Work on this volume was supported by the grant No. 401/06/0387 of the Grant Agency of the Czech Republic.

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FILOZOFICKÁ FAKULTA
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V PRAZE

Vydala Univerzita Karlova v Praze,
Filozofická fakulta,
nám. J. Palacha 2, 116 38 Praha 1,
v roce 2008 v nákladu 200 výtisků.

Tisk Reprostředisko UK MFF,
Sokolovská 83, 186 75 Praha 8-Karlín.
Objednávky vyřizuje edice@ff.cuni.cz.

Typeset by Marta Bílková

ISBN 978-80-7308-261-1

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Call for Papers

Next volume of *Miscellanea Logica* 'What is logic?' is edited by Jaroslav Peregrin. Submissions should be send to: `peregrin@ff.cuni.cz`.

Jaroslav Peregrin

Preface

Logic naturally concentrates on modeling logical consequence, often making its structural properties a central point of study. But it is not logical consequence alone that is an isolated phenomenon to study, it is an underlying inferential structure which becomes more important, a logic itself often identified with an underlying consequence relation or an inference structure behind, meaning or even a vocabulary derived from the inference structure rather then vice versa. This volume, the seventh of the series Miscellanea Logica titled 'Consequence, Inference, Structure', brings together papers dealing with such phenomena from different perspectives.

First two papers concern nonclassical logics called substructural. Consequence relations underlying such logics lack some of the properties which, however problematic, are classically considered constitutional for the very concept of logical consequence (for example monotonicity or the contraction law). Substructural logics vary the structure of consequence relation, in particular the way of combining assumptions, classically modeled simply as a set. This makes substructural logics sensitive to such phenomena as relevance or resource interpretation of assumptions.

Although it is not their original motivation, substructural logics can naturally be used (and have been used) to formalize mathematics. First contribution of the present volume explores models for substructural arithmetics based on frame semantics of underlying logics.

Frame semantics is not only a nice and transparent tool of model theory of a particular class of logics, it can have many unexpected uses. Second paper develops semantics of sub-intuitionistic logics (some of it substructural in nature) to approach an important proof complexity issue – deriving lower bounds of the length of proofs in nonclassical logics. Such semantics is however interesting itself – it reflects how assumptions on the epistemic and cognitive abilities of the creative subject influence the underlying logic.

Third paper concerns a rather technical issue of structural proof theory of modal logic – a strong version of interpolation theorem, also related to a reformulation of modal logic in a different language with a coalgebraic cover modality taken as primitive. The structure of modal proofs is considered using recently developed deep inference sequent calculi, naturally connecting concepts of normal forms, distributive laws, and uniform interpolants.

Fourth contribution develops a theory of various entailments and consequence relations in erotetic logic, a logic of questions. In present approach questions are modeled as sets of direct answers, i.e. sets of formulas. The text offers a survey of consequence relations and their basic properties, varying from more traditional entailments to pure erotetic ones, e.g. those of erotetic implication combining questions and declarative formulas, or evocation between two questions.

The last text contained in the volume is a quick quide to independence and consistency results central to modern set theory. Axiomatic set theory can be naturally seen as 'open-ended', consistency results giving space to improve our understanding of sets. The text presents an introduction to methods of proving consistency results and as such it is at the same time a very nice introduction to modern set theory in general.

The aim of the volume is not to present a representative choice of papers covering the topic, but rather to bring together different pieces of work, varying from research papers presenting new results and ideas to larger surveys and introductory texts. I hope you enjoy the choice.

Prague, November 2008

Marta Bílková

■ One

Models for Substructural Arithmetics

Greg Restall

This paper explores models for arithmetics in substructural logics. In the existing literature on substructural arithmetic, frame semantics for substructural logics are absent. We will start to fill in the picture in this paper by examining frame semantics for the substructural logics C (linear logic plus distribution), R (relevant logic) and CK (C plus weakening). The eventual goal is to find negation complete models for arithmetic in R.

1 Substructural arithmetics

Consider the Peano axioms for arithmetic.

Identity	$0 = 0$
	$\forall x \forall y (x = y \to y = x)$
	$\forall x \forall y \forall z \big(y = z \to (x = y \to x = z)\big)$
Successor	$\forall x \forall y (x' = y' \to x = y)$
	$\forall x \forall y (x = y \to x' = y')$
	$\forall x (0 \neq x')$
Addition	$\forall x (x + 0 = x)$
	$\forall x \forall y \big(x + y' = (x + y)'\big)$
Multiplication	$\forall x (x0 = 0)$
	$\forall x \forall y (xy' = xy + x)$
Induction	$A(0), \forall x \big(A(x) \to A(x')\big) \vdash \forall x A(x)$

This paper is dedicated to Professor Robert K. Meyer.

Under the standard interpretation of the language (in which quantifiers range over natural numbers) these axioms ring true. In fact, they ring true independently of the theory of the conditional '$\rightarrow$' is read. If it is a merely material conditional we have traditional Peano arithmetic. If it is an intuitionistic conditional, we have intuitionistic Peano arithmetic. If it is the conditional of the relevant logic R, then we have relevant arithmetic, studied by Meyer [1976, 1979, 1998]. If it is the conditional of a contraction free logic, then we have contraction free arithmetics Restall [1992, January 1994]. Each of these different arithmetical theories have their virtues. For any predicate logic X possessing the vocabulary $\forall, \rightarrow, \sim$, we will call the theory given by adding these axioms and rules to those of X, "$X^\sharp$."

Gödel has shown us that $K^\sharp$ is not complete (where K is classical predicate logic). As a result, neither is $X^\sharp$ for any logic X weaker than K. One way to complete $K^\sharp$ is to add the infinitary ω rule.

$$A(0),\ A(0'),\ A(0''),\ \ldots \vdash \forall x A(x)$$

We will call the arithmetic given by replacing the induction rule from $X^\sharp$ by the ω rule, "$X^{\sharp\sharp}$."[1] $K^{\sharp\sharp}$ is complete. Here is why: One can prove by induction on the complexity of A that A is provable or $\sim A$ is provable. The crucial steps are the base case—that $s = t$ is provable iff the terms s and t denote the same number, and $s \neq t$ is provable iff the terms denote a different number. The cases for conjunction, disjunction and negation are trivial. The ω rule means that we can show that if $A(n)$ is provable for every n, then $\forall x A$ is provable. And similarly, if $A(n)$ is not provable for every n, then $\sim A(n)$ is provable for some n, and hence $\sim\forall x A$ is provable. Hence we have completeness for all formulae.

So, if you think that K is a good account of the logic of quantifiers and connectives, then you have reason to believe that $K^{\sharp\sharp}$ is true arithmetic. It is a complete theory, giving you a definite answer for every arithmetic question.

The same cannot be said for $X^{\sharp\sharp}$ for other logics X. The most you can show is that (given that X satisfies a few conditions[2]) $X^{\sharp\sharp}$ is extensionally complete. That is, if A is a formula in the extensional (implication free) vocabulary, either A or $\sim A$ is provable. The proof is the same induction as before.[3]

1) It is simple to show that given the ω rule the induction rule follows.
2) In particular, that X includes distributive lattice logic and a negation satisfying the de Morgan laws.
3) This proof obtains even in the absence of the law of the excluded middle.

Now, if you endorse a logic other than K—say X—then you must answer this question: If X is the preferred logic, is $X^{\sharp\sharp}$ a good theory of arithmetic? Take the case of $R^{\sharp\sharp}$. In this theory there is no proof of $0 = 2 \rightarrow 0 = 1$,[4] and no proof of $\sim(0 = 2 \rightarrow 0 = 1)$ either.[5] However, it is a theorem of $R^{\sharp\sharp}$ that $(0 = 2 \rightarrow 0 = 1) \vee \sim(0 = 2 \rightarrow 0 = 1)$. One could rightly ask which disjunct is the case. In relevant arithmetic, is it true that if $0 = 2$ then $0 = 1$? $R^{\sharp\sharp}$ gives you no guidance on the issue.

One way to proceed is to attempt to extend arithmetic further. We ought try to find theories extending $X^{\sharp\sharp}$, which "fill in the blanks" in one way or other. One simple way to do this is to move to $K^{\sharp\sharp}$, but this is giving up the game of formalising a truly substructural arithmetic. To be sure, we can complete $X^{\sharp\sharp}$ by adding theses such as $0 = 2 \rightarrow 0 = 1$, $0 = 3 \rightarrow 0 = 2$ and so on, but to do so seems to erase all of the distinctions we wish to draw. If the conditional is to indicate some kind of relevance, then it is surely more appropriate for our theory to say that $0 = 2 \rightarrow 0 = 1$ is false. We ought have $\sim(0 = 2 \rightarrow 0 = 1)$ as a claim in our arithmetic theory.

One way to construct such a complete and genuinely substructural arithmetic involves the use of metavaluations. Meyer and I have shown that the theory $E^{\sharp\sharp}$ can be extended to a natural complete theory, which we call $TE^{\sharp\sharp}$, for "true E arithmetic" Meyer and Restall [1999]. This construction uses the technique of metavaluations to define the Truths of $TE^{\sharp\sharp}$ as follows:

- $s = t$ is True iff $s = t$.
- $A \wedge B$ is True iff both A and B are True.
- $A \vee B$ is True iff either A or B is True.
- $\sim A$ is True iff A is not True.
- $A \rightarrow B$ is True iff $A \rightarrow B \in E^{\sharp\sharp}$ and furthermore, if A is True, so is B.

The set of Truths so defined is a theory extending $TE^{\sharp\sharp}$. It is genuinely substructural, since it contains theses such as $\sim(0 = 2 \rightarrow 0 = 1)$, due

4) There is no way to reason from $0 = 2$ to $0 = 1$ using arithmetic operations alone. Reasoning classically, you can import the theorem $0 \neq 2$ to deduce $0 = 2 \wedge 0 \neq 2$ from $0 = 2$ alone, and from that contradiction you can deduce $0 = 1$. But in a logic respecting relevance, this reasoning breaks down.

5) If there were, then $K^{\sharp\sharp}$ would be inconsistent, as $R^{\sharp\sharp}$ is a subtheory of $K^{\sharp\sharp}$, and we can prove $0 = 2 \rightarrow 0 = 1$ in $K^{\sharp\sharp}$.

to the defining clause for negations. This technique produces a complete arithmetic, but it has one major shortcoming. It works for the logic E of entailment, but not for the logic R of relevant implication. In R, $\sim A$ is equivalent to $A \to f$ for a particular proposition f. This means that we cannot both give negation the extensional clause above ($\sim A$ is true iff A is not) while at the same time giving implication the intensional clause ($A \to B$ is when not only if A is true so is B, but when also $A \to B$ is in the theory $X^{\sharp\sharp}$). So, if we wish to complete $R^{\sharp\sharp}$ or theories like it, we will need to use a different technique.

In this paper I will present a different technique for constructing models for substructural arithmetics. This technique will help us shed new light on the issue of the truths of substructural arithmetics, and it be able to provide a number of complete extensions of $R^{\sharp\sharp}$. It seems, however, that these complete extensions of $R^{\sharp\sharp}$ are not completely satisfactory (we will be able to construct a consistent, complete arithmetic theory in which $\sim(0 = 2 \to 0 = 3)$, and another in which $\sim(0 = 3 \to 0 = 2)$. However, the techniques discussed in this paper does not allow us to satisfy both of them in the one model. So, I will end this paper with suggestions for future work.

2 Frame semantics

The extant literature about models for substructural arithmetic has focussed on the use of algebraic models for modelling arithmetics Meyer and Mortensen [1984]; Mortensen [1995]. The idea is simple. We take a domain of objects, and a domain of truth values. The arithmetic function symbols are interpreted by functions on the domain of objects, and the predicate of identity is interpreted by a function mapping pairs of objects in the domain to truth values in the propositional structures. The richer structure of the algebraic models of the non-classical logics choice enables us to model arithmetics with unusual properties.

The approach of this paper is different. We will not use algebraic models of our logics. We will not use three valued matrices or ever more complex algebras in order to prove interesting results about arithmetics. Instead, we will use two different kinds of frame semantics for substructural logics Restall [2000].

Frame semantics for logics like R are rather simple[6]. A reduced frame for a substructural logic is a quintuple $\mathfrak{F} = \langle P, g, R, \sqsubseteq, * \rangle$ satisfying the following conditions.

- P is a nonempty set of points, including a distinguished point g.
- R is a ternary relation on P.
- $\sqsubseteq$ is a partial order on P.
- $*$ is a one place operator on P.
- R satisfies the tonicity requirements: If $Rabc$ $a' \sqsubseteq a$, $b' \sqsubseteq b$ and $c \sqsubseteq c'$ then $Ra'b'c'$.
- g is an identity point of R: $Rgab$ if and only if $a \sqsubseteq b$.
- $*$ is order inverting: If $a \sqsubseteq b$ then $b^* \sqsubseteq a^*$.

The propositional connectives are modelled on a frame by way of an evaluation $\Vdash$ relating points and propositions satisfying the following constraints:

- If $a \Vdash A$ and $a \sqsubseteq b$ then $b \Vdash A$ too.
- $a \Vdash A \wedge B$ if and only if $a \Vdash A$ and $a \Vdash B$.
- $a \Vdash A \vee B$ if and only if $a \Vdash A$ or $a \Vdash B$.
- $a \Vdash A \to B$ if and only if for each b, c where $Rabc$, if $b \Vdash A$ then $c \Vdash B$.
- $a \Vdash {\sim}A$ if and only if $a^* \nVdash A$.

Entailment on frames is truth preservation at all points. That is, $A \vdash_{\mathfrak{F}} B$ if and only if for each $a \in \mathfrak{F}$, if $a \Vdash A$ then $a \Vdash B$. The identity point g is a witness for these entailments since we have the following semantic entailment result.

$$A \vdash_{\mathfrak{F}} B \text{ if and only if } g \Vdash A \to B \text{ in } \mathfrak{F}.$$

[6] This frame semantics for R and its neighbours was introduced by Routley and Meyer [1972, 1973] in the 1970s. For a general discussion closest to the presentation here, see An Introduction to Substructural Logics, Chapter 11, Restall [2000].

The logic of these frames is quite weak. To extend the logic to model more familiar substructural logics, conditions are imposed on R, $*$ and $\sqsubseteq$. Here are some conditions and their corresponding entailments:

$R(ab)cd \Rightarrow Ra(bc)d$	$A \to B \vdash (C \to A) \to (C \to B)$
$Rabc \Rightarrow Rbac$	$A \vdash (A \to B) \to B$
$a \sqsubseteq a^{**}$	$A \vdash {\sim}{\sim}A$
$a^{**} \sqsubseteq a$	${\sim}{\sim}A \vdash A$
$Rabc \Rightarrow Rac^*b^*$	$A \to B \vdash {\sim}B \to {\sim}A$
$Raaa$	$A \wedge (A \to B) \vdash B$
$g \sqsubseteq a$	$A \vdash B \to A$

In the first condition we use '$R(ab)cd$' as a shorthand for '$(\exists x)(Rxcd \wedge Rabx)$' and '$Ra(bc)d$' for '$(\exists x)(Raxd \wedge Rbcx)$.'

The result of adding all the conditions above the line is the substructural logic C, which is R without the contraction axiom $A \wedge (A \to B) \vdash B$. Adding this gives us R, and adding instead the 'irrelevance' condition $A \vdash B \to A$ we get the logic CK which is one of the strongest logics without the contraction axiom. For more details on these substructural logics, see Restall [2000].

If we wish to model logics without negation, then we simply rid our frame of $*$ and its conditions. Models for positive C, R and CK are given by the corresponding classes of frames.

We can extend these models to interpret quantifiers, by means of a constant domain interpretation. In the case of models of arithmetic with the ω rule our job is even easier. We may use a substitutional interpretation, as we have numerals as names for each object in the domain. We have the simple rules:

- $a \Vdash \forall x A(x)$ if and only if $a \Vdash A(n)$ for each n.
- $a \Vdash \exists x A(x)$ if and only if $a \Vdash A(n)$ for some n.

Predicates then are modelled by their extension at each point in the model, and function symbols are modelled by functions on the domain, as before. The models which result are models of the quantified logics C, R and CK. All of the theses of these logics are true at the base point g of the models appropriate for those logics. We will use these frames to construct models for the arithmetics $C^{\sharp\sharp}$, $R^{\sharp\sharp}$ and $CK^{\sharp\sharp}$.

3 The behaviour of identity

A close inspection of the axioms reveals axioms of two different kinds. First, there are those which are identities. Second, there are axioms which are at heart, implications. The only rule left is the omega rule, which is dealt with by our taking the domain of each of our frames to be the natural numbers. Let us consider the identity axioms. These are as follows (free variables are implicitly universally quantified)

$$0 = 0 \qquad x + 0 = x \qquad x + y' = (x + y)'$$

$$x0 = 0 \qquad xy' = xy + x \qquad 0 \neq x'$$

These axioms are settled in our models by the behaviour of the identity relation at the base point g, and by the interpretation of the functions. Since it is our aim to make these models look as much like standard arithmetic as possible except for the interpretation of the conditional, we take the interpretation the constant 0 to be the number zero, and the interpretation of the functions symbols for successor, addition and multiplication to be the functions addition, successor and multiplication on the domain of natural numbers. Finally, the extension of the identity relation at the base point is the standard identity relation on the natural numbers. This ensures that each of these axioms are dealt with. Except one. We have not ensured that $0 \neq x'$, as this depends not on what happens at the base point, but at the point g^*. This will have to wait for a few sections for when we discuss negation.

The interest in these models of substructural arithmetic is not what happens at the base point g, but what happens elsewhere. This is what you would expect. The aim of substructural arithmetic is not to collect together a deviant set of arithmetic facts but rather to explain different connections between the classical arithmetic facts. The conditional provides the novelty, and this is reflected by the fact that the non-standard evaluation occurs on different points of the frame. These points can be considered to be different information states, which give us the tools to extract the connections between different arithmetic facts—what would happen in a hypothetical (but of course still impossible) situation in which $0 = 2$ helps us see what follows from identifying 0 and 2.

Let us see what models must satisfy in order to be models of substructural Peano arithmetic. The implication axioms are simple.

$$x = y \to y = x \qquad x = y \to (y = z \to x = z)$$

$$x = y \to x' = y' \qquad x' = y' \to x = y$$

Let's take these one at a time to see how we can make them hold in our models. The first states, in the language of frames, that the extension of the identity predicate at each point is symmetric. For if $a \Vdash n = m$ then $a \Vdash m = n$ too.

The second axiom connects the extension of identity at different points related by R. In particular, we have the following condition:

Whenever $Rabc$, if $a \Vdash n = m$ and $b \Vdash m = l$ then $c \Vdash n = l$.

We will call this the R-transitivity of identity. We do not so much require that identity be transitive at individual points, but we do require R-transitivity.

In the presence of the contraction condition for the logic R, R-transitivity entails standard transitivity, since $Raaa$ for each a. If $a \Vdash n = m$ and $a \Vdash m = l$ then we must have $a \Vdash n = l$ too. So, in R, identity is transitive at each point in the frame. In the presence of the irrelevance condition for CK, we do not necessarily have transitivity, but since $g \sqsubseteq a$ for each a, identity must be reflexive at each point, as identity is reflexive at g.

The last two axioms connect identity at each point with the behaviour of the successor function. If $a \Vdash n = m$ then we must have $a \Vdash n' = m'$, and conversely, if $a \Vdash n' = m'$ then $a \Vdash n = m$ too. This means that the extension of identity at the point a must act like a congruence relation under successor. So at each point, the extension of identity must be symmetric, and it must be what we call a weak congruence with respect to successor. It is to the study of these symmetric weak congruences that we will now turn.

4 Symmetric weak congruences and frames

A two place relation ρ on the set of natural numbers is a symmetric weak congruence if and only if it is symmetric, and if $n\rho m$ if and only if $n'\rho m'$ for each number n and m. Symmetric weak congruences are rather simple structures, as can be seen by the following equivalence.

Fact 4.1 *The trace* Trace_ρ *of a symmetric weak congruence* ρ *is the set of all numbers n such that* $0\rho n$. *The congruence* ρ *is determined uniquely by its trace* Trace_ρ. *And conversely, any set* X *of numbers is the trace of some symmetric weak congruence* ρ_X.

The fact is quite simple to prove. If $n\rho m$ then $|n-m|$, the absolute value of the difference $n - m$ is in the trace Trace_ρ. Conversely if $n \in \mathrm{Trace}_\rho$ then $m\rho(n + m)$ and $(n + m)\rho m$ for each number m.

