The tree property at $\aleph_{\omega+2}$ with a finite gap

Sy-David Friedman,[1] *Radek Honzik,*[2] *Šárka Stejskalová*[2]

[1]Kurt Gödel Research Center for Mathematical Logic,
Währinger Strasse 25, 1090 Vienna Austria
sdf@logic.univie.ac.at

[2]Charles University, Department of Logic,
Celetná 20, Praha 1, 116 42, Czech Republic
radek.honzik@ff.cuni.cz
sarka.stejskalova@ff.cuni.cz

All authors were supported by FWF/GAČR grant I 1921-N25.

Abstract: Let n be a natural number, $2 \leq n < \omega$. We show that it is consistent to have a model of set theory where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+n}$, and the tree property holds at $\aleph_{\omega+2}$; we use a weakly compact hypermeasurable cardinal for the result. This generalises the known result with $n = 2$. We note that this the first exposition of the tree property at $\aleph_{\omega+2}$ with $\aleph_\omega$ strong limit which uses a projection-of-product analysis of the Mitchell forcing with Prikry forcing with collapses reminiscent of the analysis in Abraham [1].

Keywords: The tree property, The singular cardinal hypothesis.

AMS subject code classification: 03E35,03E55.

Date: April 12, 2017

1 Introduction

Let κ be a regular cardinal. We say that the tree property holds at κ if every κ-tree has a cofinal branch. The tree property is a compactness property which can hold at successor cardinals low in the set-theoretical hierarchy: Mitchell first showed in [14] that it is equiconsistent with the existence of a weakly compact cardinal that $\aleph_2$ has the tree property (his argument readily generalises to any κ^{++} for an infinite regular cardinal κ).

The situation is more complex when we wish to get the tree property at the double successor of singular strong limit cardinal κ. First, since one needs to have $2^\kappa > \kappa^+$ (and thus the failure of SCH, the singular cardinal hypothesis), it is known that a measurable cardinal of high Mitchell order is required. Second, a new idea is required which connects the Mitchell construction and the known ways of obtaining the failure of SCH. This was first achieved by Cummings and Foreman [4] who proved that it is consistent to have a singular strong limit cardinal κ of countable cofinality with the tree property at κ^{++}. The cardinal κ in [4] was supercompact in the ground model; without a proof, [4] claimed that κ can be collapsed to $\aleph_\omega$ using a similar argument. However, for some time no such proof had been found.

The first argument which yields a model where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2}$, and the tree property holds at $\aleph_{\omega+2}$, was given by Friedman and Halilović in [6]. The argument started with a much weaker hypothesis than [4] (an $H(\lambda)$-hypermeasurable κ for a weakly compact $\lambda > \kappa$), and used an iteration of the κ-Sacks forcing followed by the Prikry forcing and therefore was quite different in spirit from the construction in [4].

Recently, the authors of [5] found another construction which yields the same configuration (in particular with $2^{\aleph_\omega} = \aleph_{\omega+2}$). The construction is based on the Mitchell forcing followed by the Prikry forcing with collapses, and starts with a supercompact cardinal. It is of interest to note that the proof that the tree property holds at $\aleph_{\omega+2}$ proceeds directly without using a projection-of-product analysis (which is at the heart of the presentations of Abraham [1] and crucial in the analysis in [4]).

In the present paper, we formulate yet another approach to constructing a model where $\aleph_\omega$ is strong limit, violates SCH, and the tree property holds at $\aleph_{\omega+2}$. The main features of our construction are as follows:

- We start with a suitably large hypermeasurable cardinal;
- We are able to ensure that $2^{\aleph_\omega}$ is equal to $\aleph_{\omega+n}$ for any fixed $2 \leq n < \omega$;
- We use a variant of the Mitchell forcing (to ensure the large value

of $2^{\aleph_\omega}$) followed by the Prikry forcing with collapses. Our proof relies on a projection-of-product analysis and is therefore similar to the methods of [4].[1]

The paper is structured as follows.

In Section 1.1 we review the forcings which we will use: in particular, we define a variant of the Mitchell forcing which ensures the large value of $2^{\aleph_\omega}$, and provide a product analysis of the Mitchell forcing followed by the Prikry forcing with collapses which is reminiscent of the analysis in [1] and [4] (Section 1.1.3).

In Section 2, we argue that it is possible to start with a hypermeasurable cardinal κ of a suitable degree and prepare the ground model so that a further forcing with a Cohen forcing at κ (of a prescribed length) does not destroy the measurability of κ. This preparation provides a degree of indestructibility which is usually invoked in the context of a supercompact cardinal and the Laver preparation.

In Section 3, we show that over the prepared ground model, the standard Mitchell forcing followed by the Prikry forcing with collapses forces that $\kappa = \aleph_\omega$ is a strong limit cardinal, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and the tree property holds at $\aleph_{\omega+2}$.

In Section 4 we generalise the construction in Section 3 to any finite gap $2 \leq n < \omega$.

Finally, in Section 5 we mention some open questions.

[1] There is a fine difference in the proofs, though: in [4], a "Prikry-ised Mitchell forcing" is used, i.e. the Prikry part is integrated into the Mitchell forcing; we used a similar method in [7] to obtain a large value of 2^κ with the tree property at κ^{++}, κ strong-limit with countable cofinality. When the collapsing is involved, it is easy to see that the Prikry forcing with collapses must come after the Mitchell forcing, and cannot be integrated into the Mitchell part. Surprisingly, the latter method seems easier and more universal (it can be used to reprove the result of [4]).

1.1 Preliminaries

1.1.1 A variant of Mitchell forcing

We will use a variant of the standard Mitchell forcing as presented in [1].

If κ is a regular infinite cardinal and α is an ordinal greater than 0, we identify the Cohen forcing for adding α-many subsets of κ, $\mathrm{Add}(\kappa, \alpha)$, with a collection of functions p from a subset of $\kappa \times \alpha$ of size $< \kappa$ into $\{0, 1\}$. The ordering is by reverse inclusion.

Let $\kappa < \lambda$ be regular cardinals, and assume λ is inaccessible. Let $\mu > \lambda$ be an ordinal. We define a variant of the Mitchell forcing, $\mathbb{M}(\kappa, \lambda, \mu)$, as follows: Conditions are pairs (p, q) such that p is in $\mathrm{Add}(\kappa, \mu)$, and q is a function whose domain is a subset of λ of size at most κ such that for every $\xi \in \mathrm{dom}(q)$, $q(\xi)$ is an $\mathrm{Add}(\kappa, \xi)$-name, and $\emptyset \Vdash_{\mathrm{Add}(\kappa,\xi)} q(\xi) \in \mathrm{Add}(\kappa^+, 1)$. The ordering is as in the standard Mitchell forcing, i.e.: $(p', q') \leq (p, q)$ if and only if p' is stronger than p in the Cohen forcing, the domain of q' contains the domain of q and if ξ is in the domain of q, then p' restricted to ξ forces $q'(\xi)$ extends $q(\xi)$.

Lemma 1.1 *Assume GCH.*

(i) $\mathbb{M}(\kappa, \lambda, \mu)$ *is* λ*-cc.*
(ii) In $V[\mathbb{M}(\kappa, \lambda, \mu)]$*,* $2^\kappa = |\mu|$*, and the cardinals in the open interval* (κ^+, λ) *are collapsed (and no other cardinals are collapsed).*

Proof. The proof is standard. □

The following follows as in [1]:

Lemma 1.2 *(i)* $\mathbb{M}(\kappa, \lambda, \mu)$ *is a projection of* $\mathrm{Add}(\kappa, \mu) \times \mathbb{T}$*, where* $\mathbb{T}$ *is a* κ^+*-closed term forcing defined by* $\mathbb{T} = \{(\emptyset, q) \mid (\emptyset, q) \in \mathbb{M}(\kappa, \lambda, \mu)\}$*.*
(ii) $\mathbb{M}(\kappa, \lambda, \mu)$ *is equivalent to* $\mathrm{Add}(\kappa, \mu) * \dot{Q}$*, where* $\dot{Q}$ *is forced to be* κ^+*-distributive.*

PROOF. The proof is as in [1]. □

As will be apparent from the arguments in Section 3, it is also the case that if λ is weakly compact, then the tree property holds at $\lambda = \kappa^{++}$ in $V[\mathbb{M}(\kappa, \lambda, \mu)]$.

