence of a weakly compact cardinal. He used a mixed support iteration of Cohen forcings; for details see [Mit72]. Later, James Baumgartner and Richard Laver showed in [BL79] that the tree property at ω_2 can be achieved by iterating Sacks forcing for ω up to a weakly compact cardinal. In 1980, Akihiro Kanamori generalized this result to an arbitrary κ^{++}, where κ is a regular cardinal, see [Kan80]. The proof is based on the fusion property of Sacks forcing.

In this paper, we use a suitably generalized Grigorieff forcing (and Silver forcing at ω) to achieve the same results (see Section 2 for definitions).

2 Grigorieff and Silver forcing

The forcing, which we now call Grigorieff forcing, was first defined by Grigorieff in [Gri71] for $\kappa = \omega$; its generalizations for uncountable cardinals were studied extensively, see for example [HV16] and [AG09]. In this paper we focus on Grigorieff forcing at uncountable regular cardinals; we also mention Silver forcing at ω which has many similarities with Grigorieff forcing. Note that Grigorieff forcing at ω is rather specific because it is defined with respect to an ideal which is not normal; we therefore choose to use at ω Silver forcing instead. In fact, a natural generalization of Silver forcing to uncountable cardinals leads to the definition of Grigorieff forcing; see Remark 2.4 for more details.

The following definition is taken from [HV16].

Definition 2.1. Let κ be a regular cardinal and let I be a subset of $\mathcal{P}(\kappa)$. We define $\mathbb{P}_I(\kappa, 1) = (P_I(\kappa, 1), \leq)$ as

(2.1) $\quad P_I(\kappa, 1) = \{f | f \text{ is a partial function from } \kappa \text{ to } 2 \text{ and } \mathrm{Dom}(f) \in I\},$

Ordering is by reverse inclusion, i.e. for p, $q \in P_I(\kappa, 1)$, $p \leq q$ if and only if $q \subseteq p$.

[1]Jensen [Jen72] proved that the existence of a special μ^+-Aronszajn tree is equivalent to the existence of a combinatorial object called the weak square ($\square^*_\mu$). $\square^*_\mu$ is strictly weaker than the assumption $\kappa^{<\kappa} = \kappa$.

By varying I, we get Cohen forcing[2], Silver forcing and Grigorieff forcing. If I is the ideal of bounded subsets, then $\mathbb{P}_I(\kappa, 1)$ is the usual Cohen forcing $\mathrm{Add}(\kappa, 1)$. If I is a set of "coinfinite" subsets of ω, i.e. $I = \{x \subset \omega \| \omega \setminus x| = \omega\}$, then we get Silver forcing at ω. If I is an arbitrary ideal on κ, then we obtain the definition of Grigorieff forcing at κ.

Definition 2.2. Let κ be a regular cardinal and let I be an ideal on κ. We define κ-Grigorieff forcing as $\mathbb{G}_I(\kappa, 1) = \mathbb{P}_I(\kappa, 1)$.

Definition 2.3. Let $I = \{x \subset \omega \| \omega \setminus x| = \omega\}$. We define Silver forcing as $\mathbb{S}(\omega, 1) = \mathbb{P}_I(\omega, 1)$.

Remark 2.4. In principle, one can consider the following generalizations of Silver forcing at an uncountable cardinal κ. Consider $\mathbb{P}_{I_i}(\kappa, 1)$, $i < 3$, where: $I_0 = \{x \subset \kappa \| \kappa \setminus x| = \kappa\}$, $I_1 = \{x \subset \kappa | \kappa \setminus x$ is stationary$\}$ and $I_2 = \{x \subset \kappa | \kappa \setminus x$ is closed unbounded$\}$. It is easy to see that I_0 and I_1 give rise to forcing notions which are not even ω_1-closed, and tend to collapse cardinals; I_2 behaves reasonably and in fact it is Grigorieff forcing with the non-stationary ideal. The definition with I_0 is only suitable for ω.

Now we discuss the basic properties of these forcings, in particular the chain condition and the closure.

Definition 2.5. Let $\mathbb{P}$ be a forcing notion and κ a regular infinite cardinal. We say that $\mathbb{P}$ is:

- κ-cc if every antichain of $\mathbb{P}$ has size less than κ.
- κ-Knaster if for every $X \subseteq \mathbb{P}$ with $|X| = \kappa$ there is $Y \subseteq X$, such that $|Y| = \kappa$ and all elements of Y are pairwise compatible.
- κ-closed if every decreasing sequence of conditions in $\mathbb{P}$ of size less than κ has a lower bound.