Fact 4.2 *The weak congruence ρ is reflexive if and only if $0 \in \mathrm{Trace}_\rho$. It is transitive if and only if whenever $m, n \in \mathrm{Trace}_\rho$ then $m+n, |m-n| \in \mathrm{Trace}_\rho$.*

The first part of this fact is immediate. The second is almost as immediate. We have $m\rho n$ and $n\rho l$ if and only if $|m-n|, |n-l| \in \mathrm{Trace}_\rho$. Similarly, $m\rho l$ if and only if $|m-l| \in \mathrm{Trace}_\rho$. However $|m-l|$ is either equal to $|m-n|+|n-l|$ if n is between n and l, and it is $\big||m-n|+|n-l|\big|$ otherwise.

So, points in models for substructural arithmetics have corresponding symmetric weak congruences, the extension of identity at these points.

We can take the correspondence in reverse too. We can construct a frame out of symmetric weak congruences. The identity relation on natural numbers, with trace $\{0\}$ can be the base point in this frame. The congruences are related by the following ternary relation R

$$R\rho\lambda\chi \text{ if and only if whenever } n\rho m \text{ and } m\lambda l, \text{ then } n\chi l.$$

A corresponding condition on traces is simple: if $n \in X$ and $m \in Y$ then $n+m$ and $|n-m| \in Z$. This ternary relation is modelled explicitly after the connection between R and the extension of identity in any model for substructural logics.

Congruences are also ordered by an inclusion relation $\sqsubseteq$

$$\rho \sqsubseteq \lambda \text{ if and only if whenever } n\rho m \text{ then } n\lambda m.$$

This corresponds to the subset relation on traces.

Then congruence relations, together with R and $\sqsubseteq$ provide all the machinery for frames of the logic C without negation.

Fact 4.3 *The class of symmetric weak congruence relations, ordered by $\sqsubseteq$ and related by R, is a positive* C *frame. The class of* transitive *symmetric weak congruence relations is a positive* R *frame, and the class of* reflexive *symmetric weak congruence relations is a positive* CK *frame.*

There are many requirements to check. The tonicity condition falls out of the definitions. The symmetry condition $Rabc \Rightarrow Rbac$ follows from the symmetry of the congruence relations. The associativity condition is the most tedious to check, but it is a restatement of a known fact of relation algebra: the associativity of relational composition. Note that $R\rho\gamma\chi$ if and only if $(\rho\gamma) \subseteq \chi$, where $(\rho\gamma)$ is the relation defined by setting

$$n(\rho\gamma)l \text{ iff there is some } m \text{ where } n\rho m \text{ and } m\gamma l$$

Then, the associativity condition $R(ab)cd \Rightarrow Ra(bc)d$ is equivalent to checking the associativity fact

$$(\rho(\gamma\chi)) \sqsubseteq ((\rho\gamma)\chi)$$

for relational composition, and this is an immediate consequence of the definition of composition.

The identity point g in a frame is the identity relation: $(g\rho) = \rho$ for any relation ρ.For R, we have $R\rho\rho\rho$, as the congruence relations are transitive: $(\rho\rho) \sqsubseteq \rho$. For CK, we have $g \sqsubseteq \rho$ as congruence relations are reflexive.

So, frames constructed out of weak congruence relations give us models for our logics (without negation). It is only a short step to make these models of the entire language of arithmetic (except for the axiom involving negation). We take the domain to be the set of natural numbers, and we evaluate the identity predicate in the obvious way:

- $\rho \Vdash n = m$ if and only if $n\rho m$.

We interpret the function symbols in their standard way on the domain of natural numbers. This gives us a model.

Fact 4.4 *This evaluation gives us a model of all of the* positive *axioms of Peano arithmetic.*

The demonstration of this fact involves running our argument thus far in reverse. The symmetry of the relations gives us the symmetry axiom $x = y \to y = x$. The definition of the relation R gives us $x = y \to (y = z \to x = z)$. The fact that these are congruence relations ensures that $x = y \leftrightarrow x' = y'$, and the behaviour of the functions ensures that each identity axiom (for addition and multiplication) are satisfied at the base point g, as identity is interpreted there in the normal fashion.

These models, for the positive parts of $C^{\sharp\sharp}$, $R^{\sharp\sharp}$ and $CK^{\sharp\sharp}$, are genuinely substructural. In each model $g \nVdash 0 = 2 \to 0 = 1$. Take the congruence relation ρ with trace $\{0, 2, 4, \ldots\}$. We have $\rho \Vdash 0 = 2$ but $\rho \nVdash 0 = 1$. This relation is symmetric, reflexive and transitive, so it appears in models for each of $C^{\sharp\sharp}$, $R^{\sharp\sharp}$ and $CK^{\sharp\sharp}$. In fact, we have $g \nVdash 0 = m \to 0 = n$

whenever m does not divide n, as the trace $\{0, m, 2m, 3m, \ldots\}$ gives us a congruence relation in which $0 = m$ holds but $0 = n$ does not.

These models are also, to some degree, natural. The base point g models the arithmetic facts, and the other points are hypothetical arithmetic situations, in which the arithmetic truths are varied. The range of possible variations are constrained by the arithmetic laws such as the symmetry of identity. The degree of permissible variations establishes the connections between arithmetic claims. We have $0 = 2 \rightarrow 0 = 4$ in R arithmetic, since identifying 0 and 2 brings with it the identification of 0 and 4 by the transitivity of identity. We have $0 = 2 \leftrightarrow 2 = 4$, and $0 = 2 \rightarrow (2 = 4 \rightarrow 0 = 4)$, so $0 = 2 \rightarrow (0 = 2 \rightarrow 0 = 4)$ and by contraction, $0 = 2 \rightarrow 0 = 4$. This is reflected in our models by the behaviour of the appropriate congruences. Once $0\rho 2$ we must have $0\rho 4$ by transitivity. In logics without contraction, in which the number of times antecedents are used is important, we do not have $0 = 2 \rightarrow 0 = 4$. The most we have is $0 = 2 \rightarrow (0 = 2 \rightarrow 0 = 4)$. Our model provides a counterexample to $0 = 2 \rightarrow 0 = 4$, as the congruence with trace $\{2\}$ (or $\{0, 2\}$, for CK) does not contain 4, and so, we have a point at which $0 = 2$ holds but $0 = 4$ fails.

So, these models exhibit quite a degree of genuinely substructural behaviour. Unfortunately, as they stand, these models do not support negation. Remedying this is the focus of the last section of this paper.

5 Adding negation

The structure of weak congruence relations is quite rich, but is not rich enough to model negation. Let's focus on the logic R to see why. Consider the traces of symmetric and transitive weak congruences. The identity trace is $\{0\}$. There is also the empty trace $\emptyset$. Every other transitive trace must contain 0, since if $n \in \text{Trace}_\rho$, we have $0\rho n$ and $n\rho 0$, so by transitivity, $0\rho 0$. So, $0 \in \text{Trace}_\rho$. If $\{0\}$ is the identity trace, and if we wish the theory at $\{0\}$ (the set of claims which are taken to be true at $\{0\}$) to be complete, then almost every point in our model will be complete, as every for every point a, other than the empty set, we have $\{0\} \sqsubseteq a$. This is a problem when it comes to modelling negation.

To see why, we need to understand a little more about how the $*$ operator works. The operator $*$ takes points in frames to their informational "dual". The negation, $\sim A$ is true at a just when A does not hold at a^*. When $a \neq a^*$, the way is open for $A \wedge \sim A$ to be true at A, or for $A \vee \sim A$ to fail at a. This makes the semantics of substructural logics quite flexible.[7] The operator $*$ interacts with inclusion in the obvious, order inverting way. If $a \sqsubseteq b$, then we must have $b^* \sqsubseteq a^*$. A point a is consistent and complete just when $a = a^*$ (then, negation at a behaves classically). It is said to be consistent if $a \sqsubseteq a^*$, for then $a \not\Vdash A \wedge {\sim}A$. It is said to be complete if $a^* \sqsubseteq a$, for then, $a \Vdash A \vee {\sim}A$ for each A. If $*$ is of period two (that is, if $a^{**} = a$ for each a) then a is complete, a^* is consistent, and vice versa.

The following result is a straightforward consequence of our definitions:

Fact 5.1 *If $a \sqsubseteq b$ and a is consistent and complete, then b is consistent only when $a = b$ (for $b^* \sqsubseteq a^* = a \sqsubseteq b$, so $b \sqsubseteq b^*$ ensures that $b = a$). Similarly, if $b \sqsubseteq a$ and a is consistent and complete, then b is complete only when $a = b$.*

In our model of symmetric transitive weak congruences, we would like the base point g to be consistent and complete. It follows that $\emptyset$ is consistent, but not complete, and every other point is complete, but not consistent. As there are many symmetric transitive weak congruences extending the identity, it follows that no $*$ operation of period two can be defined on this class. If it could, then $\{0, 2, 4, \ldots\}^*$ and $\{0, 3, 6, \ldots\}^*$ would both be $\emptyset$. Which do you choose for $\emptyset^*$?

There is no doubt that this class of congruences has too few incomplete points to be a frame for R. Let's explore one way to attempt to remedy this deficit. (It will turn out to fail, but the manner in which it fails will be instructive, and will point to another model construction.)

Consider theories: sets of formulas closed under logical consequence. The set of formulas true at a point in a reduced R model is a theory. If T is an R-theory, then so is $T^* = \{A : {\sim}A \notin T\}$. We will call this theory the double of T.

Consider now the theory T given by adding $0 = 2$ to the set of $\mathrm{R}^{\sharp\sharp}$ truths and closing under consequence. This theory contains $0 = 2$, $0 = 4$, and $1 = 3$ and so on, together with $0 = 0$, $0 \neq 1$, $0 \neq 2$ and other arithmetic truths. We have $0 \neq 0$ since transitivity gives $0 = 2 \to (0 = 0 \to 0 = 2)$, so by contraposition, $0 = 2 \to (0 \neq 2 \to 0 \neq 0)$ and modus ponens gives $0 \neq 0$ as well.

[7] But of course, it makes the interpretation of the operator $*$ rather difficult. I will not engage in this discussion here. For more on the interpretation of negation in relevant logics and the operator $*$, I refer the reader to papers by Meyer and Martin [1986], Dunn [1994] and Restall [1999].

Consider the double T^*. It doesn't contain $0 = 0$ (as our original theory has $0 \neq 0$) or $0 = 1$ (T contains $0 \neq 0$) or $0 = 2$ (T contains $0 \neq 2$). It does contain $0 \neq 1$ and $0 \neq 3$ etc. (as T does not contain $0 = 1$, $0 = 3$). The result is as follows

$$\{0 = 0, 0 \neq 0, 0 \neq 1, 0 = 2, 0 \neq 2, 0 \neq 3, \ldots\} \subset T$$
$$\{0 \neq 1, 0 \neq 3, 0 \neq 5, \ldots\} \subset T^*$$

Similarly, if we consider the theory S given by adding $0 = 3$, it and its double look like this:

$$\{0 = 0, 0 \neq 0, 0 \neq 1, 0 \neq 2, 0 = 3, 0 \neq 3, 0 \neq 4, \ldots\} \subset S$$
$$\{0 \neq 1, 0 \neq 2, 0 \neq 4, 0 \neq 5, \ldots\} \subset S^*$$

The theories S^* and T^* are genuine $\mathrm{R}^{\sharp\sharp}$ theories (they are closed under all of the $\mathrm{R}^{\sharp\sharp}$ implications). They both contain no identities. However, they are different. They are different not in which identities they assert, but in which identities they rule out. Our models so far do not provide any way of reflecting this situation. These different theories are mirrored only in the empty point $\emptyset$. To remedy this, we would hope to extend the $\mathrm{R}^{\sharp\sharp}$ model of symmetric transitive weak congruences by replacing $\emptyset$ by a collection of new points: doubles of the existing points in our frame. However, to do this, we would need to define the inclusion relation and R on both the original points and their doubles, in such a way as to continue to validate the conditions for an R-frame.

Some of these conditions are straightforward to meet, and the others are not. Our target frame is made up of the collection of congruence relations on the class of natural numbers (non-empty symmetric transitive weak congruences are equivalence relations, and hence, congruences), together with a point a' for each congruence a—except for g, which needs no double, since the aim of the exercise is to set $g^* = g$ so that what is true at g is consistent and complete. So, we define the operator $*$ to take congruences to their doubles and back.

$$g^* = g \qquad a^* = a' \qquad a'^* = a$$

The inclusion relation places each of the doubles under each of the congruences. The doubles are ordered inversely.

$$a' \sqsubseteq b \text{ always} \qquad a \sqsubseteq b' \text{ never} \qquad a' \sqsubseteq b' \text{ iff } b \sqsubseteq a$$

This ensures that $a \sqsubseteq b$ iff $b^* \sqsubseteq a^*$. Now, for ease of presentation, we shall suppose that for every relation a—including g—we have its double a'. In the special case of g, we simply set $g' = g$. The sole difference between $'$ and * is that * is a function on the entire set of relations and doubles, whereas $'$ picks out the set of doubles (the extra objects together with g), and the original congruence relations (except for g) are not identical to a' for any a.

Now, to make this structure an R frame, we need to define the ternary relation R, and this is where the difficulties emerge. Some constraints are, again, relatively straightforward: the commutativity condition (if $Rabc$ then $Rbac$) and contraposition (if $Rabc$ then Rac^*b^*) impose some constraints on any definition of R.

$$Rab'c' \Leftrightarrow Rb'ac' \Leftrightarrow Racb$$

Similarly, the fact that each shadow a' is below g, means that $Rabc'$ holds only when $Rabg$ (by tonicity). Now, for reflexive, transitive weak congruences a and b, $Rabg$ only when $a = b = g$. But in any reduced R-frame, if $Rggx$ we must have $x \sqsubseteq g$. So, we must have.

$$Rabc' \Leftrightarrow a = b = c' = g$$

The next choice which is settled is $Ra'b'c$. in this case, if the result is to be a reduced R frame, we need $Ra'b'c$ to hold whenever $Rgb'c$ (by tonicity, since $a' \sqsubseteq g$), and then, $Rgb'c$ holds if and only if $b' \sqsubseteq c$. But this holds always, since the doubles are under the congruences. So, we have

$$Ra'b'c \text{ always.}$$

These are the simple cases. The remaining cases are for $Rab'c$, $Ra'bc$ and $Ra'b'c'$. These should be interdefinable, since $Rab'c$ iff $Rb'ac$ (by commutativity) iff $Rb'c'a'$ (by contraposition). The most straightforward constraint on the definition of $Ra'bc$ (and hence on the other clauses) is that

$$Ra'bc \text{ if either } b \sqsubseteq c \text{ or } b \sqsubseteq a$$

since if $b \sqsubseteq c$ then $Rgbc$ and hence $Ra'bc$ by tonicity. Similarly, if $b \sqsubseteq a$ then $Rc'ba$, and hence $Rbc'a$, and $Rba'c$, giving $Ra'bc$. However, this provides a necessary condition for $Ra'bc$ (and the other clauses), but not a sufficient one.

Unfortunately, no way of defining R satisfying these conditions also satisfies the associativity condition, that $R(ab)cd \Rightarrow Ra(bc)d$. The crucial case is verifying that if $R(xy')zu$ then $Rx(y'z)u$. To see why this is the case, we need to examine the structure of R a little more.

Fact 5.2 *Every* reflexive *symmetric transitive weak congruence relation is determined uniquely by the smallest nonzero number (if any) related to zero. The trace ρ is equivalent to the equivalence of natural numbers modulo $|\rho|$ for some number ρ (or ∞ when ρ is identity).*

This is a straightforward piece of elementary number theory, and a consequence of Fact 2. Consider Trace_ρ, and set $|\rho|$ to be its smallest nonzero member (and ∞ if there is none). By transitivity, ρ identifies all of the numbers modulo $|\rho|$. To see that it cannot identify any more, notice that we cannot have $m\rho n$ where the distance between m and n is less than $|\rho|$, since if $m\rho n$, then $0\rho|m-n|$, but $|\rho|$ was chosen as the smallest number identified with zero. There can be no pairs of numbers in the series related any closer than $|\rho|$ apart.

Now we can expose some more of the underlying nature of the accessibility relation in the original positive frame.

Fact 5.3 *Among the reflexive, symmetric weak congruence relations, $R\rho\gamma\chi$ if and only if both $|\rho|$ and $\gamma|$ divide $|\chi|$. Also, among inclusion relations, $\rho \sqsubseteq \gamma$ if and only if $|\gamma|$ divides $|\rho|$.*

In the statement of this fact, we have used the obvious convention that n divides ∞, which corresponds to the fact that the identity relation is included in every trace. In what follows, we will use the notation 'ρ_n' to denote the relation of equivalence modulo n. Identity is ρ_∞.

Now we can expose the failure of associativity in any frame consisting of these congruence relations and their doubles. As I mentioned, the crucial case is the step from $R(xy')zu$ to $Rx(y'z)u$. If $R(xy')zu$ then it follows that the intermediate point is either an original relation, or a shadow: we have either $Rxy'w$ and $Rwzu$ (it is a congruence) or $Rxy'w'$ and $Rw'zu$ (it is a shadow). What is supposed to follow from this is $Rx(y'z)u$. Again, the intervening point is either a congruence, or a shadow. If it is s a congruence, we have $Ry'zv$ and $Rxvu$. If it is a shadow, we have $Ry'zv'$ and $Rxv'u$. In other words, what we require is the following condition:

> If either (a) $Rxy'w$ and $Rwzu$ or (b) $Rxy'w'$ and $Rw'zu$ then there is some congruence v such that either (i) $Ry'zv$ and $Rxvu$ or (ii) $Ry'zv'$ and $Rxv'u$.

Now, we do not know exactly when $Rxy'w$ or $Rw'zu$ holds, but we do know that if $x \sqsubseteq y$ then $Rxy'w$, and that if $z \sqsubseteq u$ then $Rw'zu$. In particular, if the required condition holds, then at the very least, we have

> If either (a) $x \sqsubseteq y$ and $Rwzu$ or (b) $Rxy'w'$ and $z \sqsubseteq u$ then there is a congruence v such that either (i) $Rxvu$ or (ii) $Ry'zv'$.

We have also discarded a clause from each of (i) and (ii) as they will play no part in what follows. Now we will rewrite the conditions in such that the shadows disappear. The remaining conditions can be expressed in terms of the congruences alone

> If either (a) $x \sqsubseteq y$ and $Rwzu$ or (b) $Rxwy$ and $z \sqsubseteq u$ then there is a congruence v such that either (i) $Rxvu$ or (ii) $Rzvy$.

Now, let me abuse notation just a little, and stop considering the congruences x, y, z etc., and consider now their periods. Instead of the congruence x, we have its period, which we shall also call 'x'. Rewriting the condition, we have

> If either (a) $y \mid x$, $u \mid z$ and $u \mid w$ or (b) $y \mid x$, $y \mid w$ and $u \mid z$ then there is a some v such that either (i) $u \mid x$ and $u \mid v$ (ii) $y \mid z$ and $y \mid v$.

This condition fails when $x = y = 2$, $u = z = 3$ and $w = 6$. In this case we have both (a) $2 \mid 2$, $3 \mid 3$ and $3 \mid 6$ (b) $2 \mid 2$, $2 \mid 6$ and $3 \mid 3$, but neither (i) since $u = 3 \nmid 2 = x$ nor (ii) since $y = 2 \nmid 3 = z$.

Fact 5.4 *There is no embedding of the* R*-frame of symmetric transitive weak congruences into an* R*-frame consisting of the congruence relations and a double for each relation, satisfying* $g^* = g$.

So, this quest for a negation complete and consistent R arithmetic has failed. The failure can point our way, however, to a different kind of success. Our accessibility relation R, at least on the original points of the frame (before adding the shadows) collapses into the inclusion relation, with the equivalence

$$Rabc \Leftrightarrow a \sqsubseteq c \text{ and } b \sqsubseteq c$$

This is the characteristic frame condition corresponding to the mingle axiom.

$$A \to (A \to A)$$

Our frame for R without negation was in fact for a stronger logic—it validated R^+ together with mingle. This motivates attempting to define negation in another way: through the simple three-valued frames for RM due to Dunn [1976*a*,*b*].

6 Consistent, complete RM arithmetics

Dunn's frames for RM are simple structures. Each RM frame is a totally ordered set of points $\langle P, \sqsubseteq \rangle$.[8] The complexity required to interpret a substructural logic is found in the definition of satisfaction. The evaluation relation bifurcates into two relations $\Vdash^+$ and $\Vdash^-$. The two relations $\Vdash^+$ and $\Vdash^-$ are an evaluation on an RM frame when they satisfy the following conditions for completeness and heredity, relating points and atomic sentences:

- Either $a \Vdash^+ p$ or $a \Vdash^- p$.
- If $a \sqsubseteq b$ then if $a \Vdash^+ p$, $b \Vdash^+ p$ too. Similarly, if $a \Vdash^- p$ then $b \Vdash^- p$ too.