1.1.2 Prikry forcing with collapses

We use the forcing as it is described in Gitik's paper [8].

Here we give just a quick review to fix the notation. Let κ be a measurable cardinal, U a normal measure at κ, and $j_U : V \to M$ the ultrapower embedding generated by U. The Prikry forcing with collapses, which we denote $\mathrm{PrkCol}(U, G^g)$, is determined by U and a guiding generic G^g. G^g is a $\mathrm{Coll}(\kappa^{+n}, < j(\kappa))^M$-generic filter over M, where n typically satisfies $2 < n < \omega$ (Coll denotes the Levy collapse).

A condition r in $\mathrm{PrkCol}(U, G^g)$ has a lower part ("stem") which is a finite increasing sequence of cardinals below κ with information about collapses between the cardinals (thus the stem is an element of V_κ), and an upper part which is composed of sets A and H, where A is in U, and H is a function defined on A such that $[H]_U$, the equivalence class of H in M, belongs to G^g.

If all is set up correctly in V, the forcing $\mathrm{PrkCol}(U, G^g)$ collapses κ to $\aleph_\omega$ while preserving all cardinals above κ.

1.1.3 Mitchell followed by Prikry forcing with collapses

Assume $\kappa < \lambda < \mu$ are as above, $\mathbb{M} = \mathbb{M}(\kappa, \lambda, \mu)$ is the Mitchell forcing, and $\dot{U}$ and $\dot{G}^g$ are $\mathbb{M}$-names such that the weakest condition in $\mathbb{M}$ forces that $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ is the Prikry forcing with collapses defined with respect to $\dot{U}$ and $\dot{G}^g$.

Lemma 1.3 $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ *is* λ*-cc.*

PROOF. We know that $\mathbb{M}$ is λ-cc, and $\mathbb{M}$ forces that $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ is κ^+-cc. The lemma now follows. □

Lemma 1.4 *In $\mathbb{M}*\mathrm{PrkCol}(\dot{U},\dot{G}^g)$, conditions $((p,q),r)$, where r in the Prikry forcing depends only on the Cohen information of the Mitchell forcing and its stem is a checked name, are dense.*

Proof. This is because all conditions in the Prikry forcing exist already in the extension by the Cohen part of the Mitchell forcing (in contrast, the definition of $\mathrm{PrkCol}(\dot{U},\dot{G}^g)$ itself may require the whole $\mathbb{M}$ in order to refer to $\dot{U}$ and $\dot{G}^g$; this will be the case in our argument in Section 3.3.2). Given $((p,q),r)$ we can extend (p,q) to some (p',q') such that (p',q') forces that r is equal to some r' in the generic extension by the Cohen part of the Mitchell forcing with its stem being a ground model object (since the Cohen forcing at κ does not add bounded subsets of V_κ). $\square$

Using Lemma 1.4, we can formulate a projection-of-product analysis of the forcing $\mathbb{M}*\mathrm{PrkCol}(\dot{U},\dot{G}^g)$ reminiscent of Abraham's analysis of the Mitchell forcing in [1]. Let us define:

$$\mathbb{C}=\{((p,\emptyset),r)\,|\,((p,\emptyset),r)\in\mathbb{M}*\mathrm{PrkCol}(\dot{U},\dot{G}^g)\}, \tag{1.1}$$

where we require that r depends only on the Cohen information of the Mitchell forcing and its stem is a checked name.[2] Let us also define:

$$\mathbb{T}=\{(\emptyset,q)\,|\,(\emptyset,q)\in\mathbb{M}\}. \tag{1.2}$$

Define a function τ from $\mathbb{C}\times\mathbb{T}$ to $\mathbb{M}*\mathrm{PrkCol}(\dot{U},\dot{G}^g)$ as follows: τ applied to the pair composed of $((p,\emptyset),r)$ and $(\emptyset,q)$ is equal to the condition $((p,q),r)$.

Lemma 1.5 *(i) τ is a projection from $\mathbb{C}\times\mathbb{T}$ onto a dense part of $\mathbb{M}*\mathrm{PrkCol}(\dot{U},\dot{G}^g)$.*
(ii) $\mathbb{T}$ is κ^+-closed in V.
(iii) $\mathbb{C}$ is κ^+-cc in V.

[2]The requirement that the stem of r is a checked name is not important for Lemma 1.5, but will be useful in Section 3.3.2 when a similar analysis is performed.

Proof. (i). If $((p', \emptyset), r') \leq ((p, \emptyset), r)$ and $(\emptyset, q') \leq (\emptyset, q)$, then $((p, q'), r') \leq ((p, q), r)$, so τ is order-preserving.

Now, suppose we are given $((p', q'), r') \leq ((p, q), r)$, where

$$((p, q), r) = \tau(((p, \emptyset), r), (\emptyset, q))$$

and r' depends only on the Cohen information of the Mitchell forcing and its stem is a checked name (by Lemma 1.4 such conditions are dense). We will find q^* and r^* such that

(a) $(\emptyset, q^*) \leq (\emptyset, q)$ in $\mathbb{T}$,
(b) $((p', \emptyset), r^*) \leq ((p, \emptyset), r)$ in $\mathbb{C}$,
(c) $\tau(((p', \emptyset), r^*), (\emptyset, q^*)) = ((p', q^*), r^*) \leq ((p', q'), r')$.

In order to get (a)–(b), first define q^* so that it interprets as q' below p', and as q below conditions incompatible with p' (ensuring (a)). Since we assume that r and r' depend only on the Cohen forcing (and have checked names for their stems), we can take $r^* = r'$ (ensuring (b)). (c) is clear by the definition of q^* and r^*.

Items (ii) and (iii) are obvious. □

The existence of the projection τ in Lemma 1.5 will be useful (in a quotient setting) in Section 3.3.2.

2 Preserving measurability by Cohen forcing

In [4], the construction which yields the tree property at the double successor of a singular strong limit κ with countable cofinality starts by assuming that κ is supercompact. The reason is that we can then invoke Laver's indestructibility result [12], and assume that adding any number of Cohen subsets of κ will preserve the measurability of κ. Such an assumption tends to simplify the subsequent constructions because one can avoid the work of lifting a weaker embedding using a surgery argument, or some other methods.

A natural question is whether a "Laver-like" indestructibility is available also for smaller large cardinals. In this Section, we use an idea of Cummings and Woodin (see [2]) to argue that with a well-designed preparation it is possible to have a limited indestructibility for a $H(\mu)$-hypermeasurable[3] cardinal κ, μ regular, $\mu > \kappa^+$, in the sense that we can successively extend $V \subseteq V^1 \subseteq V^*$ in such a way that V^1 preserves the initial degree of hypermeasurability of κ, and forcing over V^* with $\mathrm{Add}(\kappa,\mu)$ yields the measurability, in fact the hypermeasurability, of κ. Note that in V^*, κ actually stops being measurable in our forcing and $\mathrm{Add}(\kappa,\mu)$ recreates its measurability.[4]

Although much weaker than the indestructibility of a supercompact κ, it may still be a useful property: as will be apparent in next section, the construction in [4] can be redone starting (roughly) with a $H(\lambda)$-hypermeasurable cardinal, where λ is the least weakly compact above κ.

2.1 Stage 1: the model V^1

Let us fix $j : V \to M$ which witnesses the $H(\mu)$-hypermeasurability of κ: in particular, the critical point of j is κ, M is closed under κ-sequences in V and $H(\mu)$ is a subset of M. Let U be the normal measure derived from j, and let $i : V \to N$ be the ultrapower embedding generated by U.

Using the fast-function forcing of Woodin, we can assume that there is $f : \kappa \to \kappa$ in V such that $j(f)(\kappa) = \mu$. Let us denote $f(\alpha)$ by μ_α; let $C(f)$ denote the closed unbounded set of the closure points of f: if $\alpha \in C(f)$, then for all $\beta < \alpha$, $f(\beta) < \alpha$.