Lemma 2.6. *Assume $2^\kappa = \kappa^+$. Then the forcing $\mathbb{P}_I(\kappa, 1)$ is κ^{++}-cc.*

Proof. This is easy observation about the size of the forcing. If $2^\kappa = \kappa^+$, then $|\mathbb{P}_I(\kappa, 1)| = \kappa^+$. Therefore $\mathbb{P}_I(\kappa, 1)$ is κ^{++}-cc. □

The properties of Grigorieff forcing depend on the properties of the given ideal. Recall the following definitions for a regular cardinal κ.

Definition 2.7. We say that an ideal I on κ is *κ-complete* if it is closed under the unions of less than κ-many elements of I.

[2]The Cohen forcing for adding a new subset of a regular cardinal κ is composed of function from κ to 2 of size less than κ with the reverse inclusion ordering. We denote the Cohen forcing as $\mathrm{Add}(\kappa, 1)$.

Definition 2.8. We say that an ideal I on κ is *normal* if it is closed under the diagonal unions of κ-many elements of I, where the diagonal union for a sequence $\langle X_\alpha \subseteq \kappa | \alpha < \kappa \rangle$ of subsets of κ is defined as follows:

$$\Sigma_{\alpha<\kappa} X_\alpha = \{\xi < \kappa | \xi \in \bigcup_{\beta<\xi} X_\beta\} \tag{2.2}$$

Lemma 2.9. *Let κ be an uncountable regular cardinal and I be a κ-complete ideal on κ. If $\alpha < \kappa$ and $\langle p_\beta | \beta < \alpha \rangle$ is a decreasing sequence in $\mathbb{G}_I(\kappa, 1)$, then $p = \bigcup_{\beta<\alpha} p_\beta \in \mathbb{G}_I(\kappa, 1)$. Therefore $\mathbb{G}_I(\kappa, 1)$ is κ-closed.*

Proof. The proof is a direct consequence of the assumption that I is a κ-complete ideal. □

By the previous results, if I is a κ-complete ideal on an uncountable regular κ and $2^\kappa = \kappa^+$ then all cardinals greater than κ^+ and all cardinals less than or equal κ are preserved by Grigorieff forcing at κ. Also if CH holds then Silver forcing preserves all cardinals greater than ω_1.

To show that κ^+ and ω_1 are also preserved by Grigorieff forcing and Silver forcing, respectively, we need to introduced the concept of a fusion sequence.

2.1 Grigorieff forcing

For the rest of the section assume that κ is an uncountable regular cardinal.

Definition 2.10. For $\alpha < \kappa$ and p, $q \in \mathbb{G}_I(\kappa, 1)$ we define

$$p \leq_\alpha q \Leftrightarrow p \leq q \text{ and } \mathrm{Dom}(p) \cap (\alpha + 1) = \mathrm{Dom}(q) \cap (\alpha + 1). \tag{2.3}$$

We say that $\langle p_\alpha | \alpha < \kappa \rangle$ is a *fusion sequence* if for every α, $p_{\alpha+1} \leq_\alpha p_\alpha$ and $p_\beta = \bigcup_{\alpha<\beta} p_\alpha$ for every limit $\beta < \kappa$.

Lemma 2.11. *Let I be a normal ideal on κ. If $\langle p_\alpha | \alpha < \kappa \rangle$ is a fusion sequence in $\mathbb{G}_I(\kappa, 1)$, then the union $p = \bigcup_{\alpha<\kappa} p_\alpha$ is a condition in $\mathbb{G}_I(\kappa, 1)$ and $p \leq_\alpha p_\alpha$ for each $\alpha < \kappa$.*

Proof. It is sufficient to show that $\bigcup_{\alpha<\kappa} \mathrm{Dom}(p_\alpha)$ is in I, or equivalently $\bigcap_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha))$ is in I^*, where I^* is the dual filter for I. Since I^* is a normal filter, the diagonal intersection $\triangle_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha)) = \{\xi < \kappa | \xi \in \bigcap_{\beta<\xi}(\kappa \setminus \mathrm{Dom}(p_\beta))\}$ is in I^* and also the set $\{\beta < \kappa | \beta \text{ is a limit ordinal}\}$ is in I^* since I extends the nonstationary ideal on κ.