The relations encode positive and negative information. Positive and negative information are exhaustive (by the lights of RM, at least). And information that we grasp is preserved as we shift from one point to another, up the ordering. Completeness and heredity jointly entail either that every point is exactly the same as far as information goes, or that as information is added, it becomes inconsistent.

An evaluation can be used to interpret the other connectives in a straightforward manner:[9]

- $a \Vdash^+ {\sim}A$ if and only if $a \Vdash^- A$.
- $a \Vdash^- {\sim}A$ if and only if $a \Vdash^+ A$.
- $a \Vdash^+ A \wedge B$ if and only if $a \Vdash^+ A$ and $a \Vdash^+ B$.
- $a \Vdash^- A \wedge B$ if and only if $a \Vdash^- A$ or $a \Vdash^- B$.
- $a \Vdash^+ A \vee B$ if and only if $a \Vdash^+ A$ or $a \Vdash^+ B$.
- $a \Vdash^- A \vee B$ if and only if $a \Vdash^- A$ and $a \Vdash^- B$.
- $a \Vdash^+ A \to B$ if and only if for each $b \sqsupseteq a$, (a) if $b \Vdash^+ A$ then $b \Vdash^+ B$, and (b) if $b \Vdash^- B$ then $b \Vdash^- A$.
- $a \Vdash^- A \to B$ if and only if either (c) $a \Vdash^+ A$ and $a \Vdash^- B$ or (d) $a \nVdash^+ A \to B$.

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[8] Compare the simplicity of the definition of a frame, compared to the relative complexity of the definition of a reduced ternary frame with negation.

[9] These clauses are taken directly from Dunn's original paper Dunn [1976*a*], with only a change in notation.

There are three distinctive features of these clauses. First, the conjunction, disjunction and negation clauses are what one would expect once we pull positive and negative information apart.[10] Second, the components (a) and (b) in the positive clause for the conditional are required for the conditional $A \to B$ to support modus ponens and contraposition (we wish $A \to B$ to give us $\sim B \to \sim A$). Third, the components (c) and (d) in the negative clause for the conditional are required in order to validate the RM inference from $A \wedge \sim B$ to $\sim(A \to B)$, and in order to ensure that completeness holds for conditional formulas.

Dunn [1976*a*] proves that completeness and heredity hold for all sentences, and not just atomic ones, and that at each point all theorems of R and the mingle axiom, are true (that is, if A is a theorem of R+mingle then $a \Vdash^+ A$ in each RM model). The fact that $\sqsubseteq$ is a total order is required in the proof to ensure heredity. Without a total order, heredity may break down.

These models for RM allow us to model negation without requiring incomplete points. Even though $A \to B \vee \sim B$ is invalid in RM, we do not need an incomplete point to invalidate it. In our model we need merely to have a point b such that $b \Vdash^- B \vee \sim B$ (by setting $b \Vdash^- B$ and $b \Vdash^+ B$—we are inconsistent about B at the point b) without $b \Vdash^- A$. This is straightforward, provided we can be inconsistent about B without making A false.

We need only to interpret quantifiers, and then we can construct our models for arithmetic. For quantifiers, we do the usual thing, with a fixed domain of quantification, the standard numbers:

- $a \Vdash^+ \forall x A(x)$ if and only if $a \Vdash^+ A(n)$ for each n.
- $a \Vdash^- \forall x A(x)$ if and only if $a \Vdash^- A(n)$ for some n.
- $a \Vdash^+ \exists x A(x)$ if and only if $a \Vdash^+ A(n)$ for some n.
- $a \Vdash^- \exists x A(x)$ if and only if $a \Vdash^- A(n)$ for each n.

Now, we have all the raw materials to define a class of models for $\mathrm{RM}^{\sharp\sharp}$.

Fact 6.1 *Any subset of numbers of* $\{1, 2, \ldots, \infty\}$ *totally ordered by the relation of divisibility determines a model for* $\mathrm{RM}^{\sharp\sharp}$, *in which the points are the congruence relations* ρ_n *for the chosen values of n, for which we set* $\rho_n \sqsubseteq \rho_m$ *if and only if* m *divides* n, *and*

- $\rho_n \Vdash^+ l = m$ *if and only if* $l \equiv m \pmod{n}$.

[10] They are shared with Priest's logic LP, Nelson's constructive negation, and so on.

- *$\rho_n \Vdash^- l \neq m$ if and only if $l \neq m$, or $n \neq \infty$.*

We require the total ordering of congruences in order for this to be an RM frame. Heredity on atomic relations is given by the definition of ordering: if $l \equiv m \pmod{n}$ and $\rho_n \sqsubseteq \rho'_n$, then n' divides n, so we have $l \equiv m \pmod{n'}$ too. Negative information changes less. At the consistent point ρ_∞, only actually distinct numbers are taken to be distinct. At all other points, even identical numbers are taken to be distinct. So heredity holds here. Completeness is immediate, since at the point ρ_∞, $l = m$ is true if and only if it is not false. At all other points, more obtains, not less, so completeness is not violated anywhere. The result is a model for $\mathrm{RM}^{\sharp\sharp}$ by the usual reasoning.

Now, consider a model in which we include the point ρ_2. At ρ_2 we have $0 = 2$, but we do not have $0 = 3$. So, at any ρ_2 and earlier point in the model, we do not have $0 = 2 \to 0 = 3$. We have $\rho_2 \Vdash^+ {\sim}(0 = 2 \to 0 = 3)$, since $\rho_2 \not\Vdash^+ 0 = 2 \to 0 = 3$ because $\rho_2 \Vdash^+ 0 = 2$ and $\rho_2 \not\Vdash^+ 0 = 3$.

Similarly, if our model contains ρ_3, we have ${\sim}(0 = 3 \to 0 = 2)$ holding at ρ_3 and at every earlier point in the model. However, no RM model contains both ρ_2 and ρ_3, since 2 doesn't divide 3 and 3 doesn't divide 2. Nonetheless, we can go fairly far.

Fact 6.2 *The model corresponding to a subset of numbers $\{1, 2, \ldots, \infty\}$ containing ∞ determines a* consistent *complete model of* $\mathrm{RM}^{\sharp\sharp}$*. Whenever ρ_n is a point in the model and $0 \not\equiv m \pmod{n}$, then at ρ_∞ in this model we have ${\sim}(0 = n \to 0 = m)$.*

The point ρ_∞ is consistent and complete. It is consistent for atomic formulas by construction. The fact that is is consistent for all formulas is a straightforward induction on the complexity of formulas. Since in our model $\rho_n \Vdash^+ 0 = n$ but $\rho_n \not\Vdash^+ 0 = m$ whenever $0 \not\equiv m \pmod{n}$, we have $\rho_\infty \not\Vdash^+ 0 = n \to 0 = m$, and hence $\rho_\infty \Vdash^+ {\sim}(0 = n \to 0 = m)$ as desired.

So, we have at last found consistent and complete models of arithmetic in R—actually, models in RM—in which irrelevant implications between identities are denied. We can model ${\sim}(0 = 2 \to 0 = 3)$ (provided we include ρ_2). And ${\sim}(0 = 3 \to 0 = 2)$ (provided we include ρ_3). Unfortunately, we cannot model both.

7 Conclusion

These constructions open up more questions than they answer. Is there a consistent and complete model of $\mathrm{R}^{\sharp\sharp}$ in which all irrelevant conditionals between identities are denied? In each of the models we have examined, is there a simple axiomatisation of the claims true at $g(=\rho_\infty)$? What else can we say about models in the logics C and CK? The symmetry of weak congruences is tied up with the symmetry axiom $x = y \to y = x$. Why not reject symmetry? (In the absence of contraction we have rejected transitivity of identity at each point, after all.) Is there any way we can keep symmetry of identity at the base point without having symmetry everywhere else? Does rejecting symmetry give us models of weaker substructural logics? What about other mathematical theories? Do these models tell us anything useful about the behaviour of identity in substructural logics?

■ Two

Semantics for Sub-intuitionistic Logics

Joost J. Joosten

1 Introduction

This paper is an exposition and exploration of various semantics for different logics. The semantics transparently reflect how assumptions on the epistemic and cognitive abilities of the creative subject influences the underlying logic. One of these semantics is used to obtain a lower bound on the length of proofs of certain intuitionistic tautologies. Most logics considered here are subsystems of intuitionistic logic. The idea is that a semantics is developed such that all but one or two of the axioms schemes or rules of intuitionistic logic hold on it.

The motivation for such a project originally came from proof complexity. In particular, the semantics that are found and exposed can be used to pursue lower bounds on the length of proofs in propositional intuitionistic logic. And, as a matter of fact, we employ one particular semantics to prove a linear lower bound for intuitionistic logic. Surely, a linear lower bound is not impressive at all, but it nicely demonstrates how semantics for sub-intuitionistic logics can be used to obtain lower bounds. This method was first applied in a recent paper (Hrubeš [2007*b*]) where an exponential lower bound for a large family of modal logics is obtained.

This paper is partly written while employed at the Mathematical Institute of the Academy of Sciences of the Czech Republic in Prague. The stay in Prague was also financed by the Netherlands Organization for Scientific Research (NWO).

This paper is rather self contained. Whatever is not explicitly mentioned here and concerns intuitionistic logic, can be found in Troelstra and van Dalen [1988] or in van Dalen [2004]. We do not only have technical applications in mind in this paper.

2 A semantic approach to lower bounds

The motivation for proof complexity comes largely from computational complexity and telling computational complexity classes apart. Some of these separations are reduced to showing super-polynomial lower bounds for general proof systems. A first step in getting such lower bounds are lower bounds for specific systems.

A general approach to the latter problem that turned out very promising is the following. A so-called Frege proof system consists of a finite set of axiom schemes, and a finite set of rules. Now, if some tautology (as a sequence) has super-polynomial proofs at least one of the axiom schemes, say $\mathcal{S}$ is likely to be included super-polynomially many times in the proofs (possibly in a subsequence, which is not problematic).

Designing a semantics that only fails this axiom scheme $\mathcal{S}$ gives a way to single out those formulas that are not provable without it. With good semantics even counting the minimal number of applications of $\mathcal{S}$ in a proof is possible. This approach has first been applied successfully in Hrubeš [2007*b*].

With this approach one can obtain lower bounds if every proof of some tautology requires a large number of applications of $\mathcal{S}$. However, the situation might be more subtle in the following sense. If τ_n is a series of tautologies which does not have polynomial size proofs, it may still be the case that certain proofs π_n of τ_n require super-polynomially many applications of $\mathcal{S}$ whereas other proofs π'_n of τ_n can do with only polynomially many. In the worst case every hard tautology exhibits this behaviour with respect to every axiom scheme.

We can summarize the above reasoning in the following easy theorem.

Theorem 2.1 *Let $\mathcal{L}$ be a Frege system of some logic. Let Γ be a subset of axioms of $\mathcal{L}$. Let $\widetilde{\models}$ be a semantical consequence relation such that $\widetilde{\models}$ is closed under all rules of $\mathcal{L}$ and sound for all axioms of $\mathcal{L}$ except for the ones in Γ.*

Let φ be a tautology of $\mathcal{L}$. If for every set Δ of instances of Γ with $|\Delta| \leq n$ we can find a structure M such that

1. $M \; \widetilde{\not\models} \varphi$,

2. $M \models \Delta$,

then φ is not provable in $\mathcal{L}$ using less than $(n+1)$ instances from Γ.

Proof of Theorem 2.1. Suppose for a contradiction that there is some proof

$$p \; : \quad p_0, p_1, \ldots, p_i, \ldots, p_m = \varphi$$

in $\mathcal{L}$ with $\leq n$ instances of Γ in p. Let M be a structure validating (in the $\widetilde{\models}$ sense) all these n instances. By induction on i we get that each p_i holds (in the $\widetilde{\models}$ sense) on M which contradicts $M \; \widetilde{\not\models} \varphi \;\; (= p_n)$. qed

Modal logics

As to illustrate the potential of the above mentioned method, let us briefly summarize the content of Hrubeš [2007*b*]. For a number of modal logical systems a semantics can be obtained where the distributivity axiom is replaced by the distributivity rule or even a weaker version of it; the so called transparency rule:

$$\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}.$$

This semantics allows one to really count the minimal number of applications of the different distributivity axioms needed. For classical tautologies of the form $\alpha(\vec{p}, \vec{r}) \; \rightarrow \; \beta(\vec{p}, \vec{s})$, where α is monotone in $\vec{p}$, it turns out that there is a very close connection between the minimal number of applications of distributivity in a proof of $\alpha(\vec{\Box p}, \vec{r}) \rightarrow \Box\beta(\vec{p}, \vec{s})$ and the number of gates in a monotone circuit interpolating $\alpha(\vec{p}, \vec{r})$ and $\beta(\vec{p}, \vec{s})$.

Having this close connection, it is possible to invoke a result of Razborov's that $\mathsf{Clique}_n^k(\vec{p}, \vec{r}) \rightarrow \neg \; \mathsf{Color}_n^{k-1}(\vec{p}, \vec{s})$ has no polynomial size monotone interpolating circuits to obtain an exponential lower bound.

Intuitionistic logic

We know that there exist faithful interpretations of propositional intuitionistic logic into modal logics, most prominently S4. The translations that are best known are the following two.

$(p)^{\Box} = \Box p$	$(p)^{\circ} = p$
$(\bot)^{\Box} = \bot$	$(\bot)^{\circ} = \bot$
$(A \wedge B)^{\Box} = A^{\Box} \wedge B^{\Box}$	$(A \wedge B)^{\circ} = A^{\circ} \wedge B^{\circ}$
$(A \vee B)^{\Box} = A^{\Box} \vee B^{\Box}$	$(A \vee B)^{\circ} = \Box(A^{\circ} \vee B^{\circ})$
$(A \rightarrow B)^{\Box} = \Box(A^{\Box} \rightarrow B^{\Box})$	$(A \rightarrow B)^{\circ} = \Box(A^{\circ}) \rightarrow B^{\circ}$

However, it seems unlikely that these translations can be used to get lower bound results for intuitionistic logic as a corollary of the lower bounds for modal logics. However, an exponential lower bound for intuitionistic propositional logic has recently been obtained by Hrubeš [2007*a*]. He uses a different kind of translation and a version of monotone interpolation for intuitionistic logic.

A more direct approach seems also fruitful. That is, again designing a semantics that makes almost all of the axioms true but fails just one or two. For example, we can consider the following set of complete axioms for intuitionistic logic, where $\neg A$ is defined as $A \rightarrow \bot$.

1. $A \rightarrow (B \rightarrow A)$
2. $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$
3. $A \rightarrow A \vee B$, $B \rightarrow A \vee B$
4. $(A \rightarrow C) \rightarrow [(B \rightarrow C) \rightarrow (A \vee B \rightarrow C)]$
5. $A \wedge B \rightarrow A$, $A \wedge B \rightarrow B$
6. $A \rightarrow (B \rightarrow A \wedge B)$
7. $\bot \rightarrow A$

The only rule involved in this system is MP, Modus Ponens. It does not really matter which Frege system we use as we know from Kojevnikov and Mints [2006] that all Frege systems for intuitionistic logic polynomially simulate each other. We shall now discuss some semantics and their behaviour with respect to the axioms.

A natural candidate

The most natural candidate to focus on when proving a lower bound, seems to be the axiom

$$[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)].$$

For, if a proof is long, then many parts of the proof have to be combined. If MP is the only rule, the only axiom to combine parts of the proof is indeed this axiom. Moreover, it is known (Statman [1979]) that the implicational fragment of intuitionistic logic is already P-SPACE complete and the above axiom seems to be the only informative axiom in this fragment.

Note the similarity of this axiom with the distributivity axiom in modal logics. In particular, by taking for $A := \top$ and viewing $\top \to \psi$ as $\Box\psi$, we syntactically recognize the distribution axiom. However, this is just a heuristic.

3 Kripke semantics for subintuitionistic logics

A semantics that is sound and complete with respect to all the axioms and rules of intuitionistic propositional logic is the well known Kripke semantics where a model is a triple $\langle W, \leq, V\rangle$. Here W is non-empty set of possible worlds, $\leq$ is a transitive reflexive relation on W, and V is a mapping that tells us which propositional variables hold at which world. In addition V has to be upwards persistent, that is, if $p \in V(a)$ and $a \leq b$, then $p \in V(b)$.

The notion of a formula A being forced in a world a is as usual inductively defined:

$a \Vdash p$	$\Leftrightarrow$	$p \in V(a)$
connectives $\bot$, $\vee$ and $\wedge$:		As usual (commuting)
$a \Vdash A \to B$	$\Leftrightarrow$	$\forall b\ (a \leq b \Vdash A \Rightarrow b \Vdash B)$

The notion of truth is upwards persistent, that is, preserved upwards. The heuristic of this semantics is that $a \leq b$ represents that b is a possible future world of a where possibly more knowledge may have been obtained by the creative subject. We shall refer to this semantics here as "classical" or "usual" Kripke semantics.

From now on we will consider only structures in which the number of possible worlds is **finite**. For classical intuitionistic logic we know that this is no restriction as we have completeness with respect to finite Kripke models.

General semantical notions

We shall now discuss some variations of Kripke semantics so as to validate almost all axioms but one or two. Again we shall define notions as the forcing relation $\Vdash$ and the like.

With $[A]$ we shall denote the set of worlds where a formula A is forced under some given definition of forcing. With $\overline{[A]}$ we shall denote the smallest set of worlds containing $[A]$ which is closed upwards. Note that in "classical" Kripke semantics we have that $[A] = \overline{[A]}$ for all A.

We say that a forcing relation $\Vdash$ is upwards persistent or just persistent if $a \Vdash A \ \& \ a \leq b \ \rightarrow \ b \Vdash A$ holds for all A.

Minimal logic

Minimal logic is as intuitionistic logic, only now omitting the axiom schema $\bot \rightarrow A$. It is well known that General Kripke semantics where $\bot$ is considered as a propositional variable is sound and complete for minimal logic.

So, this semantics is an easy example of a semantics that validates all axioms of intuitionistic logic but one. An application of an axiom $\bot \rightarrow A$ is sound on a model if $[A] \cap [\bot] = [\bot]$. Looking for a (constructive) tautology for which any proof needs a lot of applications of the axiom $\bot \rightarrow A$ is now tantamount to looking for a tautology which has models in minimal logic where it fails to hold, but where for many instances of A, we have that $[A] \cap [\bot] = [\bot]$.

Long suspense semantics

The long suspense semantics is almost as usual finite Kripke semantics. The only difference is in the truth definition of the implication $\rightarrow$. In this semantics we set:

$$a \Vdash A \rightarrow B \quad \Leftrightarrow \forall b \ (a \leq b \Vdash A \rightarrow \exists c \ b \leq c \Vdash B).$$

The heuristic is as follows. Once the creative subject knows that $A \rightarrow B$, in a future world, where he gets to know A he shall obtain B but possibly at some later time as he might need to perform some calculations.

Under this definition of $\Vdash$, it is easy to check by induction on the complexity if a formula that we still have persistency of truth. Note, that $\neg A$, which is short for $A \rightarrow \bot$, has the same semantical condition as in classical Kripke semantics.

All axioms remain valid. However, instead of Modus Ponens, we get some weaker versions of it, like:

$$\mathsf{MP}^- : \frac{A \quad A \rightarrow B}{\top \rightarrow B}; \ \mathsf{MP}_l^\top : \frac{\top \rightarrow A \quad A \rightarrow B}{\top \rightarrow B}; \ \mathsf{MP}_r^\top : \frac{\top \rightarrow A \quad \top \rightarrow (A \rightarrow B)}{\top \rightarrow B}.$$

Here, $\top$ is a logical constant satisfying the following defining axiom and rule.

$$\top$$

$$\vdash \top \to B \quad \Rightarrow \ \vdash \neg\neg B$$

It seems unlikely that this semantics is enough fine-grained to really count the number of applications of Modus Ponens that is needed in a proof of an intuitionistic tautology. In particular, we have that

$$\models_{\text{long suspense}} A \to B \quad \Leftrightarrow \ \models_i A \to \neg\neg B.$$

Short suspense semantics

This semantics is as the long suspense semantics, with the only difference that the creative subject is to calculate immediately all consequences of new facts. Let $\leq^\sharp$ be defined as

$$a \leq^\sharp b :\Leftrightarrow (a \leq b \wedge \forall c\ (a \leq c \leq b \to c = a \vee c = b)).$$

The definition of $\Vdash$ now becomes as follows.

$$a \Vdash A \to B \quad \Leftrightarrow \forall b\ (a \leq b \Vdash A \Rightarrow \exists c\ (b \leq^\sharp c \Vdash B))$$

Note that the logic corresponding to the short suspense semantics would be different if we allowed infinite structures too.