We plan to extend V by forcing to some V^1 with the same cofinalities, so that the following hold in V^1:

(A) There is $j^1 : V^1 \to M^1$ with critical point κ such that $H(\mu) \subseteq M^1$ and j^1 restricted to V is the original j.

[3] κ is $H(\mu)$-*hypermeasurable* (also $H(\mu)$-*strong*) if there is an elementary embedding $j : V \to M$ such that $j(\kappa) > \mu$, $H(\mu) \subseteq M$, and M is closed under κ-sequences in V.

[4] More details related to this construction are given in [10] (in particular, with more work, one can retain the measurability of κ in V^* if so desired).

(B) If U^1 the normal measure derived from j^1, and $i^1 : V^1 \to N^1$ is the ultrapower embedding for U^1, then in V^1 there is g which is $i^1(P)$-generic over N^1, where $P = \mathrm{Add}(\kappa, \mu)^{V^1}$. i^1 restricted to V is the original i.

The model V^1 is obtained as follows. Define P^1 in V as an Easton-supported iteration

$$\langle (P^1_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa, \alpha \text{ is measurable}, \alpha \in C(f)\rangle * \dot{Q}_\kappa, \tag{2.3}$$

where

- $\dot{Q}_\alpha$ is chosen generically[5] amongst all forcings R which satisfy the following: (a) there exists in $V[P^1_\alpha]$ a normal measure U_α on α such that the derived ultrapower embedding i_α satisfies

  $$i_\alpha : V[P^1_\alpha] \to N_\alpha[i(P^1_\alpha)] \tag{2.4}$$

  for some N_α, (b) R is the forcing

  $$i_\alpha(\mathrm{Add}(\alpha, \mu_\alpha)^{V[P^1_\alpha]}). \tag{2.5}$$

  If the above is not possible, let $\dot{Q}_\alpha$ be $\{1\}$.

- $\dot{Q}_\kappa$ is chosen generically amongst all the forcings R which satisfy the following:[6] (a) i lifts in $V[P^1_\kappa]$ to

  $$i_\kappa : V[P^1_\kappa] \to N[i_\kappa(P^1_\kappa)], \tag{2.6}$$

  (b) R is the forcing

  $$i_\kappa(\mathrm{Add}(\kappa, \mu)^{V[P^1_\kappa]}). \tag{2.7}$$

  If i does not lift to i_κ in $V[P^1_\kappa]$, set $\dot{Q}_\kappa = \{1\}$.

[5] If the forcing is chosen "generically", the related construction is often called the "lottery sum of the relevant forcing notions"; see [9] for more information.

[6] Note that at stages $\alpha < \kappa$, $\dot{Q}_\alpha$ depends on the chosen measure and the lifting of the measure ultrapower, while at stage κ we have a fixed measure (corresponding to i) and choose only the lifting i_κ (as determined by the generic filter on $i_\kappa(P^1_\kappa)$ which is formally not a part of our iteration).

Lemma 2.1 *P^1 preserves cofinalities.*

PROOF. The following suffices: Suppose P^1_α, $\alpha \leq \kappa$, preserves regularity of cardinals and the only violation of GCH below α is that for every $\beta < \alpha$, where the forcing is non-trivial, we have $2^{\beta^+} = \mu_\beta$ (this is without loss of generality). We show that this will hold for $P^1_{\alpha+1}$.

Suppose $\dot{Q}_\alpha$ is non-trivial and let G_α be P^1_α-generic. Then in $V[G_\alpha]$, $i_\alpha : V[G_\alpha] \to N_\alpha[i_\alpha(G_\alpha)]$ is a measure ultrapower embedding. It suffices to show that $R = (\dot{Q}_\alpha)^{G_\alpha}$ is α^+-closed and α^{++}-cc in $V[G_\alpha]$. Closure is obvious because $N_\alpha[i_\alpha(G_\alpha)]$ is closed under α-sequences in $V[G_\alpha]$ and R is α^+-closed in $N_\alpha[i_\alpha(G_\alpha)]$. Regarding the chain condition, notice that every element of R can be identified with the equivalence class of some function $f : \alpha \to \mathrm{Add}(\alpha, \mu_\alpha)$. For $f, g : \alpha \to \mathrm{Add}(\alpha, \mu_\alpha)$, set $f < g$ if for all $i < \alpha$, $f(i) < g(i)$; it suffices to check that the ordering $<$ on these f's is α^{++}-cc. Let A be a maximal antichain in this ordering; take an elementary substructure $\bar{M}$ in some large enough $H(\theta)$ of $V[G_\alpha]$ which contains all relevant data, has size α^+ and is closed under α-sequences. Then it is not hard to check that $A \cap \bar{M}$ is maximal in the ordering (and therefore $A \subseteq \bar{M}$), and therefore has size at most α^+. □

Lemma 2.2 *P^1 is non-trivial and forces (A) and (B) on page 8.*

PROOF. Let $G_\kappa * g$ be $P^1_\kappa * \dot{Q}_\kappa$-generic over V; let R denote for the purposes of this item the forcing $(\dot{Q}_\kappa)^{V[G_\kappa]}$. It suffices to show that the stage κ is non-trivial, i.e. R is not $\{1\}$ (as the argument carries over to all $\alpha < \kappa$ which are non-trivial). This follows immediately by the standard lifting methods, as i lifts to $V[G_\kappa]$, witnessing the measurability κ in $V[P_\kappa]$.[7]

(A). We show that for the right choice of $j(G_\kappa * g)$, j lifts to j^1 as required. In $M[G_\kappa]$, work below a condition which ensures that at stage

[7]This uses the fact that $i(P^1_\kappa)_\kappa = P_\kappa$ forces the tail iteration of $i(P^1_\kappa)$ in the interval $[\kappa, i(\kappa))$ to be κ^+-closed in $N[G_\kappa]$ (and hence also in $V[G_\kappa]$); moreover the number of antichains in the tail iteration is just κ^+ in $V[G_\kappa]$.

κ, $j(P^1_\kappa)$ chooses the forcing R.[8] Then the lifting is carried out using the standard methods.

(B). In $V[G_\kappa]$, R is a κ^+-closed forcing (and is the image of $\mathrm{Add}(\kappa,\mu)$ under the embedding i lifted to $V[G_\kappa]$), and by a Woodin lemma (see below), g is actually R-generic over $N[i^1(G*g)]$. □

Lemma 2.3 (Woodin) *Let κ be a measurable cardinal, U some measure on κ, and $i : V \to N$ the ultrapower embedding by U. Let P be a κ-closed partial ordering and let $Q = i(P)$. Let g be Q-generic over V and transfer it along i to get the lifting $i^1 : V[g] \to N[i^1(g)]$. Then g is in fact Q-generic over $N[i^1(g)]$.*

For proof, see [2], Fact 2, page 7.

2.2 Stage 2: the model V^*

Assume that κ, μ, f, and V^1, N^1, i^1, j^1 are as in Section 2.1.

We will prove the following theorem:

Theorem 2.4 *There is a forcing iteration P_κ defined in V^1 such that*

$$V^1[P_\kappa][\mathrm{Add}(\kappa,\mu)] \models \ \kappa \textit{ is measurable},$$

where $\mathrm{Add}(\kappa,\mu)$ is defined in $V[P_\kappa]$. Hence the model $V^1[P_\kappa] = V^*$ is as desired.

Proof. Define P_κ to be the following Easton-supported iteration:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa \text{ is measurable}, \alpha \in C(f)\rangle, \tag{2.8}$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathrm{Add}(\alpha,\mu_\alpha)$, and μ_α equals $f(\alpha)$.

[8] Note that in $M[G_\kappa]$, κ is measurable and the measure corresponding to $i_\kappa : V[G_\kappa] \to N[i_\kappa(G_\kappa)]$ is also an element of $M[G_\kappa]$ since $H(\mu)$ of $V[G_\kappa]$ is included in $M[G_\kappa]$; also note that in $M[G_\kappa]$, $\mu_\kappa = j(f)(\kappa) = \mu$, and therefore R is an element of $M[G_\kappa]$ and can legitimately be considered as $\dot{Q}_\kappa$ for the iteration $j(P^1_\kappa)$.