To finish the proof, it is enough to show that

$$\{\beta < \kappa | \beta \text{ is a limit ordinal}\} \cap \triangle_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha)) \subseteq \bigcap_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha)). \tag{2.4}$$

Let β be a limit ordinal in $\triangle_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha))$. Then for all $\gamma < \beta$, $\beta \notin \mathrm{Dom}(p_\gamma)$. By the limit step of the definition of fusion sequence, $\beta \notin \mathrm{Dom}(p_\beta)$. Hence β is not in $\mathrm{Dom}(p_\alpha)$ for each $\alpha > \beta$ by (2.3). Therefore β is in $\bigcap_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha))$. □

Corollary 2.12. *Let κ be an uncountable cardinal. Assume that $\kappa^{<\kappa} = \kappa$ and I is a normal ideal on κ. Then $\mathbb{G}_I(\kappa, 1)$ preserves κ^+.*

Remark 2.13. The proof of the previous corollary is a standard argument using the closure of the forcing under the fusion sequences. If $\dot{f}$ is a $\mathbb{G}_I(\kappa, 1)$-name for a function from κ to κ^+ then we construct by induction a fusion sequence such that its lower bound will force $\dot{f}$ is bounded. For the details for an inaccessible κ see Theorem 2.6 in [HV16]. If κ is a successor cardinal, a diamond-guided construction is usually invoked since it can show the preservation of κ^+ even for iterations of Grigorieff forcing (see section 2.3). However, it is easy to use a diagonal argument to show that $\mathbb{G}_I(\kappa, 1)$ preserves κ^+ even without the diamond (since $\kappa^{<\kappa} = \kappa$ implies the diamond at κ for all κ except ω_1, this observation is relevant only for $\mathbb{G}_I(\omega_1, 1)$).

Remark 2.14. The converse direction holds as well. For the proof see [HV16].

Remark 2.15. It is instructive to see the importance of having $(\alpha+1)$ and not just α in (2.3). If we required that the domains are the same on α only, it is easy to construct a fusion sequence without a lower bound.[3]

2.2 Silver forcing

The fusion argument for Grigorieff forcing at ω is more complicated since at ω we do not have the notion of a normal ideal. For more details about the case of ω, see [Gri71]. For Silver forcing, a fusion sequence can be defined as follows:

Definition 2.16. If p is a condition in $\mathbb{S}(\omega, 1)$, let n_p denote the n-th element of $\omega \setminus \mathrm{Dom}(p)$. For $n < \omega$ and p, $q \in \mathbb{S}(\omega, 1)$ we define

$$p \leq_n q \Leftrightarrow p \leq q \text{ and } \mathrm{Dom}(p) \cap (n_q + 1) = \mathrm{Dom}(q) \cap (n_q + 1). \tag{2.5}$$

We say that $\langle p_n | n < \kappa \rangle$ is a *fusion sequence* if for every n, $p_{n+1} \leq_n p_n$.

Lemma 2.17. *If $\langle p_n | n < \omega \rangle$ is a fusion sequence in $\mathbb{S}(\omega, 1)$, then the union $p = \bigcup_{n<\omega} p_n$ is a condition in $\mathbb{S}(\omega, 1)$ and $p \leq_n p_n$ for each $n < \omega$.*

[3] For instance consider the sequence $\langle p_\alpha | \alpha < \kappa \rangle$ of functions, where $\mathrm{Dom}(p_\alpha)$ is α for every $\alpha < \kappa$. If we changed the definition in (2.3) to require that the domains are equal on α only, then this is a fusion sequence without a lower bound (its greatest lower bound is a function with the domain equal to κ.)

Proof. The proof follows from (2.5) since at the n-th step we guaranteed that n_{p_n} is not in $\mathrm{Dom}(p)$. □

Corollary 2.18. *ω_1 is preserved by Silver forcing.*

2.3 Iteration

For the rest of the section, we fix an uncountable regular cardinal κ and a normal ideal I on κ. We will consider the iteration of Grigorieff forcing defined with respect to κ and I (for more details about iterations in general, see [Bau83]).

Definition 2.19. Let $\lambda > 0$ be an ordinal. Then we define $\mathbb{G}_I(\kappa, \lambda)$ by induction as follows:

(i) The forcing $\mathbb{G}_I(\kappa, 1)$ is defined as in Definition 2.2.