Again, it is not hard to see that we have persistency of $\Vdash$. And, again, instead of MP we only have MP^-. The only axiom scheme that fails is

$$[A \to (B \to C)] \to [(A \to B) \to (A \to C)].$$

In order to see that this scheme is not valid, we take $A = p$, $B = q$ and $C = r$. Next, we consider the structure

$$a \leq b \leq c \leq d$$

with

$$\begin{array}{rcl} V(d) & = & \{p, q, r\}; \\ V(c) & = & \{p, q\}; \\ V(b) & = & \{p\}; \\ V(a) & = & \varnothing. \end{array}$$

Clearly, $a \nVdash p \to r$, as $b \Vdash p$ but for no x with $b \leq^\sharp x$ we have $x \Vdash r$. However, $a \Vdash p \to q$, so $a \nVdash (p \to q) \to (p \to r)$. On the other hand, $a \Vdash q \to r$ whence also $a \Vdash p \to (q \to r)$. Consequently

$$a \nVdash (p \to (q \to r)) \to ((p \to q) \to (p \to r)).$$

It is very much unclear if this semantics allows one to control the number of applications of $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$ in an intuitionistic proof.

Weak persistent semantics

The semantics is as classical Kripke semantics, with the only difference that now we do not demand that V is persistent. Rather, we require V to be weakly persistent, that is,

$$a \leq b \to \exists\, c{\geq}b \;\; V(a) \subseteq V(c).$$

Note, that if a branch in a model has a top element, this element is maximal along this branch with respect to V. The heuristic is that the creative subject may temporarily forget something, provided that he or she later reconstructs the knowledge at some point.

We now define $\Vdash$ as usual, that is

$$a \Vdash A \to B \quad \Leftrightarrow \quad \forall b\ (a \leq b \Vdash A \Rightarrow b \Vdash B).$$

We loose upwards persistency of $\Vdash$. Rather, we have that $\Vdash$ is weakly persistent in the sense that $a \Vdash \varphi \ \wedge\ a \leq b \to \exists c\ b \leq c \Vdash \varphi$. However, $\Vdash$ is strongly persistent for formulas of the form $A \to B$. The two axioms that fail are

$$A \to (B \to A)$$

and

$$A \to (B \to (A \wedge B)).$$

However, the weak persistent semantics is sound with respect to the following two rules.

$$\vdash A \quad \Rightarrow \quad \vdash B \to A$$

and

$$\vdash A \quad \Rightarrow \quad \vdash (B \to (A \wedge B)).$$

Moreover, we do have restricted versions of the axioms. That is, if A is of the form $E \to F$, then both axioms are valid indeed. On this semantics MP holds. Note that, e.g., $\not\models (\top \to A) \leftrightarrow A$. Semantically, axioms of the form $A \to (B \to (A \wedge B))$ and of the form $A \to (B \to A)$ are equivalent and seem to be interderivable.

However, it seems that all axioms of the form $A \to (B \to A \wedge B)$ are as independent as possible. For example

$$\not\models [(p \vee q) \to (r \to ((p \vee q) \wedge r))] \to [p \to (r \to (p \wedge r))].$$

As we only consider finite frames, we do have that $A \rightarrow (B \rightarrow \neg\neg(A \wedge B))$ holds. If we allowed also infinite models this could be refuted.

A weak persistent model M being sound with respect to an axiom of the omitted form has a moderately nice characterization.

$$M \models A \rightarrow (B \rightarrow A) \quad \Leftrightarrow \quad \overline{[A]} \cap [B] \subseteq [A] \qquad (*)$$

Thus clearly, we also have the following sufficient condition.

$$[A] \text{ is upwards closed on } M \quad \Rightarrow \quad M \models A \rightarrow (B \rightarrow A) \text{ for any } B$$

A linear lower bound

We will use the weak persistent semantics to prove the following lower bound for intuitionistic propositional logic.

Theorem 3.1 *The intuitionistic tautology*

$$p_0 \rightarrow (\ldots \rightarrow (p_n \rightarrow (p_{n+1} \rightarrow p_0 \wedge \ldots p_{n+1}))$$

is not provable in intuitionistic propositional logic using less than n schemes *of the form*

$$A_i \rightarrow (B \rightarrow A_i \wedge B) \quad \textit{for fixed } A_i,$$

or

$$A_i \rightarrow (B \rightarrow A_i) \quad \textit{for fixed } A_i.$$

From now on, we will consider rooted (weak persistent) models only and denote them $\langle M, r\rangle$ where r is the root.

Definition 3.2 *If $\langle M, r\rangle$ and $\langle M', r'\rangle$ be rooted Kripke structures, e.g., weak persistent models. Let r_0 be a new variable. We define the structure $[\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle]$ by putting $r_0 \leq r$, $r_0 \leq r'$ and then taking the transitive and reflexive closure.*

Note that if A and B are weak persistent rooted models we can always find $V(r_0)$ such that $[A \nwarrow r_0 \nearrow B]$ becomes a weak persistent rooted model too. This does not hold for classical intuitionistic structures.

Whenever $V(r)$ is defined and $[A \nwarrow r \nearrow B]$ is a well defined model, we shall also use the notation $[A \nwarrow r \nearrow B]$ to refer to that model.

Lemma 3.3 *Let $\langle M, r\rangle$ be a weak persistent rooted model such that $M, r \nVdash A$. Let $\langle M', r'\rangle$ be an arbitrary other such model, and choose $V(r_0)$ such that $V(r_0) \subseteq V(r)$. If $[\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle]$ is a weak persistent model, then it holds that*

$$[\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle], r_0 \nVdash A.$$

Proof of Lemma 3.3. By an easy induction on the complexity of A. For $A = \bot$ it clearly holds. If A is a propositional variable it holds as $V(r_0) \subseteq V(r)$. The connectives $\wedge$ and $\vee$ are easy. If $A = B \to C$ and $M, r \nVdash B \to C$, we can find $r \leq \tilde{r} \Vdash B$ and $M, \tilde{r} \nVdash C$. But also $r_0 \leq \tilde{r}$ and also

$$[\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle], \tilde{r} \Vdash B \quad \text{and} \quad [\langle M, r\rangle \nwarrow r_0 \nearrow \langle M', r'\rangle], \tilde{r} \nVdash C.$$

qed

Let us phrase the following observations in a lemma.

Lemma 3.4 *If $[A]$ is upwards closed on a weak persistent model M, then the following two schemata (as schemata in B) hold on M.*

$$A \to (B \to A \wedge B)$$
$$A \to (B \to A)$$

Proof of Lemma 3.4. By the characterization stated in $(*)$ and the fact that the two schemata are semantically equivalent. qed

The proof of Theorem 3.1 now follows from Theorem 2.1 and the following lemma.

Lemma 3.5 *Let the following n axiom schemata be given.*

$$A_i \to (B \to A_i \wedge B) \quad \textit{for fixed } A_i; \quad 1 \leq i \leq n.$$

Let $\vec{q}$ be a string of variables distinct from each of $p_0, \ldots, p_{n+1}$. There exists a model $[\langle M, r\rangle \nwarrow r_0 \nearrow \langle N, r'\rangle]$ where $\vec{q}$ holds at each world, each of the n axiom schemata $A_i \to (B \to A_i \wedge B)$ holds at r_0, however,

$$r_0 \nVdash p_0 \to (\ldots \to (p_n \to (p_{n+1} \to p_0 \wedge \ldots p_{n+1})).$$

Moreover, the $\langle M, r\rangle$ and the $\langle N, r'\rangle$ are classical Kripke models.

Proof of Lemma 3.5. By induction on n. **For n=0** we consider the simple model consisting of three points $a \leq b \leq c$ where $V(a) = \{p_0, \vec{q}\}$, $V(b) = \{p_1, \vec{q}\}$ and $V(c) = \{p_0, p_1, \vec{q}\}$. Clearly, this is a weak persistent model where $p_0 \to p_1 \to p_0 \wedge p_1$ fails to hold and where, moreover, $\vec{q}$ holds at any world. It is easy to consider this model as $[\varnothing \nwarrow r_0 \nearrow \langle N, r' \rangle]$ with $\langle N, r' \rangle$ a classical Kripke model.

If we now consider **n+1 axiom schemata** $A_i \to (B \to A_i \wedge B)$ for $1 \leq i \leq n+1$, we reason as follows. First we make a case distinction.

If $\nvdash_i \vec{q} \wedge p_0 \to \bigvee_{i=1}^{n+1} A_i$ we find a classical (note that we used $\vdash_i$ in the case distinction) Kripke model $\langle M, r \rangle$ with $r \Vdash p_0, \vec{q}$ but $r \nVdash A_i$ for all i. Next, we consider the Kripke model N consisting of just two points $b \leq c$ with $V(b) = \{p_1, \ldots, p_{n+2}, \vec{q}\}$ and $V(c) = \{p_0, \ldots, p_{n+2}, \vec{q}\}$.

Finally we choose a fresh r_0, define $V(r_0) = \{p_0, \vec{q}\}$ and consider $[\langle M, r \rangle \nwarrow r_0 \nearrow N]$. We now combine Lemma 3.3 and the fact that our two building blocks are classical Kripke models to conclude that the $[A_i]$ are upwards closed on $[M, r \nwarrow r_0 \nearrow N]$. Consequently, by Lemma 3.4 we see that all the axiom schemata $A_i \to (B \to A_i \wedge B)$ hold on this model. Indeed, we also have that $r_0 \nVdash p_0 \to (\ldots \to (p_n \to (p_{n+1} \to p_0 \wedge \ldots p_{n+1}))$.

In case $\vdash_i \vec{q} \wedge p_0 \to \bigvee_{i=1}^{n+1} A_i$, by the well known disjunction/Harrop property (see e.g. Kleene [1962] or Troelstra [1973] Chapter 3, Corollary 3.1.5) we know that for some A_i we have that $\vdash_i \vec{q} \wedge p_0 \to A_i$. We assume w.l.o.g. that $\vdash_i \vec{q} \wedge p_0 \to A_{n+1}$. We now apply our induction hypothesis to the n axiom schemata $A_i \to (B \to A_i \wedge B)$ for $1 \leq i \leq n$ together with the formula $p_1 \to \ldots p_{n+2} \to p_1 \wedge \ldots \wedge p_{n+2}$. However, we now demand in our call on the induction hypothesis that all of $\vec{q}, p_0$ hold in any point of the model. By Lemma 3.4 it is clear that this model suffices for our purpose. qed

Quasi filter semantics

This semantics is reminiscent of neighborhood semantics for modal logics. A model is now a quadruple $\langle W, R, V, G \rangle$ such that $\langle W, R, V \rangle$ is a usual Kripke model and G is a set of subsets of W such that $W \in G$. The definition of $\Vdash$ is now only altered for the implication connective. To state this definition, we first need some notation. We will denote by $a\uparrow$ the set $\{b \mid a \leq b\}$. With G_a we mean the set of subsets in G intersected with $a\uparrow$. In a more general approach, the G_a could be defined separately and independent of some overall G.

With $[A]_a$ we mean the set $\{b \in a\uparrow \mid b \Vdash A\}$. Clearly these sets are defined inductively simultaneously with the forcing relation $\Vdash$.

$$a \Vdash A \to B \quad \Leftrightarrow \quad ([A]_a \in G_a \Rightarrow [B]_a \in G_a).$$

Note, that if $[A]_a \in G_a$ and $a \leq b$, then $[A]_b \in G_b$ and hence $\Vdash$ is monotone.

The semantics is closed under MP as $W \in G$. Some axioms require some closure conditions on G. Actually there is a close correspondence.

Axiom scheme	Restriction on G
$A \to (B \to A)$	no restrictions
$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	?
$A \to A \vee B$, $B \to A \vee B$	G closed under supersets
$(A \to C) \to [(B \to C) \to (A \vee B \to C)]$	?
$A \wedge B \to A, A \wedge B \to B$	G closed under supersets
$A \to (B \to A \wedge B)$	G closed under intersections
$\bot \to A$	$\emptyset \notin G$

This semantics seems rather promising with respect to direct proofs in lower bounds for two reasons. First, by altering properties of G we can make some axioms true and others not. Second, this approach is similar to the approach that was applied to modal logics in Hrubeš [2007*b*] and proved fruitful. In particular, one can say what it means for a certain instance of the axiom to hold on the model and hence the number of axioms can be controlled. A significant difference is with the finite model property. Given a finite number of variables, the canonical model in many modal logics is finite. However, this is not the case in intuitionistic logic.

Basic propositional logic

A logic that has been studied in the literature (e.g., see Ardeshir and Ruitenburg [1988]) is so-called Basic Propositional Logic (sometimes also called Visser's logic). It turns out that this logic corresponds to regular Kripke semantics where the underlying Kripke frames are not necessarily reflexive and where the meaning of $\to$ is defined as follows.

$$a \Vdash A \to B \quad \Leftrightarrow \quad \forall b\ ((a < b) \wedge b \Vdash A \ \Rightarrow b \Vdash B)$$

4 Other logics

The semantic approach to obtain lower bounds can be applied to any other logic, in particular to classical propositional logic. In the case of classical logic, we are interested in semantics that fail only some of the axioms. Classical logic is obtained by adding to the axioms of intuitionism the axiom schema

$$\neg\neg A \to A.$$

Any known sub-logic of classical logic with a good semantics can serve as a candidate to prove lower bounds for classical logic.

Intuitionistic logic

We could hope that some tautologies require a large number of axioms of the form $\neg\neg A \to A$. However, this hope is in vain as we have Glivenko's theorem which says that for propositional logical A we have:

$$\vdash_c A \leftrightarrow \vdash_i \neg\neg A.$$

Thus, every classical tautology has a proof with just one application of excluded middle of the form $\neg\neg A \to A$.

Hybrid logics

It is also possible to consider a logic where both $\neg$ and $\to$ are primitive symbols where the $\neg$ is defined classically and the $\to$ intuitionistically. Clearly, in this case we have $p \vee \neg p$ and $\neg p \not\equiv (p \to \bot)$. The relation of this logic to S4 is evident as $\Box A \wedge A$ can be defined as $\top \to A$.

Acknowledgments I would like to thank Libor Běhounek (also for a life-saving biscuit), Lev Beklemishev, Marta Bílková, Pavel Hrubeš, Dick de Jongh, Rosalie Iemhoff, Pavel Pudlák, Neil Thapen, Alan Skelley, and Albert Visser.

■ Three

Normal Forms, Distributive Laws, and Uniform Interpolants

Marta Bílková

In structural proof theory, one often thinks of a cut free proof system as providing us with a proof-search procedure based on rules of the system applied backwards. To find a proof, we reduce a formula, or a sequent, to a set of simpler sequents, for which the task to find a proof is simpler (typically axioms). This process can be also seen as a construction of an equivalent normal form based on such a set of sequents. The construction consists of steps given by the rules of the calculus applied backwards, and its soundness is obtained by inversion lemmata. Each step reduces a formula (or a sequent) to a simpler one, typically removing a connective, which is reflected by a corresponding change of the structure of resulting sequents. But the structure on the meta level of sequents translates back to formulas and thus corresponds to an application of a structural law directly on the level of formulas – typically an application of a distributive law.

This naturally relates the first two concepts - normal forms are constructed via application of distributive laws.

The work on this paper was supported by grant Dynamic Formal Systems IAA900090703, Grant agency of the Academy of Science of the Czech Republic and is a part of the research plan MSM 0021620839 that is financed by the Ministry of Education of the Czech Republic

The uniform interpolation property for a propositional logic is a strengthening of the Craig interpolation property. It states that for every formula ϕ and any choice of propositional variables $\bar{q}$, there is a post-interpolant (a right interpolant) $E_{\bar{q}}(\phi)$ depending only on ϕ and $\bar{q}$ such that for all ψ, whenever $(\phi \to \psi)$ is provable and the shared variables of ϕ and ψ are among $\bar{q}$, $(\phi \to E_{\bar{q}}(\phi))$ and $(E_{\bar{q}}(\phi) \to \psi)$ are provable. Similarly there is a pre-interpolant (a left interpolant): for every formula ψ and any choice of propositional variables $\bar{r}$ there is a formula $A_{\bar{r}}(\psi)$ depending only on ψ and $\bar{r}$ such that for all ϕ, whenever $(\phi \to \psi)$ is provable and the shared variables of ϕ and ψ are among $\bar{r}$, then $(A_{\bar{r}}(\psi) \to \psi)$ and $(\phi \to A_{\bar{r}}(\psi))$ are provable.

Thus right (left) uniform interpolants can be seen as formulas which describe, forgetting specified variables, which part of a formula ϕ is responsible for entailing (being entailed by) formulas sharing a specified sublanguage of ϕ.

Uniform interpolants often coincide with bisimulation propositional quantifiers. Such quantifiers quantify over possible values of a proposition not just in a fixed model but in any model bisimilar to it. Rather rarely they can be expressed in the propositional language itself. If so, they act as uniform interpolants, see D'Agostino [2005]; Visser [1996]. Proofs of uniform interpolation theorem based on this connection can be found in Visser [1996] (a purely semantic approach) or in Pitts [1992]; Bílková [2007, 2006] (using a proof theoretical argument).

Uniform interpolants (or equivalently bisimulation quantifiers) can often be constructed removing occurrences of specified atoms from a normal form of a given formula (as has been done e.g. in ten Cate *et al.* [2006]). Thus our strategy would be to apply certain distributive laws in a proof-search manner to a formula to construct its normal form, and from that obtain a uniform interpolant, removing occurrences of a specified atom (the one quantified over). Converting a sequent to this kind of normal form costs a lot – it is a reason why uniform interpolants constructed this way are of exponential size in general. It is an interesting open problem whether this is their substantial property, i.e., whether there are formulas with uniform interpolants of exponential size only.

Main goal of this paper is to illustrate these points using propositional modal logic **K**, to visualize construction of uniform interpolants by a suitable choice of a proof system, and to see what benefit such approach can bring. We will check a possibility of so called deep inference calculi introduced in Brünnler [2006] to visualize uniform interpolation proofs in modal logic and propose another deep inference system for basic modal logic **K** better suitable for this purpose.

1 Two examples

We start with two examples showing how uniform interpolants can be constructed forgetting a variable in a certain normal form. The first example is trivial – we concider just classical propositional logic. The other example will be more interesting, concerning basic modal logic given in a different language than usually taking a cover modality as the basic one.

In what follows we fix a modal propositional language and definition of formulas as follows (unless specified otherwise):

$$\phi \ := \ p \mid \neg p \mid \top \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Box\phi \mid \Diamond\phi$$

We use $\overline{\phi}$ to denote the dual formula to ϕ, i.e. the one where we mutually replace all occurrences of p by $\neg p$, all occurrences of $\top$ by $\bot$, all occurrences of $\wedge$ by $\vee$, and all occurrences of $\Box$ by $\Diamond$. We use capital Greek letters $\Gamma, \Delta, \ldots$ to denote finite sets of formulas, or sequents if specified so. We use capital letters $G, H, \ldots$ to denote finite sets of sets, or finite sets of sequents, $\bigcup G$ meaning the union of G, i.e. $\{\gamma | \gamma \in \Gamma \in G\}$. We adopt standard abbreviations as $\bigvee \Gamma$ (or $\bigwedge \Gamma$) to be the disjunction (the conjunction) of members of Γ (where $\bigvee \emptyset \equiv \bot$ and $\bigwedge \emptyset \equiv \top$), or $\Box\Gamma$ to be the set $\{\Box\gamma | \gamma \in \Gamma\}$.

Example 1.1 *The classical propositional logic.*

We use capital Greeks for finite sets of formulas. A sequent is of the form Γ, where Γ is a finite set of formulas. Such a sequent is interpreted as a disjunction, i.e. the characteristic formula of a sequent Γ is $\Gamma^c = \bigvee \Gamma$.

Definition 1.2 *Consider the following one-sided sequent calculus for classical propositional logic:*

$$\Gamma, p, \neg p \qquad \Gamma, \top$$

$$\wedge \, \frac{\Gamma, \phi \qquad \Gamma, \psi}{\Gamma, \phi \wedge \psi} \qquad \vee \, \frac{\Gamma, \phi, \psi}{\Gamma, \phi \vee \psi}$$

The calculus is obviously sound and complete, with the structural rules of weakening and cut admissible (for such bbasic facts reader may consult e.g. Troelstra and Schwichtenberg [1996]). A proof search tree for a sequent Γ in this calculus is constructed applying the rules backwards. It is easy to see that both the rules are invertible, which means that they preserve validity while applied backwards as well. This means that it doesn't matter in which particular order we apply the rules during a proof search. If there is no rule to be applied we close a branch. Such a tree is obviously finite since each backward application of a rule removes a connective. On its leaves there are sequents $\Gamma_1, ..., \Gamma_n$ consisting of literals only. Now $\bigwedge_{i=i}^{n} \bigvee \Gamma_i$, which equals to $\bigwedge_{i=i}^{n} \Gamma_i^c$, is a conjunctive normal form of Γ^c.