The proof uses the usual surgery argument (see [3]) with the following lemma which allows us to use the generic filter g added in V^1 (for the i^1-image of $\mathrm{Add}(\kappa,\mu)^{V^1}$) in the model $V^1[P_\kappa]$.

We will use the following result (for the proof, see Fact 2 in [2]).[9]

Fact 2.5 *Let S be a κ-cc forcing notion of cardinality κ, $\kappa^{<\kappa} = \kappa$. Then for any μ, the term forcing $Q_\mu = \mathrm{Add}(\kappa,\mu)^{V[S]}/S$ is isomorphic to $\mathrm{Add}(\kappa,\mu)$.*

Let $G_\kappa * h$ be $P_\kappa * \mathrm{Add}(\kappa,\mu)^{V^1[P_\kappa]}$-generic.

By Fact 2.5 applied to $S = P_\kappa$, $\mathrm{Add}(\kappa,\mu)^{V^1[P_\kappa]}/P_\kappa$ is isomorphic in V^1 to $\mathrm{Add}(\kappa,\mu)$ of V^1. It follows that after lifting to

$$i^1 : V^1[G_\kappa] \to N^1[i(G_\kappa)] \tag{2.9}$$

in $V^1[G_\kappa * h]$, the generic filter g constructed at stage 1 is (or more precisely, yields a generic which is)

$$i^1(\mathrm{Add}(\kappa,\mu))\text{-generic over } N^1[i(G_\kappa)]. \tag{2.10}$$

This suffices to proceed with the usual surgery argument (for details see [2]) and lift the embedding j^1 to $V^1[G_\kappa * h]$, showing that κ is measurable in $V^1[G_\kappa * h]$. □

Remark 2.6 As we will show in the following Section, the above construction can be generalised for other iterations P_κ over the model V^1: we will iterate a variant of the Mitchell forcing at α which decomposes into the Cohen forcing at α, followed by an α^+-distributive forcing.

3 The tree property with gap 3

In this section we will prove that it is consistent to have a model where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and the tree property holds at $\aleph_{\omega+2}$. It is

[9] Recall that Q_μ – mentioned in Fact 2.5 below – is the term forcing defined as follows: the elements of Q_μ are names τ such that τ is an S-name and it is forced by 1_S to be in $\mathrm{Add}(\kappa,\mu)$ of $V[S]$. The ordering is $\tau \le \sigma \leftrightarrow 1_S \Vdash \tau \le \sigma$.

relatively straightforward to generalise this construction to get a finite gap: $2^{\aleph_\omega} = \aleph_{\omega+n}$, $3 \le n < \omega$ (see Section 4).

Assume GCH. Let us fix $\kappa < \lambda$ where κ is $H(\lambda^+)$-hypermeasurable, and λ is the least weakly compact cardinal above κ. Let $j : V \to M$ be an (κ, λ^+)-extender embedding witnessing the $H(\lambda^+)$-hypermeasurability of κ, i.e.

$$M = \{j(f)(\alpha) \,|\, f : \kappa \to V \,\&\, \alpha < \lambda^+\}, \tag{3.11}$$

and

$$\lambda^+ < j(\kappa) < \lambda^{++}. \tag{3.12}$$

3.1 Preparation

Let V^1 be the model constructed in Section 2.1 with $\mu = \lambda^+$ (with j as in (3.11)). Note that instead of using the generically constructed function f in the definition of the iteration P^1 in (2.3), we can postulate directly that the α's are measurable cardinals below κ, and μ_α is the successor of the least weakly compact cardinal above α; let us denote this weakly compact cardinal as λ_α (thus $\lambda_\alpha^+ = \mu_\alpha$).

Over V^1, let us define a reverse Easton iteration P_κ as follows:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \,|\, \alpha < \kappa \text{ is measurable} \rangle, \tag{3.13}$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathbb{M}(\alpha, \lambda_\alpha, \mu_\alpha)$.

As in Section 2.2, one can show that in $V^1[P_\kappa][\mathbb{M}(\kappa, \lambda, \lambda^+)]$, κ is still measurable: In order to lift the embedding to $\mathbb{M}(\kappa, \lambda, \lambda^+)$, it suffices first to lift to $\mathrm{Add}(\kappa, \lambda^+)$ (and we have prepared for this step), and then lift the distributive part $\dot{Q}$ from Lemma 1.2(ii).

Remark 3.1 The present method can be used for $n = 2$ as well, i.e. to get $2^{\aleph_\omega} = \aleph_{\omega+2}$. In this case, we would start with a (κ, λ)-extender such that λ is the least weakly compact cardinal in the extender ultrapower.

3.2 Definition of the forcing

As we have argued in the previous section, κ is measurable in the model $V^1[P_\kappa][\mathbb{M}(\kappa, \lambda, \lambda^+)]$. In order to analyse this model, let us introduce notation for the generic filters: let $G_\kappa * H$ be a generic filter over V^1 for $P_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^+)$. By standard arguments, we can first lift to

$$j : V^1[G_\kappa] \to M^1[j(G_\kappa)], \tag{3.14}$$

and then to

$$j : V^1[G_\kappa * H] \to M^1[j(G_\kappa * H)], \tag{3.15}$$

where we use j to denote the lifted embeddings to keep the notation simple.

Note that in $M^1[j(G_\kappa)]$, $\lambda = \kappa^{++}$ and $\lambda^+ = \kappa^{+++}$ because $j(P_\kappa)$ forces with $\mathbb{M}(\kappa, \lambda, \lambda^+)$ at stage κ.

The embedding in (3.15) witnesses the measurability of κ in $V^1[G_\kappa * H]$. In fact, more is true: the lifted extender embedding j in (3.15) becomes a measure ultrapower embedding j_U in $V^1[G_\kappa * H]$, generated by the normal measure U derived from j in (3.15). Thus $M^1[j(G_\kappa * H)]$ is the measure ultrapower of $V^1[G_\kappa * H]$ via U.

In particular, we can define the Prikry forcing with collapses $\mathrm{PrkCol}(U, G^g)$ using this U and a suitable guiding generic G^g which we construct in Lemma 3.2 (the small g stands for "guiding").[10]

Let Coll denote the forcing $\mathrm{Coll}((\kappa^{+4}), < j(\kappa))^{M^1[j(G_\kappa * H)]}$.

Lemma 3.2 *In $V^1[G_\kappa * H]$, there exists an $M^1[j(G_\kappa * H)]$-generic filter for* Coll.

Proof. Consider the extender representation $j^1 : V^1 \to M^1$ ensured by the arguments in Section 2.1, where

$$M^1 = \{j^1(f)(\alpha) \mid f \in V^1 \ \& \ f : \kappa \to V^1 \ \& \ \alpha < \lambda^+\}. \tag{3.16}$$

[10] See Section 1.1.2 for more details about this forcing.

Now notice that every maximal antichain of Coll in $M^1[j(G_\kappa * H)]$ has a name of the form $j^1(f)(\alpha)$ for some $f : \kappa \to H(\kappa)^{V^1}$ and $\alpha < \lambda^+$, with the range of f being composed of P_κ-names. There are only κ^+-many such f's, and since Coll is κ^{+4}-closed in $M[j(G_\kappa * H)]$, we can built a Coll-generic filter G^g in $V^1[G_\kappa * H]$ over $M^1[j(G_\kappa * H)]$ by the standard method of grouping the antichains into κ^+ many blocks each of size at most $\lambda^+ = \kappa^{+3}$. □

Let us define in V^1:

$$\mathbb{P} = P_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g), \tag{3.17}$$

where $\dot{G}^g$ is a name for a guiding generic which we know exists by Lemma 3.2.

We plan to show that $V^1[\mathbb{P}]$ is the desired model.