(ii) $\mathbb{G}_I(\kappa, \xi+1) = \mathbb{G}_I(\kappa, \xi) * \dot{Q}_\xi$, where $\dot{Q}_\xi$ is a $\mathbb{G}_I(\kappa, \xi)$-name for the partial order $\mathbb{G}_I(\kappa, 1)$ as defined in the extension $V[\mathbb{G}_I(\kappa, \xi)]$.

(iii) For a limit ordinal ξ, $\mathbb{G}_I(\kappa, \xi)$ is the inverse limit of $\langle \mathbb{G}_I(\kappa, \zeta) | \zeta < \xi \rangle$ if $\mathrm{cf}(\xi) \leq \kappa$ and the direct limit otherwise.

We consider $\mathbb{G}_I(\kappa, \lambda)$ as the collection of functions p with domain λ such that for every $\xi < \lambda$, $p \restriction \xi \Vdash_\xi p(\xi) \in \dot{Q}_\xi$ and $|\mathrm{supp}(p)| \leq \kappa$. The ordering is defined as follows: for p, q in $\mathbb{G}_I(\kappa, \lambda)$, $p \leq q$ if and only if $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$ and for every $\xi \in \mathrm{supp}(q), p \restriction \xi \Vdash_\xi p(\xi) \leq q(\xi)$.

Lemma 2.20. *Let κ be a regular cardinal and $\lambda > \kappa$ be an inaccessible cardinal. Then $\mathbb{G}_I(\kappa, \lambda)$ has size λ and it is λ-Knaster.*

Proof. See Theorem 16.30 in [Jec03]. Theorem 16.30 is formulated for the chain condition, but it is easy to check that the reformulation of the proof for Knaster forcings actually gives Knasterness. □

The following definitions and results are analogues of the corresponding results in [Kan80] which deals with Sacks forcing. We define the notion of meet and use it to show that the iteration of Grigorieff forcing is sufficiently closed and has the fusion property.

Definition 2.21. Let α be an ordinal. If $\langle p_\beta | \beta < \alpha \rangle$ is a decreasing sequence of conditions, then the *meet* $p = \bigwedge_{\beta<\alpha} p_\beta$ is defined as follows:
(2.6)
$$\mathrm{supp}(p) = \bigcup_{\beta<\alpha} \mathrm{supp}(p_\beta) \text{ and } p \restriction \gamma \Vdash p(\gamma) = \bigcup_{\beta<\alpha} p_\beta(\gamma) \text{ for } \gamma \in \mathrm{supp}(p).$$

Lemma 2.22. *If $\alpha < \kappa$ and $\langle p_\beta | \beta < \alpha\rangle$ is a decreasing sequence in $\mathbb{G}_I(\kappa,\lambda)$, then $p = \bigwedge_{\beta<\alpha} p_\beta \in \mathbb{G}_I(\kappa,\lambda)$. Hence $\mathbb{G}_I(\kappa,\lambda)$ is κ-closed.*

Proof. See Theorem 2.5 in [Bau83]. □

Definition 2.23. Let p, $q \in \mathbb{G}_I(\kappa,\lambda)$, $X \subseteq \lambda$ with $|X| < \kappa$ and $\alpha < \kappa$. We define

$$p \leq_{X,\alpha} q \Leftrightarrow p \leq q \text{ and } p \restriction \xi \Vdash p(\xi) \leq_\alpha q(\xi) \text{ for all } \xi \in X. \tag{2.7}$$

We say that a pair $(\langle p_\xi | \xi < \kappa\rangle, \langle X_\xi | \xi < \kappa\rangle)$ is a *fusion sequence* if it satisfies the following conditions:

(i) $p_{\xi+1} \leq_{X_\xi,\xi} p_\xi$ for every $\xi < \kappa$ and $p_\zeta = \bigwedge_{\xi<\zeta} p_\xi$ for every limit $\zeta < \kappa$;

(ii) $|X_\xi| < \kappa$ and $X_\xi \subseteq X_{\xi+1}$ for every $\xi < \kappa$;

(iii) $X_\zeta = \bigcup_{\xi<\zeta} X_\xi$ for every limit $\zeta < \kappa$ and $\bigcup_{\xi<\kappa} X_\xi = \bigcup_{\xi<\kappa} \mathrm{supp}(p_\xi)$.