The steps of the previous proof search procedure corresponding to an application of the $\wedge$ rule can be seen as instances of standard distributive laws for conjunction and disjunction – applying the rule $\wedge$ backwards we use the following law: $\Gamma^c \vee (\phi \wedge \psi) \leftrightarrow (\Gamma^c \vee \phi) \wedge (\Gamma^c \vee \psi)$.

Uniform interpolation property for the classical propositional logic holds trivially. A left uniform interpolant (a universal propositional bisimulation quantifier) of a formula φ w.r.t. a propositional letter p is simply given by the formula $(\varphi[p/\top] \wedge \varphi[p/\bot])$. Equivalently we obtain the left uniform interpolant by erasing all occurrences of p and $\neg p$ from a CNF of φ. To see this consider such a CNF of φ to be $\varphi_1 \wedge \ldots \wedge \varphi_n$, each φ_i a disjunction of literals. We define φ_i^{-p} to be the disjunction of literals of φ_i except the literals $\{p, \neg p\}$, i.e. we remove all occurrences of $\{p, \neg p\}$. In case that both $\{p, \neg p\}$ occur in φ_i we remove whole φ_i from the CNF. Now we can see that

$$(\varphi_1 \wedge \ldots \wedge \varphi_n)[p/\top] \wedge (\varphi_1 \wedge \ldots \wedge \varphi_n)[p/\bot] \equiv \bigwedge_{i=i}^{n} (\varphi_i[p/\top] \wedge \varphi_i[p/\bot]).$$

If p occurs in φ_i, then

$$(\varphi_i[p/\top] \wedge \varphi_i[p/\bot]) \equiv (\top \wedge \varphi_i^{-p}) \equiv \varphi_i^{-p}.$$

If $\neg p$ occurs in φ_i, dually

$$(\varphi_i[p/\top] \wedge \varphi_i[p/\bot]) \equiv (\varphi_i^{-p} \wedge \top) \equiv \varphi_i^{-p}.$$

If neither of $\{p, \neg p\}$ occur in φ_i, the conjunct remains intact. If both $\{p, \neg p\}$ occur in φ_i then $(\varphi_i[p/\top] \wedge \varphi_i[p/\bot]) \equiv \top$. All these cases show us that removing occurrences as described above yields a formula equivalent to $(\varphi[p/\top] \wedge \varphi[p/\bot])$.

The case of the right uniform interpolant is completely dual – we use a left sided sequent calculus producing a disjunctive normal form and proceed similarly to obtain the interpolant.

Example 1.3 *Modal logic* ***K*** *given in a language with a cover modality* ∇*.*

We suppose that reader is familiar with Kripke semantics for modal logics, for basics see e.g. Blackburn *et al.* [2001]; Chagrov and Zakaryasschev [1998]. Kripke model is a triple (S, R, V) where S is a set of states, R a binary relation on it, and V assigns to each propositional variable a subset of S, inducing the $\Vdash$ relation between states and formulas in a usual way. We often understand R equivalently as a function assigning to a state t the set of states accessible from it, notation $R[t]$.

The cover modality ∇ operates on sets of formulas, so whenever Γ is a set of formulas, $\nabla\Gamma$ is a formula (we restrict ourselwes to finite sets here). Its semantics in a state t is that *Gamma* and $R[t]$ cover each other, formally:

$$\begin{array}{lll} s \Vdash \nabla\Gamma & \text{iff} & \forall t \in R[s]\ \exists\gamma \in \Gamma\ t \Vdash \gamma \\ & \text{and} & \forall\gamma \in \Gamma\ \exists t \in R[s]\ t \Vdash \gamma. \end{array} \tag{1}$$

Once the semantics is given, it is easy to check that the following mutual definition between ∇ and the two standard modalities $\Box$ and $\Diamond$ hold:

$$\begin{array}{lll} \nabla\Gamma & \equiv & \Box \bigvee \Gamma \wedge \bigwedge \Diamond\Gamma \\ \Box\phi & \equiv & \nabla\{\phi\} \vee \nabla\emptyset \\ \Diamond\phi & \equiv & \nabla\{\phi, \top\} \end{array} \tag{2}$$

This shows that the finitary ∇ modality is definable in the standard modal language, but also that ∇ itself can be used to define the standard modalities and thus modal logic can be based on language with ∇ as the only modality. For an overview of such an approach, and for more on the coalgebraic flavour of such a logic, see Venema [2006]. We consider the basic modal logic **K** (i.e. the logic of all Kripke frames) given in the language with ∇. For axiomatizations of this logic see Palmigiano and Venema [2007]; Bílková *et al.* [2008].

One particularly important feature the setting with ∇ has is that it gives us a disjunctive normal form, which is not in general available for the standard modal language. To see that, we use the propositional fragment of a tableau given by Janin and Walukiewicz [1995] – a system that has been designed for converting formulas of modal μ-calculus to a disjunctive normal form. The tableau has a form of a left-sided calculus, thus now a sequent is a finite set of formulas interpreted as a conjunction. Asserting a sequent Γ now means that the conjunction of formulas from Γ is not satisfiable, or equivalently that it entails $\bot$, the characteristic formula of a sequent Γ given by $\Gamma^c \equiv \bigwedge\Gamma \to \bot$. The propositional part of the tableau is just dual to the previous calculus for classical logic, and we add one modal rule:

$$\Gamma, p, \neg p \qquad \Gamma, \bot$$

$$\vee\ \frac{\Gamma, \phi \qquad \Gamma, \psi}{\Gamma, \phi \vee \psi} \qquad \wedge\ \frac{\Gamma, \phi, \psi}{\Gamma, \phi \wedge \psi}$$

$$\nabla\ \frac{\{\phi\} \cup \{\bigvee \Gamma' \mid \Gamma' \in G \text{ and } \phi \notin \Gamma'\}}{\{\nabla \Gamma | \Gamma \in G\}}\ \phi \in \bigcup G$$

The ∇ rule, which may seem rather complicated at first sight, is to be read as follows: Given a set of sets of formulas G, if $\{\phi\} \cup \{\bigvee \Gamma' \mid \Gamma' \in G \text{ and } \phi \notin \Gamma'\}$ entails $\bot$ for some $\phi \in \bigcup G$, then $\{\nabla\Gamma | \Gamma \in G\}$ entails $\bot$.

It can be seen from the semantic definition given above that the rule is sound, and in a sense also invertible (if the conclusion entails $\bot$, then one of the possible premises does so). For a detailed proof of this fact see e.g. Bílková *et al.* [2008]. A backward application of the rule corresponds to an interesting modal distributive law:

$$\bigwedge_{\Gamma \in G} \nabla\Gamma \ \equiv \ \bigvee_{\phi \in \bigcup G} \nabla\{\phi \wedge \bigvee \Gamma' | \Gamma' \in G \ \text{ and } \ \phi \notin \Gamma'\}.$$

The tableau can be used to produce a disjunctive normal form of a given formula: we start with the formula and decompose it applying the rules backwards until there is no rule to be applied. An application of the ∇ rule backwards corresponds to an application of the corresponding distributive law, pushing conjunctions down the formula. In classical steps we reduce the sequent to a simpler one ($\wedge$ steps), or to a disjunction of two simpler sequents ($\vee$ steps), or, in the case of the ∇ rule, to a disjunction of nablas of simpler sequents. Such a procedure terminates, since each rule while applied backwards simplifies the sequent (it always removes a connective or a modality). We end up with a disjunctive normal form.

This way it is possible to show that each formula is equivalent to some formula in the restricted language given below, where π is an arbitrary propositional (non modal) formula and Γ is a finite set of formulas of the language:

$$\phi \ = \ \pi \mid \phi \vee \psi \mid \pi \wedge \nabla\Gamma$$

Given a formula in this language, finding its uniform interpolant becomes simple since we have only restricted occurrences of conjunction. Now we are looking for a right uniform interpolant, or equivalently we want to simulate the existential bisimulation quantifier. This quantifier, over **K**, commutes with disjunction and nabla modality. Moreover it can be shown to commute with the entire normal form, thus, the same way as before, removing occurrences of the quantified atom we actually obtain the desired interpolant. For a detailed proof of this fact see ten Cate *et al.* [2006].

We have followed the setting of Janin and Walukiewicz, treating nabla as the basic modality. But all we have just used can be easily rewritten dually, using a right-sided system, the dual triangle modality $\triangle$, and a conjunctive normal form instead. This setting occurs in the section 3.

2 Deep inference calculi

So called deep inference calculi are generalizations of Gentzen style sequent proof systems. In contrast to the usual setting, they enable one to work also deep inside formulas. Deep sequents therefore incorporate a relevant part of the tree structure of formulas in various ways, e.g. the depth of nested conjunctions and disjunctions in the classical case, or the depth of nested modalities in the case of modal logics.

Let's consider cut-free sequent or normalized natural deduction proofs and think of them as dealing with derivations from assumptions. If one moves from natural deduction proofs to sequent proofs, what happens is that all the assumptions of a derivation are stored in a sequent, e.g., in each line of the proof. One needn't to look backwards to see the corresponding assumptions. Now to move from sequent proofs to deep sequent proofs is very much alike - we moreover store the relevant tree structure of formulas in each line of the proof and needn't to look backwards to extract it from the corresponding sub-proof where the formula was derived.

On the other hand, this enforces us to add more structure to sequents since they consist no more just of sets, multisets or sequences of formulas - now sequents code trees. What happens is that we move the tree structure of formulas to the meta-level of sequents (as we do with connectives - conjunction corresponds to comma on the left side of a sequent, disjunction to comma on the right side, and implication corresponds to the sequent arrow) and treat it there. To this aim we use various brackets as structure signs to code the tree-depth. Once we move the relevant structure of formulas to the meta level and store it in a sequent, we can work deep inside formulas without decomposing them first. Such calculi have been studied for various logics, including cassical propositional logic or modal logics – for extensive literature on the topic consult the web page Guglielmi [2008].

Our next goal is to check the ability of deep inference systems for modal logic proposed in Brünnler [2006] to visualize uniform interpolation proofs, or rather construction of uniform interpolants. Such an idea seems promising since the depth structure enables us to built in the recursive definition of interpolants into sequents and, perhaps, proceed exactly like shown in Example 1.3, i.e., construct a normal form and than simply erase all occurrences of a quantified atom.

However, we will see that the solution is not quite like that and that the structure of interpolants is more sensitive and therefore needs more structure incorporated in sequents. It turns out that the corresponding structural pattern needed is given rather by the triangle modality (which is the dual of the cover modality nabla) which reflects the balance between the box and the diamond part of recursion. It is not surprising since over **K** the triangle modality provides a normal form which commutes with the universal bisimulation quantifier.

From now on we use capital Greeks to denote deep sequents as well as finite sets of formulas, it will be clear from the context if we deal with a deep sequent or with a set of formulas (which is however a special case of a deep sequent). The comma denotes the set union, thus e.g. Γ, ϕ is $\Gamma \cup \{\phi\}$. We use capital letters $G, H, \ldots$ to denote finite sets of sequents (or sets of sets). Not to get lost in brackets we use the semicolon to denote the union on this level, thus $\Gamma; \Delta$ is an abbreviation of $\{\{\Gamma\}, \{\Delta\}\}$.

Deep sequents (wee call them just sequents for short and) are of the form $\phi_1, \ldots, \phi_m, [\Delta_1], \ldots, [\Delta_n]$ where the comma denotes the set union and all Δ_i are deep sequents. The characteristic formula of such a deep sequent is $\phi_1 \vee \ldots \vee \phi_m \vee \Box\Delta_1^c \vee \ldots \vee \Box\Delta_n^c$, where each Δ_i^c is the characteristic formula of Δ_i. A sequent context is a sequent with exactly one occurrence of the symbol $\big\{\big\}$, the hole, which never occurs inside formulas. The sequent $\Gamma\big\{\Delta\big\}$ is obtained by replacing the hole with a sequent Δ.

Definition 2.1 *The deep inference modal rules Brünnler [2006]:*

$$\Gamma\big\{p, \neg p\big\} \qquad \Gamma\big\{\top\big\}$$

$$\wedge\,\frac{\Gamma\big\{\phi\big\} \qquad \Gamma\big\{\psi\big\}}{\Gamma\big\{\phi \wedge \psi\big\}} \qquad \vee\,\frac{\Gamma\big\{\phi, \psi\big\}}{\Gamma\big\{\phi \vee \psi\big\}} \qquad \Box\,\frac{\Gamma\big\{[\phi]\big\}}{\Gamma\big\{\Box\phi\big\}}$$

$$k\,\frac{\Gamma\{\Diamond\phi, [\Delta, \phi]\}}{\Gamma\{\Diamond\phi, [\Delta]\}} \qquad t\,\frac{\Gamma\{\Diamond\phi, \phi\}}{\Gamma\{\Diamond\phi\}} \qquad 4\,\frac{\Gamma\{\Diamond\phi, [\Delta, \Diamond\phi]\}}{\Gamma\{\Diamond\phi, [\Delta]\}}$$

$$\Box l\,\frac{\Gamma\{\Box\phi, [\Diamond\bar{\phi}, \phi]\}}{\Gamma\{\Box\phi\}}$$

A deep inference system for the logic **K**, notation K_d, consists of the boolean rules plus the **k** rule and the □ rule. We will concentrate just on this logic in this paper.

Deep inference systems for the other logics are obtained as follows: a deep inference system for the logic **T** is the extension of K_d by the **t** rule, a deep inference system for the logic **GL** consists of the boolean rules plus the **4**, the **k** rule, and the □l rule. The system GL_d, our proposal of a deep inference system for **GL**, is introduced through a special box rule since it is rather the proof theoretical behavior of box then an interplay of diamond and box contexts to be defined by the rule. However, the solution in not quite satisfactory – since it is a one-sided system, it doesn't mirror the two sided nature of box behaviour in **GL** as given by the Löb's axiom.

Structural rules of weakening and cut, and the necessitation rule are admissible in the calculi (for a proof of this fact for K, T, K4 see Brünnler [2006]).

Uniform interpolation

Let us for simplicity consider just the logic **K**, for which the interpolation theorem is rather simple. The uniform interpolation property formulated in a deep inference manner is the following:

Theorem 2.2 *For every sequent* Γ *and a variable* p *there exists a single formula* $A_p(\Gamma)$ *such that:*

(i) $Var(A_p(\Gamma)) = Var(\Gamma) \setminus p$

(ii) $\vdash_{K_d} \overline{A_p(\Gamma)}, \Gamma$

(iii) *Whenever a sequent* $\Pi\{\Gamma\}$ *is provable where* Π *doesn't contain* p, *then the sequent* $\Pi\{A_p(\Gamma)\}$ *is provable as well.*

The part (ii) of the theorem is an analog of an axiom of specification while the part (iii) can be seen as a form of a deep inference rule for the universal propositional quantifier.

Sketch of proof. To construct a uniform interpolant of a given sequent Γ we first convert the sequent to a normal form and then we write the normal form down in a certain way, as a formula, omitting all occurrences of p. However, we will need recursive calls of the procedure for the diamond subformulas to construct the interpolant. Such a normal form consists of the conjunction of sequents occurring on leaves in a proof-search tree for Γ. In case of deep inference system, we create such a tree applying the rules backwards as follows: the **k** rule is applied only if $\phi \notin \Delta$, i.e. only if the sequent in the premiss differs from the sequent in the conclusion. Having in mind that all the rules are invertible, and thus it does not matter in which particular order we apply them, we continue until all sequents are irreducible. We call such sequents least sequents. Then we use the conjunction of such least sequents to create a desired uniform interpolant. Note that such sequents, due to the **k** rule, contain diamond formulas.

The difference between this construction and proofs contained in Bílková [2006] is not substantial – the earlier proofs use a standard two-sided calculus and carry out a normal form in a different way, alternating invertible parts with critical modal steps which are not in general invertible. The proof presented here can be seen as a more compact way how to present the earlier proof.

The exact procedure constructing, for a given sequent Γ, its uniform interpolant $A_p(\Gamma)$ is the following:
First we create a proof search tree applying the rules bottom up, with the proviso that the **k** rule is applied only if the application changes the entire set of formulas Δ. We proceed until each sequent is irreducible, i.e. there is no rule to be applied. We call the set of such least sequents occurring on leaves in a proof search tree for Γ a closure of Γ, notation $Cl(\Gamma)$.

Second we define, for Γ not being a least sequent, its uniform interpolant:

$$A_p(\Gamma) \equiv \bigwedge_{\Gamma' \in CL(\Gamma)} A_p(\Gamma').$$

For a least sequent Γ' of the form $\Pi, [\Delta_1], \ldots, [\Delta_n], \Diamond\phi_1, \ldots, \Diamond\phi_m$ where Π consists of literals only, we proceed as follows: define Π^{-p} be Π without occurrences of literals $p, \neg p$, in the case that Π contains both p and $\neg p$ we define Π^{-p} to be $\{\top\}$. Take for the interpolant the following formula, defined recursively on the depth of Γ':

$$A_p(\Gamma') \equiv \bigvee \Pi^{-p} \bigvee_{i=1}^{n} \Box A_p(\Delta_i) \bigvee_{j=1}^{m} \Diamond A_p(\phi_j).$$

Note that the pattern of the recursive step is very close to have the form of the triangle modality – the dual modality to nabla. Notice that the construction is however not of the form of creating a normal form and simply writing it down, omitting all occurrences of literals $p, \neg p$. We are writing down the corresponding formula of the sequent (a normal form in a sense), omitting all occurrences of $p, \neg p$, and adding recursive calls of A_p for all the diamond formulas. The latter is unfortunately inevitable, since the use of box brackets to deal with the structure of sequents enables us to built in just the box part of the recursive construction of interpolants.

Next we should prove that this definition of uniform interpolant works. However, since the construction itself is not quite satisfactory and indeed the earlier proofs in Bílková [2007, 2006]; ten Cate *et al.* [2006] are significantly simpler, we find it useless to do the whole proof here. We rather move to our proposed improvement of the definition of a deep sequent calculus better suited for such proofs, and we present the full proof using the new calculus. qed

3 Another calculus

We have seen that for a construction of uniform interpolants, or of a normal form producing a uniform interpolant, the standard deep inference calculi are not ideal. This motivates the next improvement of the deep inference calculus incorporating more structure to sequents. We do not claim that this should be the calculus, but we think that the ability to create, by a proof search, a normal form which can be transformed to an interpolant is of some proof theoretical significance. Reader can take it simply as a tableau constructing a uniform interpolant.

An intuition behind the new calculus we propose here is that it is rather the nabla (or the dual to nabla) modality which produces a normal form commuting with uniform interpolants and their construction. So why do not we try to base the very structure of sequents on such a modality? To keep the formulation of uniform interpolation theorem analogous to the previous case we construct the left interpolant (the universal quantifier) again, using the triangle modality and a right sided calculus. However, all can be dually rewritten, using the nabla modality, to a right sided sequent system to capture the case of the right uniform interpolant.

First we recall some properties of the triangle modality $\triangle$. The semantic definition of $\triangle$ which as nabla operates on finite sets of formulas, is the following:

$$\begin{array}{lll} s \Vdash \triangle\Gamma & \text{iff} & \exists t \in R[s]\ \forall \gamma \in \Gamma\ t \Vdash \gamma \\ & \text{or} & \exists \gamma \in \Gamma\ \forall t \in R[s]\ t \Vdash \gamma. \end{array} \tag{3}$$

Using the semantics it is easy to check that the mutual definitions between usual modalities and the $\triangle$ modality are now as follows:

$$\begin{array}{lll} \triangle\Gamma & \equiv & \Diamond \bigwedge \Gamma \vee \bigvee \Box\Gamma \\ \Box\phi & \equiv & \triangle\{\phi, \bot\} \\ \Diamond\phi & \equiv & \triangle\{\phi\} \wedge \triangle\emptyset \end{array} \tag{4}$$

From now on we will use the triangle modality as an abbreviation given by the above equation, keeping the language as defined before with the standard modalities only.

We define a deep inference calculus for **K** adding more structure to sequents - instead of using box structure to build a tree and dealing with diamonds as in the rule **k** of the previous calculus K_d, we use the triangle structure on the meta level of sequents. We use different structure brackets $\lfloor, \rfloor$ to simulate the triangle modality on the meta level. Sequents are now of the form $\Delta_1, \ldots, \Delta_n, \lfloor G_1 \rfloor, \ldots \lfloor G_m \rfloor$, each $G_i = \{\Gamma_{i,1}, \ldots, \Gamma_{i,k_i}\}$ a set of sequents. The characteristic formula of such a sequent is $\Delta_1^c \vee \ldots \vee \Delta_n^c \vee \triangle\{\Gamma_{1,1}^c, \ldots, \Gamma_{1,k_1}^c\} \vee \ldots \vee \triangle\{\Gamma_{m,1}^c, \ldots, \Gamma_{m,k_m}^c\}$. The propositional part is the same as before, we spell out just the modal rules.