3.3 Verifying the tree property with gap 3

Now we show that the tree property holds with gap 3. See Section 4 for a generalisation for any $3 \leq n < \omega$.

Theorem 3.3 *The forcing $\mathbb{P}$ in (3.17) forces $\kappa = \aleph_\omega$, $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and the tree property holds at $\lambda = \aleph_{\omega+2}$.*

By Lemma 1.1 and standard facts about the Prikry forcing with collapses, it suffices to check that we have the tree property at $\aleph_{\omega+2}$.

The argument has two parts. In Part 1 (Section 3.3.1), we show that if there were in $V^1[P_\kappa]$ an $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$-name $\dot{T}$ for an $\aleph_{\omega+2}$-Aronszajn tree, we could find a suitable β, $\lambda < \beta < \lambda^+$, and define a "truncation" $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ of the original forcing which forces that that there is an $\aleph_{\omega+2}$-Aronszajn tree (witnessed by $\dot{T}$). Then in Part 2 (Section 3.3.2), we show that in fact this cannot be the case, i.e. we show that $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ forces the tree property at $\aleph_{\omega+2}$. This will yield the final contradiction, finishing the proof of Theorem 3.3.

3.3.1 Part 1

Lemma 3.4 *$\mathbb{P}$ has the λ-cc.*

Proof. Follows by Lemma 1.3. □

Assume for contradiction that there is some condition in $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ which forces over $V^1[G_\kappa]$ that there is an $\aleph_{\omega+2}$-Aronszajn tree in $V^1[\mathbb{P}]$ with a name $\dot{T}$. To simplify the presentation, we will assume that the weakest condition forces that $\dot{T}$ is an Aronszajn tree (if not, work below a condition which forces it). Let us view $\dot{T}$ as a nice name for a subset of λ: i.e. $\dot{T}$ is of the form $\bigcup\{\{\check{\alpha}\} \times A_\alpha \mid \alpha < \lambda\}$, where each A_α is an antichain in $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$; by Lemma 3.4, we can assume that each A_α has size less than λ.

Lemma 3.5 *There is $\lambda \leq \bar{\beta} < \lambda^+$ such that for every $\alpha < \lambda$ and every $((p, q), r)$ in A_α, p is in $\mathrm{Add}(\kappa, \bar{\beta})$, q is a function with domain included in $\bar{\beta}$, and r refers only to conditions in $\mathrm{Add}(\kappa, \bar{\beta})$.*

Proof. This is a simple counting argument using the fact that each A_α has size less than λ, and therefore $\bigcup_{\alpha<\lambda} A_\alpha$ has size at most λ. The claim regarding r follows from Lemma 1.4. □

We wish to find $\bar{\beta} < \beta < \lambda^+$ which allows us to define a suitable truncation of $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ to β which we will analyse in Part 2.

Work in $V^1[G_\kappa]$. Using standard arguments, construct an elementary submodel $\mathscr{A}$ of $H(\theta)$ for some large enough regular θ so that $\mathscr{A}$ satisfies the following conditions:

(i) $|\mathscr{A}| = \lambda$, and $\mathscr{A}$ is closed under $< \lambda$-sequences,
(ii) $\bar{\beta} + 1 \subseteq \mathscr{A}$,
(iii) $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, $\dot{U}$, $\dot{G}^g$, $\dot{T}$ are elements of $\mathscr{A}$,
(iv) $\mathscr{A} \cap \lambda^+ = \beta$ for some β of cofinality κ^+, $\bar{\beta} < \beta$,
(v) There is an $\mathbb{M}(\kappa, \lambda, \beta)$-name $\dot{U}_\beta$ which is forced by $\mathbb{M}(\kappa, \lambda, \lambda^+)$ to be a normal measure and a restriction of the measure $\dot{U}$ to $V^1[P_\kappa][\mathbb{M}(\kappa, \lambda, \beta)]$.

The last item (v) follows as in [4].

Let $c : \mathscr{A} \to \bar{\mathscr{A}}$ be the transitive collapse. The following hold (the proofs are routine):

(i) $c(\lambda^+) = \beta$,
(ii) $c(\mathbb{M}(\kappa, \lambda, \lambda^+)) = \mathbb{M}(\kappa, \lambda, \beta)$,
(iii) $c(\dot{U})$ is forced by $\mathbb{M}(\kappa, \lambda, \beta)$ to be equal to $\dot{U}_\beta$,
(iv) $c(\dot{G}^g)$, which we denote by $\dot{G}^g_\beta$, is forced by $\mathbb{M}(\kappa, \lambda, \beta)$ to be a guiding generic with respect to $\dot{U}_\beta$, and therefore $\mathbb{M}(\kappa, \lambda, \beta)$ forces that $\mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ is a Prikry forcing with collapses,
(v) $\mathbb{M}(\kappa, \lambda, \lambda^+)$ forces that $\mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ is a regular subforcing of $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$,
(vi) $c(\dot{T}) = \dot{T}$ is forced by $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ to be an $\aleph_{\omega+2}$-Aronszajn-tree.

By elementarity, $c(\dot{T}) = \dot{T}$ is forced in $\bar{\mathscr{A}}$ by the forcing $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ to be a λ-Aronszajn tree. This by itself would not be enough to conclude that $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ adds such a tree over the universe $V^1[G_\kappa]$. However, since the collapse $c(\dot{T})$ is equal to $\dot{T}$, and (v) holds, any $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$-generic filter $h * x$ yields an $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$-generic filter $h' * x'$ over $V^1[G_\kappa]$ (and therefore over $\bar{\mathscr{A}}$) such that $(\dot{T})^{h*x} = (\dot{T})^{h'*x'}$. It follows that $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ forces over $V^1[G_\kappa]$ that $\dot{T}$ is a λ-Aronszajn tree. In Part 2 we show this is not possible, and this will be the desired contradiction.

3.3.2 Part 2

Let $\mathbb{M}$ denote the forcing $\mathbb{M}(\kappa, \lambda, \beta)$, where β is as in Part 1. Let us work in $V^1[P_\kappa]$.

Let

$$k : \mathscr{M} \to \mathscr{N} \tag{3.18}$$

be a weakly compact embedding with critical point λ such that $\mathscr{M}$ and $\mathscr{N}$ are transitive models of size λ, $\mathscr{M} \in \mathscr{N}$, $k \in \mathscr{N}$, $\beta < k(\lambda)$, and $\mathscr{M}$

contains all relevant information (in particular, β, $\mathbb{M} * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$, $\dot{U}_\beta$ and $\dot{G}^g_\beta$ are elements of $\mathscr{M}$). Let $\mathbb{M}^*$ denote $k(\mathbb{M}(\kappa, \lambda, \beta))$, which is equal to $\mathbb{M}(\kappa, k(\lambda), k(\beta))$. Such a $k : \mathscr{M} \to \mathscr{N}$ exists by an argument in [3], Theorem 16.1.[11]

Let h^* be $\mathbb{M}^*$-generic over $V^1[P_\kappa]$; use h^* to define h which is $\mathbb{M}$-generic over $V^1[P_\kappa]$ and $k``h \subseteq h^*$. Now lift to

$$k : \mathscr{M}[h] \to \mathscr{N}[h^*]. \tag{3.19}$$

Let us abuse notation a little and write $U = (\dot{U}_\beta)^h$ and $G^g = (\dot{G}^g_\beta)^h$ instead of U_β and G^g_β (to simplify notation).

In $\mathscr{N}[h^*]$, consider $U^* = k(U)$, and $G^{g*} = k(G^g)$, and the forcing $\mathrm{PrkCol}(U^*, G^{g*})$. Note that by elementarity $U \subseteq U^*$ (since $k(X) = X$ for every $X \in U$), and all functions F whose equivalence class is in G^g appear in the forcing $\mathrm{PrkCol}(U^*, G^{g*})$ (since $k(F) = F$ for every $F : \kappa \to V_\kappa^{\mathscr{M}[h]}, F \in \mathscr{M}[h]$), and $k(\mathrm{PrkCol}(U, G^g)) = \mathrm{PrkCol}(U^*, G^{g*})$.[12]

It follows that k is a regular embedding:

$$k : \mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g) \to \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}), \tag{3.20}$$

as by the λ-cc of $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, if A is a maximal antichain in $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, then $k(A) = k``A$ is a maximal antichain in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$.