Lemma 2.24. *Let $\lambda > 0$ be an ordinal. If $(\langle p_\beta | \beta < \kappa\rangle, \langle X_\beta | \beta < \kappa\rangle)$ is a fusion sequence, then $p = \bigwedge_{\beta<\kappa} p_\beta$ is in $\mathbb{G}_I(\kappa,\lambda)$.*

Proof. We prove the lemma by induction on $\xi \leq \lambda$ and we show that for each $\xi \leq \lambda$, $p \restriction \xi \in \mathbb{G}_I(\kappa,\xi)$.

If $\xi = 0$, then $p(\xi)$ is in $\mathbb{G}_I(\kappa,1)$ by Lemma 2.11.

If $\xi = \zeta + 1$, then we want to show that $p \restriction \zeta \Vdash_\zeta p(\zeta) \in \dot{Q}_\zeta$. Since $p \restriction \zeta \leq p_\beta \restriction \zeta$ for all $\beta < \kappa$, it is clear that $p \restriction \zeta \Vdash_\zeta$ "$\langle p_\beta(\zeta) | \beta < \kappa\rangle$ is a decreasing sequence in $\dot{Q}_\zeta$".

If ζ is not in $\mathrm{supp}(p)$, then we are done, since $p \restriction \zeta \Vdash_\zeta p(\zeta) = \check{1} \in \dot{Q}_\zeta$.

If $\zeta \in \bigcup_{\xi<\kappa} \mathrm{supp}(p_\xi)$, then by the definition of meet, we know that $p \restriction \zeta \Vdash p(\zeta) = \bigcup_{\beta<\kappa} p_\beta(\zeta)$. Now we use the properties of fusion sequence to show $p \restriction \zeta \Vdash \bigcup_{\beta<\kappa} p_\beta(\zeta) \in \dot{Q}_\zeta$. Since $\bigcup_{\beta<\kappa} X_\beta = \bigcup_{\beta<\kappa} \mathrm{supp}(p_\beta)$, there is $\alpha < \kappa$ and X_α such that $\zeta \in X_\alpha$. As the sequence $\langle X_\beta | \beta < \kappa\rangle$ is increasing and $p \restriction \zeta \leq p_\beta \restriction \zeta$ for all $\beta < \kappa$, we have that $p \restriction \zeta \Vdash p_{\beta+1}(\zeta) \leq_\beta p_\beta(\zeta)$ for all $\alpha \leq \beta < \kappa$. Therefore $p \restriction \zeta \Vdash \bigcup_{\alpha\leq\beta<\kappa} p_\beta(\zeta) \in \dot{Q}_\zeta$ by Lemma 2.11. Since $p \restriction \zeta \Vdash_\zeta$ "$\langle p_\beta(\zeta) | \beta < \kappa\rangle$ is a decreasing sequence in $\dot{Q}_\zeta$", $p \restriction \zeta \Vdash \bigcup_{\alpha\leq\beta<\kappa} p_\beta(\zeta) = \bigcup_{\beta<\kappa} p_\beta(\zeta) \in \dot{Q}_\zeta$.

If ξ is a limit ordinal, then the claim is clear. □

The fusion property is used to show that κ^+ is preserved in the extension by $\mathbb{G}_I(\kappa,\lambda)$.

Fact 2.25. *Assume that either κ is inaccessible or that $\diamondsuit_\kappa$ holds. Then $\mathbb{G}_I(\kappa, \lambda)$ preserves κ^+.*

Proof. Follows from [Kan80] by adapting the argument with the fusion defined for Grigorieff forcing. □

3 Forcing the tree property

In this section, let us assume that κ is an uncountable regular cardinal and I is a normal ideal on κ.

3.1 Fusion and not adding branches

This section is based on the paper [FH15] where a general notion of fusion was defined. Both Grigorieff and Silver forcing satisfy this general notion, and we can therefore use a criterion from [FH15] to argue that new branches are not added to certain trees. To prove Fact 3.6, we need to apply the criterion to the iteration $\mathbb{G}_I(\kappa, \lambda)$ for an arbitrary uncountable regular κ. To illustrate the method, we will assume that κ is inaccessible and the iteration has length 1. Longer iterations for an inaccessible κ are more complicated notationally, but do not introduce new ideas. If κ is a successor cardinal, a diamond-guided construction must be used.