Definition 3.1 *Calculus $K\triangle_d$, the modal rules:*

$$k\Box\ \frac{\Gamma\{\lfloor\{\{\sigma\} | \sigma \in \Sigma\}; \emptyset\}}{\Gamma\{\Box\Sigma\}} \qquad k\Diamond 1\ \frac{\Gamma\{\lfloor\{\{\Lambda, \varphi\} | \Lambda \in G\}\rfloor\}}{\Gamma\{\Diamond\varphi, \lfloor\{\Lambda | \Lambda \in G\}\rfloor\}}$$

$$k\Diamond 2\ \frac{\Gamma\{\lfloor\Sigma\rfloor\} \qquad \Gamma\{\lfloor\ \rfloor\}}{\Gamma\{\Diamond\Sigma\}}$$

The second diamond rule is included for an effective proof search to be possible – when there is no triangle brackets in the context of a diamond formula, we apply the second rule backwards.

Lemma 3.2 *Calculus $K\triangle_d$ is sound and complete.*

Proof of Lemma 3.2. To establish soundness and completeness, we first prove that the three new modal rules are sound and invertible. Completeness then easily follows – each sequent can be equivalently rewritten using the rules (each of them simplifies the sequent while applied backwards so this procedure terminates) to a set of irreducible sequents. The original sequent is provable if and only if each of such sequents is provable. Since irreducible sequents consist of literals and structure marks only we can easily decide whether they are of the form of an axiom or not.

Soundness and invertibility of the rules can be obtained by proving, for each rule, that the characteristic formula of the premiss is equivalent with the characteristic formula of the conclusion of the rule.

For the $\mathbf{k}\Box$ rule it is sufficient to show that $\bigvee \Box\Sigma \equiv \triangle\{\Sigma, \bot\}$ which follows immediately from the definition of triangle in the language with box and diamond. For the $\mathbf{k}\Diamond 2$ rule $\bigvee \Diamond\Sigma \equiv \triangle\Sigma \wedge \triangle\emptyset$ follows from the same definition.

For the $\mathbf{k}\Diamond 1$ rule we need to show that $\Diamond\phi \vee \triangle\{\Gamma^c | \Gamma \in G\}$ is equivalent to $\triangle\{\Gamma^c \vee \phi | \Gamma \in G\}$:

Direction from left to right: fix a world s and suppose $s \Vdash \Diamond\phi \vee \triangle\{\Gamma^c | \Gamma \in G\}$. We show that $s \Vdash \triangle\{\Gamma^c \vee \phi | \Gamma \in G\}$. Suppose $s \Vdash \Diamond\phi$, then there is a world $t \in R[s]$ such that $t \Vdash \phi$. But then $t \Vdash \phi \vee \Gamma^c$ for each $\Gamma \in G$ and thus the conclusion follows. Now suppose that $s \Vdash \triangle\{\Gamma^c | \Gamma \in G\}$. Then obviously, by the semantic definition, $s \Vdash \triangle\{\Gamma^c \vee \phi | \Gamma \in G\}$ as well.

Direction from right to left: fix a world s and suppose now that $s \Vdash \triangle\{\Gamma^c \vee \phi | \Gamma \in G\}$. We show that $s \Vdash \Diamond\phi \vee \triangle\{\Gamma^c | \Gamma \in G\}$. Suppose there is a world $t \in R[s]$ such that for each $\Gamma \in G$, $t \Vdash \Gamma^c \vee \phi$. Then either t satisfies all Γ^c, and thus $s \Vdash \triangle\{\Gamma^c | \Gamma \in G\}$, or t satisfies ϕ and thus $s \Vdash \Diamond\phi$. The other case is that there is $\Gamma \in G$ such that for each $t \in R[s]$, $t \Vdash \Gamma^c \vee \phi$. Then either all such worlds t satisfy Γ^c, and thus $s \Vdash \triangle\{\Gamma^c | \Gamma \in G\}$, or some of the worlds above s satisfies ϕ and thus $s \Vdash \Diamond\phi$. qed

Next we show some simple structural properties of the calculus we will need later to prove the uniform interpolation theorem. We shall use admissibility of the following rules of weakening and necessitation:

$$\text{w}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \text{Nec}\,\frac{\Gamma}{\lfloor\{\Gamma\}\rfloor}$$

Warning!: in the following, the former is **not** a correct instance of the weakening rule while the latter is:

$$\frac{\Gamma\{\lfloor\Sigma\rfloor\}}{\Gamma\{\lfloor\Sigma;\Pi\rfloor\}} \qquad \frac{\Gamma\{\lfloor\Sigma;\emptyset\rfloor\}}{\Gamma\{\lfloor\Sigma;\Pi\rfloor\}}$$

Lemma 3.3 *Weakening and necessitation rules are admissible in $K\triangle_d$.*

Proof of Lemma 3.3. We can show, by induction on the depth of the proof of the premiss, that we can prove the conclusion exactly the same way as the premiss, only starting with appropriate axioms – we can weaken by Δ in all the axioms already and it does not affect the proof. This is always possible because no rule application affects the context (in contrast to usual sequent calculi for modal logics). The same holds for necessitation, we simply repeat the original proof inside the triangle brackets. In fact, we obtain depth preserving admissibility in both cases. qed

Lemma 3.4 $\vdash_{K\triangle_d} \Gamma\{\Delta\}$ *iff* $\vdash_{K\triangle_d} \Gamma\{\{\Delta^c\}\}$

Proof of Lemma 3.4. We reason by induction on the complexity of Δ. If Δ consists of literals only, the conclusion follows by the invertibility of the disjunction rule. The case for boolean connectives uses the induction hypothesis and the invertibility of the rules. We spell out the only interesting case, which is the triangle modality brackets. It is sufficient to show that $\vdash_{K\triangle_d} \Gamma\{\lfloor\Delta_1;\ldots;\Delta_n\rfloor\}$ iff $\vdash_{K\triangle_d} \Gamma\{\Diamond \bigwedge_{i=1}^{n} \Delta_i^c, \Box\Delta_1^c,\ldots,\Box\Delta_n^c\}$.

By the invertibility of the modal rules we obtain the following equivalences:

$$\vdash_{K\triangle_d} \Gamma\{\Diamond \bigwedge_{i=1}^{n} \Delta_i^c, \Box\Delta_1^c,\ldots,\Box\Delta_n^c\}$$

iff

$$\vdash_{K\triangle_d} \Gamma\{\Diamond \bigwedge_{i=1}^{n} \Delta_i^c, \lfloor\{\Delta_1^c\};\ldots;\{\Delta_n^c\};\varnothing\rfloor\}$$

iff

$$\vdash_{K\triangle_d} \Gamma\{\lfloor\{\Delta_1^c, \bigwedge_{i=1}^{n} \Delta_i^c,\};\ldots;\{\Delta_n^c, \bigwedge_{i=1}^{n} \Delta_i^c,\};\{\bigwedge_{i=1}^{n} \Delta_i^c,\}\rfloor\}$$

iff

$$\vdash_{K\triangle_d} \Gamma\{\lfloor\{\Delta_1^c\};\ldots;\{\Delta_n^c\}\rfloor\}$$

iff, by the i.h.,

$$\vdash_{K\triangle_d} \Gamma\big\{\lfloor\Delta_1;\ldots;\Delta_n\rfloor\big\}.$$

Where in the third equivalence, the down direction follows by invertibility of the conjunction rule, and the up direction follows by applications of the conjunction rule (to build up the occurrences of $\bigwedge_{i=1}^{n} \Delta_i^c$), and then by admissibility of weakening (to add the singleton $\{\bigwedge_{i=1}^{n} \Delta_i^c\}$).

qed

The following Lemma is an easy observation showing in which sense the closure of a sequent is equivalent to the sequent. We skip the proof since it easily follows from the soundness and the invertibility of the rules.

Lemma 3.5 *Let* $\{\Gamma_1,\ldots,\Gamma_n\} = Cl(\Gamma)$. *For any* Σ, *Then*

(i) *the sequents* $\Sigma\big\{\Gamma_1\big\},\ldots,\Sigma\big\{\Gamma_n\big\}$ *prove the sequent* $\Sigma\big\{\Gamma\big\}$

(ii) $\Sigma\big\{\Gamma\big\}$ *proves each of the sequents* $\Sigma\big\{\Gamma_i\big\}$.

Now, let us turn to the uniform interpolation theorem. Its statement is similar to the previous case:

Theorem 3.6 *For every sequent* Γ *and a variable* p *there exists a single formula* $A_p(\Gamma)$ *such that:*

(i) $Var(A_p(\Gamma)) = Var(\Gamma) \setminus p$

(ii) $\vdash_{K\triangle_d} \overline{A_p(\Gamma)}, \Gamma$

(iii) *Whenever a sequent* $\Pi\big\{\Gamma\big\}$ *is provable where* Π *doesn't contain* p, *then the sequent* $\Pi\big\{A_p(\Gamma)\big\}$*is provable as well.*

Proof of Theorem 3.6. The construction of $A_p(\Gamma)$ is analogous to the previous case. We create a proof search tree applying the rules of $K\triangle_d$ backwards to Γ. Again, since every rule application simplifies the sequent to which it is applied, such a tree is finite. We end up with a set of least sequents – the closure of Γ. Moreover, least sequents do not contain formulas of the form $\Diamond\phi$ now, they consist of literals only. Thus we end up with a real normal form. For a sequent Γ which is not a least sequent we put

$$A_p(\Gamma) \equiv \bigwedge_{\Gamma' \in Cl(\Gamma)} A_p(\Gamma').$$

For a least sequent Γ we proceed as follows. Let Γ be of the form $\Pi, \lfloor G_1 \rfloor, \ldots, \lfloor G_n \rfloor$, recall that it consists of literals and structure marks only. For a set of literals Π, we write Π^{-p} to denote the set $\Pi \setminus \{p, \neg p\}$, only in the case Π contains both p and $\neg p$ we define Π^{-p} to be $\{\top\}$. We write Γ^{-p} for the sequent obtained from Γ replacing each set of literals Π at each level by Π^{-p}. Now

$$A_p(\Gamma) \equiv (\Gamma^{-p})^c,$$

we simply write down the characteristic formula of Γ removing all occurrences of literals $\{p, \neg p\}$ as specified above. Now part (i) of Theorem 3.6 immediately follows.

Proof of part (ii) We show $\vdash_{K\triangle_d} \overline{A_p(\Gamma)}, \Gamma$ by induction on the depth of the sequent Γ. If Γ is a set of literals, $A_p(\Gamma)$ is just $\bigvee \Gamma^{-p}$. Thus we want to show $\vdash_{K\triangle_d} \bigwedge \overline{\Gamma^{-p}}, \Gamma$, i.e. for each literal $q \in \Gamma^{-p}$ we need $\vdash_{K\triangle_d} \neg q, \Gamma$, and for each literal $\neg r \in \Gamma^{-p}$ we need $\vdash_{K\triangle_d} r, \Gamma$. But this is always the case. We only need to check what happens if $\Gamma^{-p} = \emptyset$ – but then $\bigwedge \overline{\Gamma^{-p}} \equiv \top$ and clearly $\vdash_{K\triangle_d} \bigwedge \overline{\Gamma^{-p}}, \Gamma$.

Next consider Γ is not a least sequent. Then

$$A_p(\Gamma) \equiv \bigwedge_{\Gamma_i \in Cl(\Gamma)} A_p(\Gamma_i).$$

By the induction hypothesis, each of sequents $\overline{A_p(\Gamma_i)}, \Gamma_i$ is provable, and so are the sequents $\bigvee \overline{A_p(\Gamma_i)}, \Gamma_i$. But $\bigvee \overline{A_p(\Gamma_i)} \equiv \overline{\bigwedge A_p(\Gamma_i)} \equiv \overline{A_p(\Gamma)}$ and using Lemma 3.5 we obtain $\overline{A_p(\Gamma)}, \Gamma$ provable.

Now consider Γ is a least sequent and it is of the form $\Pi, \lfloor D_1 \rfloor, \ldots, \lfloor D_n \rfloor$. Notice, that now we want to prove by induction on Γ the following:

$$\vdash_{K\triangle_d} \overline{(\Gamma^{-p})^c}, \Gamma.$$

$\overline{(\Gamma^{-p})^c}$ is a conjunction, so we want to show $\vdash_{K\triangle_d} \gamma, \Gamma$ for each of its conjuncts γ. We start with $\vdash_{K\triangle_d} \overline{(\Pi^{-p})^c}, \Pi$: this has been already done in the basic step of the inductive argument. Next we show, for each $i = 1 \ldots n$, that

$$\vdash_{K\triangle_d} \overline{\triangle\{(\Delta^{-p})^c | \Delta \in D_i\}}, \lfloor D_i \rfloor.$$

Unravelling the triangle modality and converting it to its dual, this means to show

$$\vdash_{K\triangle_d} \Box \bigvee_{\Delta \in D_i} \overline{(\Delta^{-p})^c} \wedge \bigwedge_{\Delta \in D_i} \Diamond \overline{(\Delta^{-p})^c}, \lfloor D_i \rfloor.$$

This is equivalent to showing each of the following two sequents provable:

(1)

$$\vdash_{K\triangle_d} \Box \bigvee_{\Delta\in D_i} \overline{(\Delta^{-p})^c}, \lfloor D_i \rfloor,$$

which is by Lemma 3.4 equivalent to

$$\vdash_{K\triangle_d} \Box \bigvee_{\Delta\in D_i} \overline{(\Delta^{-p})^c}, \Diamond \bigwedge_{\Delta\in D_i} \Delta^c, \{\Box\Delta^c | \Delta \in D_i\}.$$

Here is the proof, first line given by the induction hypothesis:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{(\Delta^{-p})^c}, \Delta^c, \text{ for each } \Delta \in D_i}{\bigvee_{\Delta\in D_i} \overline{(\Delta^{-p})^c}, \bigwedge_{\Delta\in D_i} \Delta^c}\,\vee \text{ and } \wedge \text{ inferences}}{\lfloor\{\bigvee_{\Delta\in D_i} \overline{(\Delta^{-p})^c}, \bigwedge_{\Delta\in D_i} \Delta^c\}\rfloor}\,\text{Nec}}{\Diamond \bigwedge_{\Delta\in D_i} \Delta^c, \lfloor\{\bigvee_{\Delta\in D_i} \overline{(\Delta^{-p})^c}\}\rfloor}\,\Diamond\text{k}}{\Diamond \bigwedge_{\Delta\in D_i} \Delta^c, \Box \bigvee_{\Delta\in D_i} \overline{(\Delta^{-p})^c}}\,\Box\text{k}$$

Now the conclusion follows by weakening.

(2) For each $\Delta \in D_i$,

$$\vdash_{K\triangle_d} \Diamond\overline{(\Delta^{-p})^c}, \lfloor D_i \rfloor,$$

which is by Lemma 3.4 equivalent to

$$\vdash_{K\triangle_d} \Diamond\overline{(\Delta^{-p})^c}, \Diamond \bigwedge_{\Delta\in D_i} \Delta^c, \{\Box\Delta^c | \Delta \in D_i\}.$$

The proof is fully analogous to the previous one.

Proof of part (iii). Suppose Π doesn't contain p, we show that whenever $\vdash_{K\triangle_d} \Pi\{\Gamma\}$, then $\vdash_{K\triangle_d} \Pi\{A_p(\Gamma)\}$ as well. But this is equivalent, by the definition of A_p and Lemma 3.4, to showing that whenever $\vdash_{K\triangle_d} \Pi\{\Gamma\}$, then $\vdash_{K\triangle_d} \Pi\{\Gamma^{-p}\}$ as well.

We reason by induction on the complexity of Γ. First suppose Γ is a set of literals and $\vdash_{K\triangle_d} \Pi\{\Gamma\}$. We show that $\vdash_{K\triangle_d} \Pi\{\Gamma^{-p}\}$. There are three possibilities why $\Pi\{\Gamma\}$ can be provable: either $\Pi\{\emptyset\}$ is provable and the conclusion follows by weakening, or there are literals q and $\neg q$

in Γ and then Γ^{-p} itself is provable (no matter if $q = p$ or not), or there is one of the literals q or $\neg q$ in Γ and the other in Π. In the last case q must be distinct from p. Let us say that $q \in \Gamma$. But then Γ^{-p} contains q and $\vdash_{K\triangle_d} \Pi\big\{\Gamma^{-p}\big\}$ follows. The remaining case is symmetrical.

Now suppose Γ is not a least sequent and $\vdash_{K\triangle_d} \Pi\big\{\Gamma\big\}$. Now by Lemma 3.5 $\vdash_{K\triangle_d} \Pi\big\{\Gamma_i\big\}$ for each $\Gamma_i \in Cl(\Gamma)$. Then by the induction hypothesis, $\vdash_{K\triangle_d} \Pi\big\{\{A_p(\Gamma_i)\}\big\}$ for each $\Gamma_i \in Cl(\Gamma)$. By the conjunction rule $\vdash_{K\triangle_d}$ $\Pi\big\{\{\bigwedge A_p(\Gamma_i)\}\big\}$, but this is $\vdash_{K\triangle_d} \Pi\big\{\{A_p(\Gamma)\}\big\}$.

Finally suppose that Γ is a least sequent (thus consisting of literals and structure marks only) and $\vdash_{K\triangle_d} \Pi\big\{\Gamma\big\}$. We want to show that $\vdash_{K\triangle_d}$ $\Pi\big\{\Gamma^{-p}\big\}$. If Γ is just a set of literals, the conclusion follows as shown above. Now observe that the same argument shows, for Γ consisting of literals, that if

$$\vdash_{K\triangle_d} \Pi\big\{\Gamma, \lfloor D_1\rfloor, \ldots, \lfloor D_n\rfloor\big\}$$

then

$$\vdash_{K\triangle_d} \Pi\big\{\Gamma^{-p}, \lfloor D_1\rfloor, \ldots, \lfloor D_n\rfloor\big\},$$

no matter if p occur in D_i or not, and it also shows that if

$$\vdash_{K\triangle_d} \Pi\big\{\Gamma, \lfloor D_1\rfloor, \ldots, \lfloor D_n\rfloor\big\}$$

then

$$\vdash_{K\triangle_d} \Pi\big\{\Gamma^{-p}, \lfloor D_1^{-p}\rfloor, \ldots, \lfloor D_n^{-p}\rfloor\big\}.$$

The reason is that the pair of literals which makes the former sequent provable cannot be separated by the triangle brackets, it must occur within the same context hole (within the same sequent) on the same level. We substantially use the fact that p does not occur in Π here, since if it did, this would be the only possibility how p could occur in Γ and $\neg p$ in Π (or vice versa), and removing one of the literals would destroy provability. This finishes the proof. qed

■ Four

Consequence Relations in Inferential Erotetic Logic

Michal Peliš

1 Introduction

The aim of this text is to study consequence relations where both declaratives and interrogatives appear together. That is, we will discuss inferential structures in the field of logic of questions. It will be seen in the next subsection that many theories belong to the range of logic of questions. Our interest is restricted to inferential erotetic logic (IEL, for short) and erotetic consequences introduced by it.

IEL was established by Andrzej Wiśniewski and his collaborators in the 1990s. The book Wiśniewski [1995] and article Wiśniewski [2001] contain a well-written presentation of Wiśniewski's approach. We utilize the framework of this theory and show some properties, relationships, and possible generalizations.

The work on this paper was supported by grant Dynamic Formal Systems IAA900090703, Grant agency of the Academy of Science of the Czech Republic. Simultaneously, I want to thank Libor Běhounek, Marta Bílková, Jeroen Groenendijk, Radek Honzík, Ondrej Majer, Jaroslav Peregrin, and Andrzej Wiśniewski for valuable discussions and comments.

Logic and questions

Referring to Kubiński and Gornstein in Harrah [2002], David Harrah mentions the long history of formal-logical approaches to interrogatives. F. Cohen (1929) and R. Carnap seem to be the first logicians working with the idea of interrogatives formalized inside logic. The real "boom" of such attempts appeared in 1950s and continued in 1960s. The reader can find very nice overviews of the history of erotetic logic in Harrah [2002], [Wiśniewski, 1995, chapter 2], and Groenendijk and Stokhof [1997].

Logic of questions is multiparadigmatic. There are many branches given by different ways of formalization of interrogatives. This is nicely illustrated by Harrah's examples of "(meta)axioms" (cf. [Harrah, 1997, pp. 25–26]). He groups them into three sets according to the acceptance by erotetic logicians.

The first group includes (meta)axioms accepted in almost all systems. Harrah calls them absolute axioms and examples are:

- Every question has at least one partial answer.
- (In systems with negation) For every statement P, there exists a question Q whose direct answers include P and the negation of P.
- Every question Q has a presupposition P such that: P is a statement, and if Q has any true direct answer, then P is true.