Let x^* be $\mathrm{PrkCol}(U^*, G^{g*})$-generic over $V^1[P_\kappa][h^*]$; the pull-back of x^* via k^{-1} is a generic filter x for $\mathrm{PrkCol}(U, G^g)$ such that $k``x \subseteq x^*$. Let us lift k further to

$$k : \mathscr{M}[h][x] \to \mathscr{N}[h^*][x^*]. \tag{3.21}$$

By (3.20) and (3.21), we can define in $\mathscr{N}[h^*][x^*]$ a generic filter $h * x$ for $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g) \in \mathscr{N}$ using the inverse of k (by our assumptions

[11] To ensure $\beta < k(\lambda)$, define E in the proof of Theorem 16.1 so that it also codes a well-ordering of β of type λ: then $\mathscr{N} \models |\beta| = \lambda$ and therefore $k(\lambda) > \beta$ since by elementarity, $k(\lambda)$ is in $\mathscr{N}$ a limit cardinal greater than λ.

[12] However, note that the equivalence classes of a fixed F with respect to U and U^* may be different objects (after the transitive collapse).

in (3.18), k is an element of $\mathscr{N}$). By standard arguments for complete Boolean algebras it follows that there is a projection π,

$$\pi : \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \to \mathrm{RO}^+(\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)). \tag{3.22}$$

Notice that if $((p,q),r)$ is a condition in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$, then we can identify $\pi(p) = \pi((p,\emptyset), 1_{\mathrm{PrkCol}(\dot{U}^*,\dot{G}^{*g})})$ with $(k^{-1})"p$, i.e. with

$$p \restriction (\kappa \times \lambda) \cup \{((\gamma,\alpha),i) \mid \gamma < \kappa, \alpha \in [\lambda,\beta), i \in \{0,1\}, ((\gamma,k(\alpha)),i) \in p\}. \tag{3.23}$$

In the analysis of the quotient determined by π, it will be important to control the names for the conditions in $\mathrm{PrkCol}(U^*, G^{g*})$. Recall that by Lemma 1.4, we can adopt the following convention:

Convention. We now view $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$ as consisting of conditions $((p,q),r)$, where r depends only on Cohen information, and its stem is a checked name (such conditions are dense by Lemma 1.4). With this convention in mind, let us denote the quotient determined by π in (3.22) as $\mathbb{Q}_\pi$:

$$\mathbb{Q}_\pi = \{((p,q),r) \in \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \mid \pi(((p,q),r)) \in h * x\}, \tag{3.24}$$

where we identify $h*x$ with the generic filter for the associated complete Boolean algebra.

The following product analysis reformulates the analysis in Section 1.1.3 in a quotient setting.

Define:

$$\mathbb{C} = \{((p,\emptyset),r) \mid ((p,\emptyset),r) \in \mathbb{Q}_\pi\}, \tag{3.25}$$

The ordering is the one inherited from $\mathbb{Q}_\pi$.

Define:

$$\mathbb{T} = \{(\emptyset,q) \in \mathbb{M}^* \mid (\emptyset,q)\restriction\lambda \in h\}. \tag{3.26}$$

The ordering is the one inherited from $\mathbb{M}^*$.

Define a function τ from $\mathbb{C} \times \mathbb{T}$ to $\mathbb{Q}_\pi$ as follows: τ applied to the pair composed of $((p, \emptyset), r)$ and $(\emptyset, q)$ is equal to the condition $((p, q), r)$. Note that if $((p, \emptyset), r)$ is in $\mathbb{C}$ and $(\emptyset, q)$ is in $\mathbb{T}$, then $((p, q), r)$ is a condition in the quotient $\mathbb{Q}_\pi$ since $\pi((p, q), r)$ is the infimum of $\pi((p, \emptyset), r)$ and $((\emptyset, q) \restriction \lambda, 1_{\mathrm{PrkCol}(\dot{U}, \dot{G}^g)})$ in $\mathrm{RO}^+(\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g))$.

Lemma 3.6 is proved exactly as Lemma 1.5(i):

Lemma 3.6 *τ is a projection from $\mathbb{C} \times \mathbb{T}$ onto $\mathbb{Q}_\pi$.*

The following lemma is obvious:

Lemma 3.7 *$\mathbb{T}$ is κ^+-closed in $\mathscr{N}[h]$.*

Finally, we analyse the chain condition of $(\mathbb{C})^2$ in $\mathscr{N}[h][x]$.

Lemma 3.8 *$(\mathbb{C})^2$ is κ^+-cc in $\mathscr{N}[h][x]$.*

Proof. Assume for contradiction that A is an antichain in $(\mathbb{C})^2$ in $\mathscr{N}[h][x]$ of size κ^+. Denote the elements of A by $(1a_i, 2a_i)$ for $i < \kappa^+$. By thinning out the antichain if necessary, we can choose a condition $((p, q), r)$ in $h * x$ which forces that A is an antichain and also forces that stems of all conditions $1a_i$, $i < \kappa^+$, are the same and the stems of all conditions $2a_i$, $i < \kappa^+$, are the same (they may not equal each other, but they are compatible; denote them $1t$, and $2t$). Now choose $((p_i, q_i), r_i)$ in $h * x$ which decide the $1a_i$'s and $2a_i$'s; let us write $1a_i = ((1p_i^*, \emptyset), 1r_i^*)$ and $2a_i = ((2p_i^*, \emptyset), 2r_i^*)$, $i < \kappa^+$.

By further thinning and extending the stems if necessary, we may assume that the stems of $((p, q), r)$ and $((p_i, q_i), r_i)$, $i < \kappa^+$, are all the same; denote this stem s. Note that s extends both $1t$ and $2t$.

Now, we need to handle together $1a_i$ and $2a_i$, for all $i < \kappa^+$, to get mutual compatibility in Claim 3.11 below: Let $((1p_i^{**}, 1q_i^{**}), 1r_i^{**})$ be a lower bound of $((1p_i^*, \emptyset), 1r_i^*)$, $((p_i, q_i), r_i)$, and $((p, q), r)$ with stem s such that $\pi(1p_i^{**})$[13] is in the Cohen part of the generic $h * x$ (this can be done since such conditions are dense). Analogously, let $((2p_i^{**}, 2q_i^{**}), 2r_i^{**})$ be a lower bound of $((2p_i^*, \emptyset), 2r_i^*)$, $((p_i, q_i), r_i)$, and $((p, q), r)$ with stem s such that $\pi(2p_i^{**})$ is in the Cohen part of the generic $h * x$. Note that in particular $\pi(1p_i^{**})$ is compatible with $\pi(2p_i^{**})$.

Using a Delta-system argument, find $i < j$ such that $1p_i^{**}$ is compatible with $1p_j^{**}$ and $2p_i^{**}$ is compatible with $2p_j^{**}$. Let us define:

$$1(*) \text{ is the greatest lower bound (glb) of } ((1p_i^{**}, \emptyset), 1r_i^{**}) \text{ and } ((1p_j^{**}, \emptyset), 1r_j^{**}) \tag{3.27}$$

and

$$2(*) \text{ is the greatest lower bound (glb) of } ((2p_i^{**}, \emptyset), 2r_i^{**}) \text{ and } ((2p_j^{**}, \emptyset), 2r_j^{**}). \tag{3.28}$$

Note that both $1(*)$ and $2(*)$ have the same stem s.

Denote

$$p' = \pi(1p_i^{**}) \cup \pi(1p_j^{**}) \cup \pi(2p_i^{**}) \cup \pi(2p_j^{**}).$$

Note that p' is correctly defined by the construction of the $1p_i^{**}$'s and $2p_i^{**}$'s. Let $((\bar{p}, \bar{q}), \bar{r})$ denote the glb of the conditions $((p', \emptyset), \emptyset)$, $((p, q), r)$, $((p_i, q_i), r_i)$, $((p_j, q_j), r_j)$. Note that $((\bar{p}, \bar{q}), \bar{r})$ has stem s.