Definition 3.1. Let $\mathbb{P}$ be a forcing notion and G a $\mathbb{P}$-generic filter. We say that a sequence of ground-model objects $x = \langle a_i | i < \kappa \rangle$ in $V[G]$ is *fresh* if for every $\alpha < \kappa$, $x \restriction \alpha$ is in V, but x is in $V[G] \setminus V$.

Lemma 3.2. *Let $\mathbb{P}$ be a forcing notion and let the weakest condition of $\mathbb{P}$ force that $\dot{f}$ is a fresh κ-sequence. Then for every p_0 and p_1 in $\mathbb{P}$ and every $\delta < \kappa$ there are $r_0 \leq p_0$, $r_1 \leq p_1$ and $\gamma \geq \delta$ such that r_0 and r_1 force contradictory information about $\dot{f}$ at level γ.*

Proof. Let p_0, p_1 and $\delta < \kappa$ be given. Since $\dot{f}$ is a fresh sequence there are q^0, $q^1 < p_0$ and $\gamma > \delta$ such that q^0 and q^1 force contradictory information about $\dot{f}$ at γ. Also there is $r_1 \leq p_1$ which decides the value of $\dot{f}$ at γ to be some element of the ground model a. Since q^0 and q^1 force contradictory information about $\dot{f}$ at γ, at least one of them has to force $\dot{f}(\gamma) \neq a$. Chose r_0 to be the one with smaller upper index which forces this. □

Definition 3.3. Assume $\kappa^{<\kappa} = \kappa$. We say that $\mathbb{G}_I(\kappa, 1)$ *does not decide fresh κ^+-sequences in a strong sense* if the following hold: whenever $\dot{f}$ is a name for a fresh sequence of length κ^+, i.e

(3.1) $\qquad \mathbb{G}_I(\kappa, 1) \Vdash$ "$\dot{f}$ is a name for a fresh sequence of length κ^+, " then for every $p \in \mathbb{G}_I(\kappa, 1)$, every $\alpha < \kappa$ and every $\delta < \kappa^+$, there are $p_0 \leq_\alpha p$ and $p_1 \leq_\alpha p$ and γ, with $\delta < \gamma < \kappa^+$, such that whenever $r_0 \leq p_0$ and $r_1 \leq p_1$ and

$$r_0 \Vdash \dot{f} \restriction \gamma = \check{f}_0 \text{ and } r_1 \Vdash \dot{f} \restriction \gamma = \check{f}_1 \tag{3.2}$$

Then

$$f_0 \neq f_1. \tag{3.3}$$

That means, r_0 and r_1 force contradictory information about $\dot{f}$ restricted to γ.

Theorem 3.4. *Let κ be an inaccessible cardinal. If $\mu \geq \kappa$ is such that $2^\kappa > \mu$, then $\mathbb{G}_I(\kappa, 1)$ does not add cofinal branches to μ^+-trees.*

Proof. We use Theorem 3.4 from [FH15], which says that it is enough to verify that Grigorieff forcing $\mathbb{G}_I(\kappa, 1)$ does not decide κ^+-sequence in a strong sense.

Assume that $1 \Vdash$ "$\dot{b}$ is a fresh sequence of length κ^+". Now we need to show that for any $\alpha < \kappa$, $\delta < \kappa^+$, and condition p, there are conditions p_0, p_1 and ordinal γ such that $p_0 \leq_\alpha p$, $p_1 \leq_\alpha p$, $\delta < \gamma < \kappa^+$ and whenever $r_0 \leq p_0$ and $r_1 \leq p_1$ are such that

$$r_0 \Vdash \dot{b} \restriction \gamma = \check{b}_0 \text{ and } r_1 \Vdash \dot{b} \restriction \gamma = \check{b}_1. \tag{3.4}$$

Then

$$b_0 \neq b_1. \tag{3.5}$$

Denote $A = \{(f, g) | f, g \in {}^{\alpha+1}2 \text{ and } f \leq p \restriction \alpha + 1 \text{ and } g \leq p \restriction \alpha + 1\}$. Since κ is inaccessible, the size of A is less than κ.