The second group—standard axioms—is often accepted, but not in all systems.

- Every question has at least one direct answer.
- Every direct answer is a statement.
- Every partial answer is implied by some direct answer.
- Every question is expressed by at least one interrogative.
- Each interrogative expresses exactly one question.
- Given an interrogative I there is an effective method for determining the direct answers to the question expressed by I.

The last group is called excentric axioms. Thus, the following examples of such axioms are accepted only in some interrogative systems.

- If two questions have the same direct answers, then the two questions are identical.

- Every question Q has a presupposition that is true just in case some direct answer to Q is true.

We will not attempt to provide an overview of interrogative theories, nor shall we discuss formal approaches to a question analysis. We entirely omit "pragmatical" approaches and theories based on logical analysis of natural language[1)] as well as epistemic-imperative approaches.[2)] We will, however, answer the following problems:

- What is the formal shape of questions?
- What kind of inferences are studied?

An important part of erotetic logic working with inferential structures consists of models of inquiry processes. Hintikka's interrogative model of inquiry is the best known case (cf. Hintikka *et al.* [2002]). Also IEL is presented as an alternative to Hintikka's approach (cf. Wiśniewski [2001, 2003]). Since we want to focus on consequence relations only, this topic will not be explicitly studied here.

Another important issue to solve is the question—answer relationship. Inferential erotetic logic is based on explicit set-of-answers methodology (SAM, for short), which seems to provide one of the possible solutions to this problem as well as to the problem of formal "shape" of questions.

Set-of-answers methodology

This kind of methodology is often connected with Hamblin's postulates:

1. An answer to a question is a statement.
2. Knowing what counts as an answer is equivalent to knowing the question.
3. The possible answers to a question are exhaustive set of mutually exclusive possibilities.

The vague formulation evokes discussions and enables to formulate various kinds of SAM (cf. Harrah [2002]). Inferential erotetic logic accepts

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1) Such approach can be found in Pavel Tichý's transparent intensional logic; see Tichý [1978] and Peliš [2004].

2) It must be emphasized that many terms have their origin just in these theories and we use them in an informal way.

only the first two postulates and tries to keep maximum of the (classical) declarative logic and its consequence relation.[3)]

First, let us define a (general) erotetic language $\mathcal{L}_Q$. A (general) declarative language $\mathcal{L}$ is extended by curly brackets ($\{,\}$) and question mark (?).[4)]

Simultaneously, we will use the following metavariables:

- small Greek letters $\alpha, \beta, \varphi, \ldots$ for declarative sentences,
- $Q, Q_1, \ldots$ for questions,
- capital Greek letters $\Gamma, \Delta, \ldots$ for sets of declaratives, and
- $\phi, \phi_1, \ldots$ for sets of questions.

Second, a question Q is the following structure

$$?\underbrace{\{\alpha_1, \alpha_2, \ldots\}}_{dQ}$$

In the correspondence with the first and second postulate, a question is a set of declarative sentences, which are called direct answers. The symbol dQ is used for a set of direct answers to a question Q. In the case of a question $?\{\alpha_1, \alpha_2, \ldots\}$ we expect one of the following answers:

It is the case that α_1.
It is the case that α_2.
$\vdots$

Although we have not posed any restriction on dQ and, generally, every set of declaratives is a question, it is useful to postulate that a question should have length at least two, i.e., $|dQ| = 2$. In case of finite versions of questions $?\{\alpha_1, \ldots, \alpha_n\}$ we suppose that the listed direct answers are semantically non-equivalent. The class of finite questions corresponds to the class of questions of the first kind in Wiśniewski [1995]. Some of them are important in our future examples and counterexamples. Let us mention two abbreviations and terms that are very frequent in this paper.

[3)] The full acceptance can be found in an intensional-semantic approach presented by Jeroen Groenendijk and Martin Stokhof (cf. Groenendijk and Stokhof [1990, 1997]).
[4)] In the text we use only propositional examples in the language with common connectives ($\wedge, \vee, \rightarrow, \neg$).

- Simple yes-no questions are of the form $?\alpha$, which is an abbreviation for $?\{\alpha, \neg\alpha\}$. If α is an atomic formula, then the term atomic yes-no question is used.
- A conjunctive question $?|\alpha, \beta|$ requires the answer whether α (and not β), or β (and not α), or neither α nor β, or both (α and β). It is an abbreviation for $?\{(\alpha \wedge \beta), (\neg\alpha \wedge \beta), (\alpha \wedge \neg\beta), (\neg\alpha \wedge \neg\beta)\}$. Similar versions are $?|\alpha, \beta, \gamma|$, $?|\alpha, \beta, \gamma, \delta|$, and so on.

By using the word explicit for set-of-answers methodology in inferential erotetic logic, we emphasize the role of listed direct answers.[5)] Explicit SAM brings out the answerhood conditions of questions: the meaning of interrogatives resides in the background logic and in the structure of a set of direct answers.[6)]

Consequence relations in IEL

Having answered the question What is a question in our system?, we have to move on to inferences. We believe that consequence relations are the central point of logic.[7)] Inferential erotetic logic adds three new consequence relations (mixing declaratives and interrogatives) to the standard one.

- Evocation is a binary relation $\langle\Gamma, Q\rangle$ between a set of declaratives and a question.
- Erotetic implication is a ternary relation $\langle Q, \Gamma, Q_1\rangle$ between an initial question Q and an implied question Q_1 with respect to a set of declaratives Γ.
- Reducibility is a ternary relation $\langle Q, \Gamma, \phi\rangle$ between an initial question Q and a set of questions ϕ with respect to a set of declaratives Γ.

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5) In the original version of IEL, questions are not identified with sets of direct answers: questions belong to an object-level language and are expressions of a strictly defined form, but the form is designed in such a way that, on the metalanguage level (and only here), the expression which occurs after the question mark designates the set of direct answers to the question. Questions are defined in such a way that sets of direct answers to them are explicitly specified. The general framework of IEL allows for other ways of formalizing questions. [personal communication with Andrzej Wiśniewski]

6) This connection of the meaning of questions and answerhood conditions are stated in Belnap's answerhood requirement, cf. [Groenendijk and Stokhof, 1990, p. 3].

7) Declarative logic can be defined by its consequence relation as a set of pairs $\langle\Gamma, \Delta\rangle$, where Γ and Δ are sets of formulas and Δ is usually considered to be a singleton.

Motivations and natural-language examples of both the evocation and erotetic implication can be found in Wiśniewski [1995, 2001]. Reducibility is studied in Leśniewski and Wiśniewski [2001]; Wiśniewski [1995, 2006].

The chosen shape of questions in IEL makes it possible to compare questions in the sense of their answerhood power. Inspired by Groenendijk and Stokhof [1990] and [Wiśniewski, 1995, section 5.2.3] we will examine the relationship of "giving an answer" of one question to another, which is a generalisation of Kubiński's term "weaker question".

Our aim is to study erotetic consequence relations in a very general manner, independently of the logic behind. The definitions of IEL consequences are based on the semantic entailment and the model approach relative to the chosen logical background.

Model-based approach

Let us introduce the set of all models for a declarative language as follows:

$$\mathcal{M}_{\mathsf{L}} = \{\mathbf{M} \mid \mathbf{M} \text{ is a (semantic) } \mathsf{model} \text{ for } \mathcal{L}\}.$$

The term model varies dependently on a background logic L. If L is classical propositional logic (CPL, for short), then $\mathcal{M}_{CPL}$ is a set of all valuation. In case of predicate logic it is a set of all structures with a realizations of non-logical symbols. Because of the possibility of adding some other constraints for models we will deal with (e.g., finiteness, preferred models, etc.), let us generally use a set $\mathcal{M} \subseteq \mathcal{M}_{\mathsf{L}}$. If necessary, the background logic and restrictions posed on models will be stated explicitely.

Speaking about tautologies of a logic L we mean the set of formulas

$$\textsc{Taut}_{\mathsf{L}} = \{\varphi \mid (\forall \mathbf{M} \in \mathcal{M}_{\mathsf{L}})(\mathbf{M} \models \varphi)\}.$$

If a restricted set of models $\mathcal{M}$ is in use, we speak about $\mathcal{M}$-tautologies

$$\textsc{Taut}_{\mathsf{L}}^{\mathcal{M}} = \{\varphi \mid (\forall \mathbf{M} \in \mathcal{M})(\mathbf{M} \models \varphi)\}.$$

All semantic terms will be relativized to $\mathcal{M}$. Each declarative sentence φ (in $\mathcal{L}$) has its (restricted) set of models

$$\mathcal{M}^{\varphi} = \{\mathbf{M} \in \mathcal{M} \mid \mathbf{M} \models \varphi\}$$

and similarly for a set of sentences Γ

$$\mathcal{M}^{\Gamma} = \{\mathbf{M} \in \mathcal{M} \mid (\forall \gamma \in \Gamma)(\mathbf{M} \models \gamma)\}.$$

(Semantic) entailment

Let us recall the common (semantic) entailment relation. For any set of formulas Γ and any formula ψ:

$$\Gamma \models \psi \text{ iff } \mathcal{M}^{\Gamma} \subseteq \mathcal{M}^{\psi}.$$

In case $\Gamma = \{\varphi\}$ we write only $\varphi \models \psi$.

$$\varphi \models \psi \text{ iff } \mathcal{M}^{\varphi} \subseteq \mathcal{M}^{\psi}$$

Now, we introduce multiple-conclusion entailment (mc-entailment, for short).

$$\Gamma \mid\models \Delta \text{ iff } \mathcal{M}^{\Gamma} \subseteq \bigcup_{\delta \in \Delta} \mathcal{M}^{\delta}$$

If $\mathcal{M}^{\Gamma} = \mathcal{M}^{\Delta}$, let us write $\Gamma \equiv \Delta$.[8)]

Mc-entailment is reflexive ($\Gamma \mid\models \Gamma$), but it is neither symmetric nor transitive relation:

Example 1.1 *Let* $\Gamma \subseteq \textsc{Taut}_{\mathsf{L}}$*,* Δ *be a set of sentences containing at least one tautology and at least one contradiction, and* Σ *be such that* $\bigcup_{\sigma \in \Sigma} \mathcal{M}^{\sigma} \subset \mathcal{M}_{\mathsf{L}}$*. Then* $\Gamma \mid\models \Delta$ *and* $\Delta \mid\models \Sigma$*, but* $\Gamma \mid\not\models \Sigma$*.*

Entailment is definable by mc-entailment:

$$\Gamma \models \varphi \text{ iff } \Gamma \mid\models \{\varphi\}$$

On the other hand, mc-entailment is not definable by entailment. In this context, the following theorem could be surprising at the first sight.[9)]

Theorem 1.2 *Entailment (for* L*) is compact iff mc-entailment (for* L*) is compact.*

The proof can be found in [Wiśniewski, 1995, pp. 109–110].

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8) In case of the semantic equivalence of formulas φ and ψ it will be only written $\varphi \equiv \psi$. On the other hand, two different sets of models do not imply the existence of two different sets of sentences (in $\mathcal{L}$).

9) We say that mc-entailment is compact iff for each $\Gamma \mid\models \Delta$ there are finite subsets $G \subseteq \Gamma$ and $D \subseteq \Delta$ such that $G \mid\models D$.

Basic properties of questions

As soon as we have introduced the term question and the model-based approach, we can mention some basic properties of questions. The term soundness is one of the most important terms in IEL.

Definition 1.3 *A question Q is* sound in $\mathbf{M}$ *iff* $\exists\alpha \in dQ$ *such that* $\mathbf{M} \models \alpha$.

For all IEL consequence relations, it is important to state the soundness of a question with respect to a set of declaratives.

Definition 1.4 *A question Q is* sound relative to Γ *iff* $\Gamma \mid\models dQ$.

The sum of all classes of models of each direct answer α, i.e., $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$, is called semantic range of a question Q. Considering semantic range, our liberal approach admits some strange questions; one of them is a completely contradictory question that has only contradictions in its set of direct answers, its semantic range being just $\emptyset$. Another type is a question with a tautology among its direct answers, then the semantic range expands to the whole $\mathcal{M}$. Questions with such a range are called safe.[10)] Of course, it need not be any tautology among direct answers for to be a safe question.

Definition 1.5

- *A question Q is* safe *iff* $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \mathcal{M}$.
- *A question Q is* risky *iff* $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} \subset \mathcal{M}$.

Questions $?\alpha$, $?|\alpha, \beta|$ are safe in CPL, but neither is safe in Bochvar logic. $?\{\alpha, \beta\}$ is risky in CPL. Neither $?\alpha$ nor $?|\alpha, \beta|$ are safe in intuitionistic logic, but there are safe questions in this logic; just each question with at least one tautology among direct answers. Simple yes-no questions are safe in logics that accept the law of excluded middle.

It is good to emphasize that the set of direct answers of a safe question is mc-entailed by every set of declaratives. On the other hand, knowing a question to be sound relative to every set of declaratives implies its safeness.

Fact 1.6 *Q is safe iff* $(\forall\Gamma)(\Gamma \mid\models dQ)$.

Specially, safe questions are sound relative to $\Gamma = \emptyset$.

[10)] This term originates from Nuel Belnap.

The road we are going to take

Since a background logic determines the properties of entailment, we do not want to pose any restriction on it. This model-based approach was inspired by minimal erotetic semantics from Wiśniewski [2001].

After introducing evocation and presupposition in section 2, we will show the role of maximal and prospective presuppositions in the relationship to semantic range of questions. Some classes of questions will be based on it. One could be surprised that we are not going to discuss answers in this section; in fact, there is not much to say about them. It turns out that various types of answers do not play any special role in the inferential structures.

Section 3 is crucial from the chosen viewpoint. We investigate erotetic implication and reducibility there. An important part is devoted to a discussion of the role of an auxiliary set of declaratives. We will demonstrate some variants of erotetic implication and their properties. All three consequence relations will be studied under their mutual influence and the relation “giving answer” will be stressed.

Questions will be considered as independent structures not being combined by logical connectives. Going through the definitions of erotetic implication and reducibility we can recognize their “both-sidedness” and just reducibility can substitute such combination of questions.

2 Questions and declaratives

In this section, we introduce two terms: evocation and presupposition. The first one will provide a consequence relation between a set of declaratives and a question. The second one is an important term in almost all logics of questions. In IEL there are some classes of questions based on it.

Evocation

Consider the following example: after a lecture, we expect a lecturer to be ready to answer some questions that were evoked by his or her talk. Thus, evocation seems to be the most obvious relationship among declarative sentences and questions. (Of course, next to the connection question—answer.) Almost every information can give rise to a question. What is the aim of such a question?

First, it should complete our knowledge in some direction. Asking a question we want to get more then by the conclusion based on a background knowledge. A question Q should be informative relative to Γ, it means, there is no direct answer to Q which is a conclusion of Γ.

Second, after answering an evoked question, no matter how, the answer must be consistent with the evoking knowledge. Moreover, transmission of truth into soundness is required: if an evoking set of declaratives has a model, there must be at least one direct answer of the evoked question that is true in this model. An evoked question should be sound relative to an evoking set of declaratives (see Definition 1.4).[11)]

The definition of evocation is based on the previous two points (cf. Wiśniewski [1995, 2001]). A question Q is evoked by a set of declaratives Γ if Q is sound and informative relative to Γ.

Definition 2.1 *A set of declarative sentences Γ evokes a question Q (let us write $\Gamma \models Q$) iff*

1. $\Gamma \mid\models dQ$,
2. $(\forall \alpha \in dQ)(\Gamma \not\models \alpha)$.

In our model-based approach we can rewrite both conditions this way:

1. $\mathcal{M}^\Gamma \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$
2. $(\forall \alpha \in dQ)(\mathcal{M}^\Gamma \not\subseteq \mathcal{M}^\alpha)$

In some special cases (e.g., dQ is finite or entailment is compact) we can define evocation without the link to mc-entailment. The first condition is of the form: there are $\alpha_1, \ldots, \alpha_n \in dQ$ such that $\Gamma \models \bigvee_1^n \alpha_i$.

Evocation yields some clear and useful properties of both a set of declaratives and an evoked question. The following fact lists some of them.

Fact 2.2 *If $\Gamma \models Q$, then*

- *Γ is not a contradictory set,*
- *there is no tautology in dQ, and*
- *Q is not a completely contradictory question.*

[11)] For now, as we do not discuss epistemic issues, we shall not use the word "knowledge" but use the phrases "set of declarative(s) (sentences)" or "database" instead.

However, by Fact 1.6, we obtain a less intuitive conclusion: every safe question is evoked by any Γ that does not entail any direct answer to it. It underlines the special position of safe questions and their semantic range. When we restrict the definition of evocation to risky questions only, we get the definition of generation. [Wiśniewski, 1995, chapter 6]

Generation does not solve all problems with irrelevant and inefficient evoked questions either. We can accept another restriction to avoid questions that have direct answers which are incompatible with declaratives in Γ. Borrowing an example from De Clercq and Verhoeven [2004], $\Gamma = \{\alpha \vee \beta, \gamma\}$ evokes also $?\{\alpha, \beta, \neg\gamma\}$. To eliminate this, the consistency of each direct answer with respect to Γ could be required, i.e., we could add the third condition to Definition 2.1:

$$(\forall \alpha \in dQ)(\mathcal{M}^{\Gamma} \cap \mathcal{M}^{\alpha} \neq \emptyset)$$

Some solutions of the problem of irrelevant and inefficient questions based on a semantics in the background are discussed in the just mentioned paper De Clercq and Verhoeven [2004]. For our purpose we keep Definition 2.1 unchanged.

Back to safe questions, let us mention the following fact:

Fact 2.3 *If* $\emptyset \models Q$*, then* Q *is safe.*

As a conclusion of semantic definition of evocation we have the following expected behavior of evocation: semantically equivalent databases evoke the same questions.

Fact 2.4 *For every* Γ, Δ *and* Q*, if* $\Gamma \equiv \Delta$*, then* $\Gamma \models Q$ *iff* $\Delta \models Q$*.*

If mc-entailment is compact and $\Gamma \models Q$, then we are still not allowed to conclude that there is a finite subset $\Delta \subseteq \Gamma$ such that $\Delta \models Q$. See the first item in the following fact.

Fact 2.5 *If* $\Gamma \models Q$ *and* $\Delta \subseteq \Gamma \subseteq \Sigma$*, then*

- $\Delta \models Q$ *if* $\Delta \mid\models dQ$*,*
- $\Sigma \models Q$ *if* $(\forall \alpha \in dQ)(\Sigma \not\models \alpha)$*.*

The second item points out the non-monotonicity of evocation (in declaratives). Considering questions as sets of answers, evocation is non-monotonic in interrogatives as well, see section 3.

Fact 2.6 *If* $\Gamma \models Q$ *and the entailment is compact, then* $\Delta \models Q_1$ *for some finite subset* dQ_1 *of* dQ *and some finite subset* Δ *of* Γ.

These and some more properties of evocation (and generation) are discussed in the book Wiśniewski [1995].

Presuppositions

Many properties of questions are based on the term presupposition. Everyone who has attended a basic course of research methods in social sciences has heard of importance to consider presuppositions of a question in questionaries.

What is presupposed must be valid under each answer to a question. Moreover, an answer to a question should bring at least the same information as presupposition does. The following definition (originally given by Nuel Belnap) is from Wiśniewski [1995].

Definition 2.7 *A declarative formula* φ *is a* ***presupposition*** *of a question* Q *iff* $(\forall \alpha \in dQ)(\alpha \models \varphi)$.

A presupposition of a question is entailed by each direct answer to the question. Let us write $\mathsf{Pres}Q$ for the set of all presuppositions of Q.

At the first sight, the set $\mathsf{Pres}Q$ could contain a lot of sentences. Let us have a question $Q = ?\{\alpha_1, \alpha_2\}$, the set of presuppositions (e.g., in CPL) contains $(\alpha_1 \vee \alpha_2)$, $(\alpha_1 \vee \alpha_2 \vee \varphi)$, $(\alpha_1 \vee \alpha_2 \vee \neg\varphi)$, etc. Looking at the very relevant member $(\alpha_1 \vee \alpha_2)$ it is useful to introduce the concept of maximal presupposition. Formula $(\alpha_1 \vee \alpha_2)$ entails each presupposition of the question Q.

Definition 2.8 *A declarative formula* φ *is a* ***maximal presupposition*** *of a question* Q *iff* $\varphi \in \mathsf{Pres}Q$ *and* $(\forall \psi \in \mathsf{Pres}Q)(\varphi \models \psi)$.

The model-theoretical view brings it about in a direct way. The definition of presupposition gives $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \mathcal{M}^\varphi$, for each $\varphi \in \mathsf{Pres}Q$, which means

$$\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcap_{\varphi \in \mathsf{Pres}Q} \mathcal{M}^\varphi = \mathcal{M}^{\mathsf{Pres}Q}$$

and the set $\mathcal{M}^{\mathsf{Pres}Q}$ is a model-based counterpart to the definition of maximal presuppositions.