We need the following claims to finish the proof.

Claim 3.9 *Assume $((p, q), r)$ is a condition in $\mathbb{M}^*\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ and $((p^*, \emptyset), r^*)$ is a condition in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$ and the following conditions are satisfied:*

(i) Stems of r and r^* are checked names,
(ii) $p \leq \pi(p^*)$,
(iii) The length of the stems of r and r^* is the same and the ordinals on the stems coincide,
(iv) The collapsing information in the stem of r extends the collapsing information in the stem of r^*.

[13]See (3.23) for definition.

Then $((p,q),r)$ *does not force* $((p^*,\emptyset),r^*)$ *out of the quotient* $\mathbb{C}$.

Proof. It suffices to find a generic filter $h' * x'$ for $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$ such that $\pi((p,q),r)$ is in the $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$-generic filter obtained by π from $h' * x'$. Any filter $h' * x'$ containing a lower bound of $((p,q),r)$ and $((p^*,\emptyset),r^*)$ (such a lower bound exists by conditions (i)-(iv)) satisfies this. □

Recall that the Prikry forcing with collapses satisfies the Prikry condition: any statement in the forcing language is decidable by a direct extension (note that a direct extension does not lengthen the stem, but is allowed to extend the collapsing information).

Claim 3.10 *Let* $((p,q),r)$ *and* $((p^*,\emptyset),r^*)$ *are as in Claim 3.9. Then there is a direct extension* $((p',q'),r')$ *of* $((p,q),r)$ *which forces* $((p^*,\emptyset),r^*)$ *into* $\mathbb{C}$.

Proof. By the Prikry property, there is a direct extension of $((p,q),r)$ which decides the statement "$((p^*,\emptyset),r^*)$ is in $\mathbb{C}$". The negative decision contradicts Claim 3.9 (when applied to the direct extension); it follows that the decision must be positive. □

Returning to our proof, we get:

Claim 3.11 *There is a direct extension of* $((\bar{p},\bar{q}),\bar{r})$ *which forces* $1(*)$ *and* $2(*)$ *into* $\mathbb{C}$.

Proof. $((\bar{p},\bar{q}),\bar{r})$ and $1(*)$ satisfy the conditions in Claim 3.9, and therefore by Claim 3.10, there is a direct extension $a_1 \leq ((\bar{p},\bar{q}),\bar{r})$ which forces $1(*)$ into $\mathbb{C}$. a_1 and $2(*)$ satisfy the conditions in Claim 3.9, and therefore by Claim 3.10 there is a direct extension $a_2 \leq a_1$ as desired. □

This finishes the proof since a_2 forces that $(1(*),2(*))$ is in $\mathbb{C}$ a witness for compatibility of $(1a_i, 2a_i)$ and $(1a_j, 2a_j)$ in the antichain A. As a_2 is below $((p,q),r)$, it also forces that A is an antichain. Contradiction. □

The good chain condition of $(\mathbb{C})^2$ and the closure of $\mathbb{T}$ are enough to argue that $\mathbb{C} \times \mathbb{T}$, and therefore $\mathbb{Q}_\pi$, do not add branches to λ-trees, finishing the argument in the standard way.

For completeness we state the relevant facts below.

The following Fact is implicit in Mitchell's [14], and stated in Unger's [17].

Fact 3.12 *Suppose γ is a regular infinite cardinal and P adds a subset x of γ such that x is not in V but $x \cap \alpha$ is in V for all $\alpha < \gamma$. Then $P \times P$ is not γ-cc.*

Proof. It suffices to show that if G is P-generic then P is not γ-cc in $V[G]$. In $V[G]$ let $x = (\dot{x})^G$ be a subset of γ as in the hypothesis and choose a sequence $\langle p_i \,|\, i < \gamma \rangle$ of conditions in G and an increasing sequence of ordinals $\langle \alpha_i \,|\, i < \gamma \rangle$ less than γ such that p_i fixes $\dot{x} \cap \alpha_i$ (i.e. forces it to equal a specific element of V) but does not fix $\dot{x} \cap \alpha_{i+1}$. This is possible as $x \cap \alpha$ is fixed by some condition in G for each $\alpha < \gamma$ but x itself is fixed by no condition in G. Now choose q_{i+1} extending p_i to disagree with p_{i+1} about $\dot{x} \cap \alpha_{i+1}$. This is possible as p_i does not fix $\dot{x} \cap \alpha_{i+1}$. But then the q_{i+1}'s form an antichain as any condition extending q_{i+1} disagrees with p_{i+1} (and therefore with p_j for all $j > i$) about $\dot{x}$ and therefore cannot extend q_{j+1} for $j > i$, as q_{j+1} extends p_j. □

The following Fact is due to Silver:

Fact 3.13 *Suppose γ is an infinite cardinal and P is γ^+-closed. Suppose $\mu > \gamma$ is a regular cardinal and T is a tree of height μ which has all levels of size less than 2^γ. Then P does not add branches to T.*

Finally, the following Fact is due to Unger [16], generalising Fact 3.13:

Fact 3.14 *Suppose γ is an infinite cardinal, P is γ^+-cc and preserves γ, Q is γ^+-closed, and $2^\gamma > \gamma^+$. If T is a γ^{++}-tree in $V[P]$, then in $V[P][Q]$, T has no new branches.*

Suppose T is a λ-tree in the model $\mathscr{N}[h][x]$, where $\lambda = \kappa^{++}$. By Lemma 3.8 and Fact 3.12, $\mathbb{C}$ does not add new branches to T. As $\mathrm{PrkCol}(U, G^g) * \mathbb{C}$ has the κ^+-cc in $\mathscr{N}[h]$, we can apply Fact 3.14 over $\mathscr{N}[h]$ (with Q being $\mathbb{T}$), and conclude that $\mathbb{T}$ does not add branches to trees in $\mathscr{N}[h][\mathrm{PrkCol}(U, G^g) * \mathbb{C}]$, and therefore $\mathbb{C} \times \mathbb{T}$ does not add new branches to trees in $\mathscr{N}[h][x]$.

This finishes the proof of Theorem 3.3.

4 The tree property with a finite gap

We would like to generalise the result of the previous section to a finite gap m, i.e. obtain the tree property at $\aleph_{\omega+2}$ and have $2^{\aleph_\omega} = \aleph_{\omega+m}$ for any $2 < m < \omega$. To this end, we need to do some straightforward modifications to definitions and lemmas we used to obtain gap 3. To simplify indexing of the forcing notions, we will use the index n, where $m = n + 2$ (thus gap 3 is obtained with $n = 1$).

As in the previous section, let κ be the large cardinal which gets collapsed to $\aleph_\omega$, and λ the least weakly compact cardinal above κ.

We now list the modifications we need to do:

- In Section 2.1, we choose $\mu = \lambda^{+n}$ so that the preparation ensures that κ stays measurable after adding μ-many Cohen subsets of κ. Let us denote the resulting model as V^1.
- The definition of P_κ in (3.13) is to be modified as follows:

  (4.29) $$P^n_\kappa = \langle (P^n_\alpha, \dot{Q}^n_\alpha) \mid \alpha < \kappa \text{ is measurable} \rangle,$$

  where $\dot{Q}^n_\alpha$ denotes the forcing $\mathbb{M}(\alpha, \lambda_\alpha, \lambda_\alpha^{+n})$.
- Let $G_\kappa * H$ be a generic filter for $P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n})$, and let $j : V^1[G_\kappa * H] \to M^1(j(G_\kappa * H))$ be the lifting of j as in (3.15).
- Let Coll^n denote the forcing $\mathrm{Coll}((\kappa^{+3+n}, < j(\kappa)))^{M^1[j(G_\kappa * H)]}$. As in Lemma 3.2, we can fix a guiding generic G^g for Coll^n over $M^1[j(G_\kappa * H)]$.