We will construct by induction on $|A|$ two $\leq_\alpha$-decreasing sequences continuous at limits $\langle p_0^i | i < |A| \rangle$ and $\langle p_1^i | i < |A| \rangle$ which satisfy

$$p_0^i \restriction \alpha + 1 = p_1^i \restriction \alpha + 1 = p \restriction \alpha + 1 \tag{3.6}$$

for all $i < |A|$; p_0 will be the infimum of $\langle p_0^i | i < |A| \rangle$ and p_1 the infimum of $\langle p_1^i | i < |A| \rangle$. We will also construct an increasing sequence of ordinals continuous at limits $\langle \gamma_i | i < |A| \rangle$. The desired γ will be the supremum of this sequence. Enumerate $A = \{(f, g)_i | i < |A|\}$.

Set $p_0^0 = p$ and $p_1^0 = p$ and $\gamma_0 > \delta$.

For $m < |A|$, assume p_j^m, for $j \in \{0, 1\}$, and γ_m were already constructed. To construct the $m+1$-st element of the sequences, and also γ_{m+1}, consider $(f, g) = (f, g)_m$.

Consider the conditions $p_0^m \cup f$ and $p_1^m \cup g$. By Lemma 3.2, find $s_0 \leq p_0^m \cup f$ and $s_1 \leq p_1^m \cup g$ such that s_0 and s_1 force contradictory information about $\dot{b}$ at level β for some $\beta > \gamma_m$. Set p_0^{m+1} to be $p_0^m \cup s_0 \restriction [\alpha+1, \kappa)$ and p_1^{m+1} to be $p_1^m \cup s_1 \restriction [\alpha+1, \kappa)$ and $\gamma_{m+1} = \beta$.

At limit stages, take the infimum of the conditions and the supremum of the ordinals.

We now verify that $p_0 = \bigwedge \langle p_0^i | i < |A| \rangle$, $p_1 = \bigwedge \langle p_1^i | i < |A| \rangle$, and $\gamma = \sup \langle \gamma_i | i < |A| \rangle$ are as desired. Let $r_0 \leq p_0$ and $r_1 \leq p_1$ be given. We can assume that both r_0 and r_1 are defined on $\alpha + 1$. Then there is some $(f, g)_m \in A$ such that $r_0 \leq p_0^{m+1} \cup f$ and $r_1 \leq p_1^{m+1} \cup g$, and so r_0 and r_1 decide $\dot{b}$ differently at $\gamma_{m+1} < \gamma$. □

Remark 3.5. Note that the previous proof can be easily modified for Silver forcing at ω and its definition of fusion.

Fact 3.6. *Assume that either κ is inaccessible or that $\Diamond_\kappa$ holds. Let $\lambda > 0$ be an ordinal. If $\mu \geq \kappa$ is such that $2^\kappa > \mu$, then $\mathbb{G}_I(\kappa, \lambda)$ does not add cofinal branches to μ^+-trees.*

Remark 3.7. Note that for $\kappa = \xi^+ > \omega_1$, we just need to assume $2^\xi = \xi^+$, since this ensures $\Diamond_\kappa$.

3.2 The tree property

We showed in the previous section that under GCH, $\mathbb{G}_I(\kappa, \lambda)$ preserves all cardinals smaller or equal to κ (by κ-closure) and cardinals greater or equal to λ (by λ-cc). Moreover, under an additional assumption, κ^+ is preserved due to the fusion property.

Now we show that cardinals in the interval (κ^+, λ) are collapsed.

Lemma 3.8. *Assume that either κ is inaccessible or that $\Diamond_\kappa$ holds. Let $\lambda > \kappa$ be an inaccessible cardinal. Then $V[\mathbb{G}_I(\kappa, \lambda)] \models \lambda = \kappa^{++}$.*

Proof. The preservation of κ^+ follows by Fact 2.25, and the collapse of λ to become the second successor of κ follows by the more general fact which says that Cohen forcing at κ^+ is regularly embedded to any κ-support iteration of non-trivial forcing notions of length (at least) κ^+. □

Now we have everything that we need to prove the main theorem of this paper.

Theorem 3.9. *Assume GCH. Assume κ is regular uncountable. If there exists a weakly compact cardinal $\lambda > \kappa$, then in the generic extension by $\mathbb{G}_I(\kappa, \lambda)$, the following hold:*

(i) $2^\kappa = \lambda = \kappa^{++}$*;*

(ii) κ^{++} *has the tree property.*

Proof. For simplicity, we assume that λ is measurable.[4]

Ad (i). It is easy to see that $2^\kappa = \lambda$ and $\lambda = \kappa^{++}$ follows from Lemma 3.8.