- The definition of the forcing $\mathbb{P}$ in (3.17) is modified as follows:

  (4.30) $$\mathbb{P}^n = P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n}) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g),$$

  where $\dot{U}$ is a name for a normal measure and $\dot{G}^g$ is a name for a guiding generic (defined with respect to $\dot{U}$).

Now we get the following generalisation of Theorem 3.3:

Theorem 4.1 *Let $1 \leq n < \omega$ be fixed. The forcing $\mathbb{P}^n$ in (4.30) forces $\kappa = \aleph_\omega$, $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2+n}$, and the tree property holds at $\lambda = \aleph_{\omega+2}$.*

Proof. The basic strategy of the proof is to reduce the general case to gap 3.

Let G_κ be a P^n_κ-generic filter over V^1. Let us denote $\mathbb{M} = \mathbb{M}(\kappa, \lambda, \lambda^{+n})$. Let us assume for simplicity that the weakest condition of the forcing $\mathbb{M}*\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ forces in $V^1[G_\kappa]$ that $\dot{T}$ is a λ-Aronszajn tree (otherwise work below a condition which forces it).

Let $\mathscr{A}$ be an elementary substructure of large enough $H(\theta)^{V^1[G_\kappa]}$ which has size λ^+, is closed under λ-sequences, and contains the name $\dot{T}$ and the forcing $\mathbb{M}*\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$. Let $c : \mathscr{A} \to \bar{\mathscr{A}}$ be the transitive collapse. Then the following hold:

(i) $c(\lambda^{+n})$ is an ordinal between λ^+ and λ^{++}, let us denote this ordinal as β.
(ii) $c(\mathbb{M}(\kappa, \lambda, \lambda^{+n})) = \mathbb{M}(\kappa, \lambda, \beta)$.
(iii) $c(\dot{U})$ is forced by $\mathbb{M}(\kappa, \lambda, \lambda^{+n})$ to be a normal ultrafilter on κ in the generic extension of $V^1[G_\kappa]$ by $\mathbb{M}(\kappa, \lambda, \beta)$.
(iv) $c(\dot{G}^g)$ is forced by $\mathbb{M}(\kappa, \lambda, \beta)$ to be a guiding generic with respect to $c(\dot{U})$, and therefore $\mathbb{M}(\kappa, \lambda, \beta)$ forces that $\mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ is a Prikry forcing with collapses.
(v) $c(\dot{T})$ is forced by $\mathbb{M}(\kappa, \lambda, \beta)*\mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ to be a λ-Aronszajn tree.

In contrast to the analogous construction in Section 3.3.1, we cannot claim now that $c(\dot{T})$ is equal to $\dot{T}$. However, since this time the model

$\bar{\mathscr{A}}$ has size λ^+ and is closed under λ sequences, the forcing $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ adds a λ-Aronszajn tree not only over $\bar{\mathscr{A}}$ (which follows by elementarity), but also over $V^1[G_\kappa]$. The reason is that by λ-closure of $\bar{\mathscr{A}}$, a name for a cofinal branch in $c(\dot{T})$ would appear already in $\bar{\mathscr{A}}$.

Let f be any bijection between β and λ^+ which is the identity on λ. This bijection extends in $V^1[G_\kappa]$ into an isomorphism between $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ and $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}_{\lambda^+}, \dot{G}^g_{\lambda^+})$, where $\dot{U}_{\lambda^+}$ and $\dot{G}^g_{\lambda^+}$ are names obtained naturally from f.

This is a contradiction since we can argue as in Theorem 3.3 that the forcing $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}_{\lambda^+}, \dot{G}^g_{\lambda^+})$ does not add a λ-Aronszajn tree over $V^1[G_\kappa]$. $\square$

5 Open questions

The following questions are not solved by the methods of this paper:

Q1. Can we obtain an infinite gap at $2^{\aleph_\omega}$? More precisely, given an $\omega \leq \alpha < \omega_1$, is there a model where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+\alpha+1}$, and the tree property holds at $\aleph_{\omega+2}$?

It seems that an entirely different method is required for this configuration (perhaps based on the methods of Magidor [13] and Shelah [15]).

Q2. Can we obtain a similar result for $\aleph_{\omega_1}$? Or in general, for any uncountable cofinality?

The last question is more general. Let $P * \dot{Q}$ be a forcing notion and assume P is κ^+-cc for a cardinal κ. Then the following are equivalent:

(i) $P * \dot{Q}$ is κ^+-cc.
(ii) $P \Vdash$ "$\dot{Q}$ is κ^+-cc".

Consider Lemma 3.8 where we argued that $(\mathbb{C})^2$ is κ^+-cc in $\mathscr{N}[h][x]$, which is useful by Fact 3.12 for showing that certain trees do not get new branches in a generic extension by $\mathbb{C}$. It would be helpful in various contexts to have a generalisation of the equivalence (i) and (ii) for the "square-cc condition" for some – rich enough – class of forcing notions.

Q3. Let $P * \dot{Q}$ be a forcing notion. Assume that

($*$) $P * \dot{Q}$ is κ^+-Knaster (and therefore P is κ^+-Knaster as well).

Is there a useful characterisation of the forcings $P * \dot{Q}$ for which ($*$) already implies $P \Vdash$ "$(\dot{Q})^2$ is κ^+-cc"?

Note that this cannot hold for all $P * \dot{Q}$ by the following example: Work in a model where MA (Martin's Axiom) holds and assume P is $\mathrm{Add}(\omega, 1)$ and $\dot{Q}$ is a name for the Souslin tree constructed from the generic filter for P (see Jech [11] for details). Then $P * \dot{Q}$ is $\aleph_1$-Knaster by MA, and yet P forces that $(\dot{Q})^2$ is not $\aleph_1$-cc.

References

[1] Uri Abraham. Aronszajn trees on $\aleph_2$ and $\aleph_3$. *Annals of Pure and Applied Logic*, 24(3):213–230, 1983.

[2] James Cummings. A model in which GCH holds at successors but fails at limits. *Transactions of the American Mahematical Society*, 329(1):1–39, 1992.

[3] James Cummings. Iterated forcing and elementary embeddings. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*, volume 2. Springer, 2010.

[4] James Cummings and Matthew Foreman. The tree property. *Advances in Mathematics*, 133(1):1–32, 1998.

[5] James Cummings, Sy-David Friedman, Menachem Magidor, Assaf Rinot, and Dima Sinapova. The eightfold way. Submitted.

[6] Sy-David Friedman and Ajdin Halilović. The tree property at $\aleph_{\omega+2}$. *The Journal of Symbolic Logic*, 76(2):477–490, 2011.

[7] Sy-David Friedman, Radek Honzik, and Šárka Stejskalová. The tree property at the double sucessor of a singular cardinal with a larger gap. Submitted, 2016.

[8] Moti Gitik. The negation of singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$. *Annals of Pure and Applied Logic*, 43:209–234, 1989.

[9] Joel David Hamkins. The lottery preparation. *Annals of Pure and Applied Logic*, 101:103–146, 2000.

[10] Radek Honzik. Preserving measurability with Cohen iterations. To appear in *Acta Univ. Carolinae*.

[11] Tomáš Jech. *Set Theory*. Springer, 2003.

[12] Richard Laver. Making the supercompactness of κ indestructible under κ directed closed forcing. *Israel Journal of Mathematics*, 29(4):385–388, 1978.

[13] Menachem Magidor. On the singular cardinals problem I. *Israel Journal of Mathematics*, 28:1–31, 1977.

[14] William J. Mitchell. Aronszajn trees and the independence of the transfer property. *Annals of Mathematical Logic*, 5(1):21–46, 1972/1973.

[15] Saharon Shelah. The Singular Cardinals Problems: Independence results. In A. R. D. Mathias, editor, *Surveys in Set Theory*, pages 116–134. Cambridge University Press, 1983. London Math. Soc. Lecture Note Ser. 87.

[16] Spencer Unger. Fragility and indestructibility of the tree property. *Archive for Mathematical Logic*, 51(5–6):635–645, 2012.

[17] Spencer Unger. Aronszajn trees and the successors of a singular cardinal. *Archive for Mathematical Logic*, 52(5–6):483–496, 2013.