Ad (ii). Let G be a $\mathbb{G}_I(\kappa, \lambda)$-generic filter over V. Since λ is measurable in V, there is an elementary embedding $j : V \to M$ with critical point λ and ${}^\lambda M \subseteq M$, where M is a transitive model of ZFC.

In M, the forcing $j(\mathbb{G}_I(\kappa, \lambda))$ is the iteration of $\mathbb{G}_I(\kappa, 1)$ of length $j(\lambda)$ with κ-support by the elementarity of j. The forcing $\mathbb{G}_I(\kappa, j(\lambda))^M$ is forcing equivalent to $(\mathbb{G}_I(\kappa, \lambda) * \dot{\mathbb{G}}_I(\kappa, [\lambda, j(\lambda)))^M$. As j is the identity below λ, $\mathbb{G}_I(\kappa, \alpha) = \mathbb{G}_I(\kappa, \alpha)^M$, for $\alpha < \lambda$ and since we take direct limit at λ, $\mathbb{G}_I(\kappa, \lambda) = \mathbb{G}_I(\kappa, \lambda)^M$. Hence G is also $\mathbb{G}_I(\kappa, \lambda)^M$-generic over M.

Let H be $\mathbb{G}_I(\kappa, [\lambda, j(\lambda)))^{M[G]}$-generic over $V[G]$, and let us work in $V[G][H]$. Since we have $j[G] \subseteq G * H$, we can use Silver lifting lemma (see Proposition 9.1 in [Cum10]) and lift j to $j^* : V[G] \to M[G][H]$.

Assume T is a λ-tree in $V[G]$; we show that T has a cofinal branch in $V[G]$, and therefore there is no λ-Aronszajn tree in $V[G]$.

We can consider T as a subset of λ. Let $\dot{T}$ be a nice name for T in V. As $\dot{T}$ is an element of $H(\lambda^+)$, $\dot{T}$ is in M, and hence T is in $M[G]$. By elementarity of j^*, $j^*(T)$ is a $j^*(\lambda)$-tree in $M[G][H]$, hence it has a node b of length λ in $M[G][H]$. As j^* is the identity below λ, $j^*(T) \restriction \lambda = T$; therefore b is a cofinal branch trough T in $M[G][H]$.

By Fact 3.6, $\mathbb{G}_I(\kappa, [\lambda, j(\lambda)))^{M[G]}$ does not add cofinal branches to λ-trees over $M[G]$. Therefore b is in $M[G]$, and hence in $V[G]$. □

Remark 3.10. As we noted above (see Remark 3.5), the Silver forcing at ω satisfies the criterion for not adding branches from [FH15]; therefore it is easy to show (as in Theorem 3.9) that $\mathbb{S}(\omega, \lambda)$ forces the tree property at ω_2 if λ is a weakly compact cardinal.

Remark 3.11. We say that an uncountable μ^+ has the *weak tree property* if there are no special μ^+-Aronszajn trees. One can show that whenever GCH holds and κ is regular, $\mathbb{G}_I(\kappa, \lambda)$ and $\mathbb{S}(\omega, \lambda)$ force the weak tree property at κ^{++} and $\aleph_2$, respectively, whenever λ is a Mahlo cardinal greater than κ.

[4]If λ is just a weakly compact cardinal, we modify the argument as follows. If $\dot{T}$ is a nice name for a λ-tree, fix $j : M \to N$ so that M is a transitive model of ZFC$^-$ of size λ closed under $< \lambda$-sequences which contains as elements the forcing $\mathbb{G}_I(\kappa, \lambda)$ and $\dot{T}$, j has critical point λ, N has size λ, is closed under $< \lambda$-sequences and $M \in N$ (in particular, $\dot{T}$ is in N). The existence of such j follows from the weak compactness of λ. Then apply the argument below to this j.

The proof is a variant of the argument in Theorem 3.8; for more details, see [Mit72].

3.3 Open question

Q1. As in [Ung12], one may ask about the indestructibility of the tree property in the models obtained by Silver and Grigorieff forcing. For instance, one can ask: Is the tree property at κ^{++} obtained by Grigorieff forcing indestructible under Cohen forcing at κ?

Q2. Or more generally, one may study the indestructibility over models with the tree property obtained by forcings which satisfy some kind of fusion (Sacks, Grigorieff, Silver, axiom-A forcing notions, etc.).

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