If the background logic has tautologies, each of them is in $\mathsf{Pres}Q$.

$$\textsc{Taut}_{\mathsf{L}}^{\mathcal{M}} \subseteq \mathsf{Pres}Q$$

Considering safe questions we get

Fact 2.9 *If Q is safe, then* $\mathsf{Pres}Q = \textsc{Taut}_{\mathsf{L}}^{\mathcal{M}}$.

This fact says that if Q is safe, then $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \mathcal{M}^{\mathsf{Pres}Q}$. In classical propositional logic the disjunction of all direct answers of a question is a presupposition of this question and if $\mathsf{Pres}Q = \textsc{Taut}_{\mathsf{CPL}}^{\mathcal{M}}$, then Q is safe. This evokes a (meta)question whether the implication from right to left is valid. If Q is not safe, then we know that $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$ is a proper subset of $\mathcal{M}$. But what about $\mathcal{M}^{\mathsf{Pres}Q}$? After introducing a class of normal questions in section 2 it will be valid $\mathcal{M}^{\mathsf{Pres}Q} \subset \mathcal{M}$ as well as the implication from right to left (see Fact 2.14).

A presupposition can be seen as an information which is announced by asking a question, without answering it. Such information is relatively small. The semantic range of all maximal presuppositions is wider than the range of a question. Looking at finite CPL example where the disjunction of all direct answers forms just the semantic range of the question brings us to the idea of prospective presupposition. It is a presupposition which a question Q is sound relative to.

Definition 2.10 *A declarative formula φ is a prospective presupposition of a question Q iff $\varphi \in \mathsf{Pres}Q$ and $\varphi \,||\!\!\models dQ$. Let us write $\varphi \in \mathsf{PPres}Q$.*

All prospective presuppositions of a question are equivalent:

Lemma 2.11 *If $\varphi, \psi \in \mathsf{PPres}Q$, then $\varphi \equiv \psi$.*

Proof of Lemma 2.11. If $\mathbf{M} \models \varphi$, then there is $\alpha \in dQ$ such that $\mathbf{M} \models \alpha$. Since $\psi \in \mathsf{Pres}Q$, $\alpha \models \psi$ and it gives $\mathbf{M} \models \psi$. We got $\varphi \models \psi$.

The proof of $\psi \models \varphi$ can be done the same way. qed

A prospective presupposition forms exactly the semantic range of a question.

$$\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} = \mathcal{M}^{\mathsf{PPres}Q}$$

If Q has a prospective presupposition φ, it can be understood as the "strongest" presupposition because φ entails each presupposition of Q.

Two questions with the same sets of presuppositions have the same prospective presuppositions.

Lemma 2.12 *If $\mathsf{Pres}Q = \mathsf{Pres}Q_1$ and both $\mathsf{PPres}Q$ and $\mathsf{PPres}Q_1$ are not empty, then $\mathsf{PPres}Q = \mathsf{PPres}Q_1$.*

Proof of Lemma 2.12. We show that if $\varphi \in \mathsf{PPres}Q$ and $\psi \in \mathsf{PPres}Q_1$, then $\varphi \equiv \psi$.

$\varphi \in \mathsf{PPres}Q$ implies $\mathcal{M}^\varphi = \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$ and $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \mathcal{M}^\psi$, because $\psi \in \mathsf{Pres}Q$. It gives $\mathcal{M}^\varphi \subseteq \mathcal{M}^\psi$ and $\varphi \models \psi$.

The proof that $\psi \models \varphi$ is similar. qed

Presuppositions of evoked questions are entailed by the evoking set of declaratives.

Fact 2.13 *If* $\Gamma \models Q$, *then* $\Gamma \models \varphi$, *for each* $\varphi \in \mathsf{Pres}Q$.

The implication from right to left does not hold. If we only know $\mathcal{M}^\Gamma \subseteq \mathcal{M}^{\mathsf{Pres}Q}$, we are not sure about $\mathcal{M}^\Gamma \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$ as required by the first condition of evocation. Clearly, the informativeness must be ensured as well. Let us note that it cannot be improved by replacing of $\mathsf{Pres}Q$ by $\mathsf{PPres}Q$. We will return to this in the next subsection at the topic of normal questions. To sum up all general conditions of an evoked question (by Γ) and its presuppositions let us look at this diagram:

$$\boxed{\mathcal{M}^\Gamma \quad \subseteq \quad \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha = \quad \mathcal{M}^{\mathsf{PPres}Q} \quad \subseteq \quad \mathcal{M}^{\mathsf{Pres}Q}}$$

Classes of questions based on presuppositions

Using the term presupposition we can define some classes of questions.[12]

Normal questions A question Q is called normal if it is sound relative to its set of presuppositions ($\mathsf{Pres}Q \mid\models dQ$).

- $Q \in$ NORMAL iff $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha = \mathcal{M}^{\mathsf{Pres}Q}$

Model-based approach introduces normal questions as questions with semantic range delimitated by models of maximal presuppositions. Working with finite sets of direct answers and in logical systems with the "classical" behavior of disjunction (each direct answer entails the disjunction of all direct answers) we do not leave the class NORMAL. Non-normal questions can be found in classical predicate logic.

Two announced facts follow. They continue on the discussions at Fact 2.9 and Fact 2.13.

Fact 2.14 *If* $\mathsf{Pres}Q = \textsc{Taut}_{\mathsf{L}}^{\mathcal{M}}$ *and* Q *is normal, then* Q *is safe.*

[12] Names and definitions of the classes are from Wiśniewski [1995].

Let us only add the clear fact, that the class of safe questions is a subset of the class of normal questions ($\textsc{safe} \subseteq \textsc{normal}$).

Fact 2.15 *If* $\Gamma \models \varphi$*, for each* $\varphi \in \mathsf{Pres}Q$*, and* $\Gamma \not\models \alpha$*, for each* $\alpha \in dQ$ *of a normal question* Q*, then* $\Gamma \models Q$*.*

Regular questions Each question with the non-empty set of prospective presuppositions is regular.

- $Q \in \textsc{regular}$ iff $(\exists\varphi \in \mathsf{Pres}Q)(\varphi \mid\models dQ)$

Regularity of Q gives $\mathcal{M}^{\mathsf{Pres}Q} \subseteq \mathcal{M}^{\varphi} \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha}$ and it holds

$$\textsc{regular} \subseteq \textsc{normal}$$

If entailment is compact, both classes are equal.

Normal questions are sound relative to $\mathsf{Pres}Q$ and regular questions are sound relative to $\mathsf{PPres}Q$. The following example shows an expected fact that it is still not sufficient for evocation.

Example 2.16 (in CPL) *Let* $Q = ?\{(\alpha \vee \beta), \alpha\}$*. This question is normal and regular, the formula* $(\alpha \vee \beta)$ *is a prospective presupposition of* Q*, but* $\mathsf{Pres}Q \not\models Q$*.*

If there is a set of declaratives Γ such that $\Gamma \models Q$, then normal (regular) questions are sound as well as informative relative to $\mathsf{Pres}Q$ ($\mathsf{PPres}Q$). This is summed up by

Lemma 2.17 *Let* $\Gamma \models Q$*, for some set of declaratives* Γ*. Then*

1. $Q \in \textsc{normal}$ *implies* $\mathsf{Pres}Q \models Q$*.*
2. $Q \in \textsc{regular}$ *implies* $\varphi \models Q$*, for* $\varphi \in \mathsf{PPres}Q$*.*

Proof of Lemma 2.17. For the first item, only informativeness (relative to $\mathsf{Pres}Q$) must be showed. But if it is not valid, then Fact 2.13 causes non-informativeness of Q relative to Γ.

The second item is provable by the same idea. qed

Self-rhetorical questions Another special class of questions are self-rhetorical questions. They have at least one direct answer entailed by the set of presuppositions.

- $Q \in$ SELF-RHETORICAL iff $(\exists\alpha \in dQ)(\mathsf{Pres}Q \models \alpha)$

From this definition, it is clear that self-rhetorical questions are normal. However, do we ask such questions? This class includes such strange questions as completely contradictory questions that have only contradictions in the set of direct answers, and questions with tautologies among direct answers.

An evoked question is not of this kind.

Lemma 2.18 *If there is* Γ *such that* $\Gamma \models Q$*, then* Q *is not self-rhetorical.*

Proof of Lemma 2.18. From Fact 2.13. qed

Proper questions Normal and not self-rhetorical questions are called proper. Proper questions are evoked by their set of presuppositions.

- $Q \in$ PROPER iff $\mathsf{Pres}Q \models Q$

Evoked normal questions are proper (compare both Lemma 2.17 and Lemma 2.18).

3 Questions and questions

This section is devoted to various inferential structures in which questions appear on both sides (erotetic implication, reducibility of questions to sets of questions) and to relations between two questions based on their sets of direct answers. This second point focuses on "answerhood power" of questions formalized by set-of-answers methodology.

Erotetic implication

Now, we extend the class of inferences by "implication" between two questions with a possible assistance of some set of declaratives. Let us start with an easy and a bit tricky example. If I ask

> Q: What is Peter a graduate of: faculty of law or faculty of economy?

then I can be satisfied by the answer

He is a lawyer.

even if I did not ask

Q_1: What is Peter: lawyer or economist?

The connection between both questions could be shown by the following set of declaratives:

Someone is a graduate of a faculty of law iff he/she is a lawyer.
Someone is a graduate of a faculty of economy iff he/she is an economist.

The first question Q can be formalized by $?\{\alpha_1, \alpha_2\}$ and the latter one, speaking of Peter's position, can be $?\{\beta_1, \beta_2\}$. Looking at the questions there is no connection between them. The relationship is based on the set of declaratives $\Gamma = \{(\alpha_1 \leftrightarrow \beta_1), (\alpha_2 \leftrightarrow \beta_2)\}$. Now, we say that Q implies Q_1 on the basis of Γ and write $\Gamma, Q \models Q_1$.

This relation is called erotetic implication (e-implication, for short) and the following definition is from Wiśniewski [1995].

Definition 3.1 *A question Q implies a question Q_1 on the basis of a set of declaratives Γ iff*

1. $(\forall \alpha \in dQ)(\Gamma \cup \alpha \mid\models dQ_1)$,
2. $(\forall \beta \in dQ_1)(\exists \Delta \subset dQ)(\Delta \neq \emptyset$ *and* $\Gamma \cup \beta \mid\models \Delta)$.

Returning to the introductory example, both questions are even erotetically equivalent with respect to Γ: $\Gamma, Q \models Q_1$ as well as $\Gamma, Q_1 \models Q$.

The definition requires a little comment. The first clause should express the soundness of an implied question relative to each extension of Γ by $\alpha \in dQ$. This transmission of truth/soundness into soundness has the following meaning: if there is a model of Γ and a direct answer to Q, then there must be a direct answer to Q_1 that is valid in this model. If Q_1 is safe, then this condition is always valid (see Fact 1.6).

The second clause requires direct answers to Q_1 to be cognitively useful in restricting the set of direct answers of the implying question Q.

In comparison with evocation, the role of the set of declaratives is a bit different. Γ plays, especially, the auxiliary role; e-implication is monotonic in declaratives and it gives the following [Wiśniewski, 1995, p. 173]:

Fact 3.2 *Let* $\Gamma, Q \models Q_1$. *Then* $\Delta, \Gamma, Q \models Q_1$, *for any set of declaratives* Δ.

This could be called weakening in declaratives. From this, it is clear that $\bot, Q \models Q_1$, for each Q and Q_1.

We will say a word or two about auxiliary sets of declaratives in the next subsection.

Pure erotetic implication

Pure e-implication is e-implication with the empty set of declaratives. In our semantic approach, Γ includes only tautologies of a chosen logical system. From Fact 3.2, whenever two questions are in the relation of pure e-implication, then they are in the relation of e-implication for each set of declaratives.

If $Q \models Q_1$, then (pure) e-implication says that both questions have the same semantic range.

Lemma 3.3 *If* $Q \models Q_1$, *then* $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha = \bigcup_{\beta \in dQ_1} \mathcal{M}^\beta$.

Proof of Lemma 3.3. From the first condition of Definition 3.1

$$\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_1} \mathcal{M}^\beta$$

and from the second one

$$\bigcup_{\beta \in dQ_1} \mathcal{M}^\beta \subseteq \bigcup_{\Delta} \bigcup_{\alpha \in \Delta} \mathcal{M}^\alpha \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha.$$

qed

From this we can conclude that classes of safe and risky questions are closed under pure e-implication for both implied and implying questions.

Fact 3.4 *If* $Q \models Q_1$, *then* Q *is safe (risky) iff* Q_1 *is safe (risky).*

The same semantic range of questions linked together by pure e-implication does not form an equivalence relation on questions (see non-symmetry in Example 3.12 and non-transitivity in Example 3.13 in subsection 3). On the other hand, pure e-implication has some important conclusions for classes of presuppositions. [Wiśniewski, 1995, p. 184]

Lemma 3.5 *If* $Q \models Q_1$, *then* $\mathsf{Pres}Q = \mathsf{Pres}Q_1$.

Proof of Lemma 3.5. First, let us prove $\mathsf{Pres}Q \subseteq \mathsf{Pres}Q_1$. Let $\varphi \in \mathsf{Pres}Q$, so $\bigcup_{\alpha \in dQ} \mathcal{M}^\alpha \subseteq \mathcal{M}^\varphi$. Simultaneously, we know that from the second condition of the definition of pure e-implication there is a non-empty $\Delta \subset dQ$, for each $\beta \in dQ_1$, such that $\mathcal{M}^\beta \subseteq \bigcup_{\alpha \in \Delta} \mathcal{M}^\alpha \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$. Thus, $\mathcal{M}^\beta \subseteq \mathcal{M}^\varphi$, for each $\beta \in dQ_1$.

Second, for proving $\mathsf{Pres}Q_1 \subseteq \mathsf{Pres}Q$ suppose $\varphi \in \mathsf{Pres}Q_1$. The following inclusions are valid $\mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_1} \mathcal{M}^\beta \subseteq \mathcal{M}^\varphi$, for each $\alpha \in dQ$. qed

The claim of Lemma 3.5 is not extendable to the general term of e-implication (cf. Example 3.8).

On this lemma we can base the following statement about an influence of pure e-implication on classes of normal and regular questions.

Theorem 3.6 *If* $Q \models Q_1$, *then* Q *is normal iff* Q_1 *is normal.*

Proof of Theorem 3.6. If Q is normal, then

$$\bigcup_{\beta \in dQ_1} \mathcal{M}^\beta = \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha = \mathcal{M}^{\mathsf{Pres}Q} = \mathcal{M}^{\mathsf{Pres}Q_1}$$

The first equation is from Lemma 3.3, the second one is from the normality of Q, and the third one is from Lemma 3.5. qed

If we recall Lemma 2.12, then it is easy to say a similar fact for regular questions.

Theorem 3.7 *If* $Q \models Q_1$, *then* Q *is regular iff* Q_1 *is regular.*

Both theorems have similar results we have got for safe (risky) questions in Fact 3.4. Classes of normal and regular questions are closed to pure e-implication. Normal (regular) questions purely imply only normal (regular) questions and they are purely implied by the same kind of questions. [Wiśniewski, 1995, pp. 185–186]

Concerning classes of questions in relationship with e-implication, let us add that whenever $Q \models Q_1$, then Q is completely contradictory question iff Q_1 is.

Note on auxiliary sets of declaratives in e-implication Let us remind the introductory example on page 68 to emphasize the importance of declaratives for e-implication. Similarly, the following example will point out the role of implicitly and explicitly expressed presuppositions.

Example 3.8 (in CPL) *Let* $Q_1 = ?\{\alpha, \beta, \gamma\}$ *and* $Q_2 = ?\{\alpha, \beta\}$*, then neither* $Q_1 \models Q_2$ *nor* $Q_2 \models Q_1$ *(see the different semantic ranges of both questions). On the other hand, if we would know that it must be* $(\alpha \vee \beta)$*, then* $(\alpha \vee \beta), Q_1 \models Q_2$*.*

Keeping the context of this example: the question Q_2 is normal as well as regular, then $\mathsf{PPres}Q_2 \mid\models dQ_2$ and, in addition, there is Δ, non-empty proper subset of dQ_1, such that $\mathsf{PPres}Q_2 \mid\models \Delta$. It gives $\mathsf{PPres}Q_2, Q_1 \models Q_2$. If the set $\mathsf{PPres}Q_2$ is explicitly expressed, the implication from Q_1 to Q_2 is justified.

But now, back to the general approach. In the following fact we summarize when we can say that two questions and a set of declaratives are in the relationship of e-implication.

Fact 3.9 *Let us have* Γ *and two questions* Q_1 *and* Q_2*. In order to conclude* $\Gamma, Q_1 \models Q_2$ *it is sufficient to have* $\Gamma \mid\models dQ_2$ *and* $\Gamma \mid\models \Delta$*, where* Δ *is a non-empty proper subset of* dQ_1*.*

This fact can be formulated in this form: if Q_2 is sound relative to Γ and Γ gives a partial answer to Q_1, then Q_1 implies Q_2 with respect to Γ. We will add some points to this discussion in sections 3 and 3.

Regular erotetic implication

A special kind of e-implication arises if there is exactly one direct answer in each Δ, then we say that Q regularly implies Q_1 (on the basis of Γ). The following definition originates from Wiśniewski [2001].

Definition 3.10 *A question* Q regularly implies *a question* Q_1 *on the basis of a set of declaratives* Γ *iff*

1. $(\forall \alpha \in dQ)(\Gamma \cup \alpha \mid\models dQ_1)$,
2. $(\forall \beta \in dQ_1)(\exists \alpha \in dQ)(\Gamma \cup \beta \models \alpha)$.

Because of the special importance of this relation let us use the symbol $\vdash$ for it (so we write $\Gamma, Q \vdash Q_1$).

In the case of pure regular e-implication, both conditions are changed into the form:

1. $(\forall\alpha \in dQ)(\alpha \,||\!\!= dQ_1)$,
2. $(\forall\beta \in dQ_1)(\exists\alpha \in dQ)(\beta \models \alpha)$.

If $Q \vdash Q_1$ such that we can answer Q_1, then we have an answer to Q. The relationship of pure regular e-implication between two questions says that the implied question is "stronger" than the implying one in the sense of answerhood (see also section 3).

Regularity can be enforced by the minimal number of direct answers of an implying question: if $Q \models Q_1$ and $|dQ| = 2$, then $Q \vdash Q_1$.

Basic properties of erotetic implication

In this subsection, we are going to be interested in such properties as reflexivity, symmetry, and transitivity of e-implication.

Erotetic implication is a reflexive relation.

Fact 3.11 *$\Gamma, Q \models Q$, for each Γ and Q.*

Even if there are examples of the symmetric behavior of e-implication, it is not a symmetric relation, generally.

Example 3.12 (in CPL) *Let $Q_1 = ?\{(\alpha \vee \beta), \alpha\}$ and $Q_2 = ?\{\alpha, \beta\}$. Then $Q_1 \models Q_2$, but $Q_2 \not\models Q_1$.*

In this example there is no non-empty proper subset of dQ_2 for the formula $(\alpha \vee \beta)$ to fulfil the second condition in the definition of e-implication. Moreover, it is useful to add that Q_1 regularly implies Q_2.

Erotetic implication is not transitive either.

Example 3.13 (in CPL) *$?(\alpha\wedge\beta) \vdash ?|\alpha, \beta|$ and $?|\alpha, \beta| \models ?\alpha$, but $?(\alpha\wedge\beta) \not\models ?\alpha$.*

On the other hand, if we consider regular e-implication only, the following theorem is valid.

Theorem 3.14 *If $Q_1 \vdash Q_2$ and $Q_2 \vdash Q_3$, then $Q_1 \vdash Q_3$.*

Proof of Theorem 3.14. The first condition of Definition 3.10 is done by Lemma 3.3.

The second clause of this definition is based on regularity that gives $(\forall\gamma \in dQ_3)(\exists\alpha \in dQ_1)(\mathcal{M}^\gamma \subseteq \mathcal{M}^\alpha)$. qed

We can do a cautious strengthening by the following fact:

Fact 3.15 *If* $\Gamma, Q_1 \vdash Q_2$ *and* $Q_2 \vdash Q_3$*, then* $\Gamma, Q_1 \vdash Q_3$.

As a final remark, let us add that presuppositions of an implied question are entailed by each direct answer of an implying question (with respect to an auxiliary set of declaratives).

Fact 3.16 *Let* $\Gamma, Q \models Q_1$. *Then*

1. $(\forall \alpha \in dQ)(\forall \varphi \in \mathsf{Pres}Q_1)(\Gamma \cup \alpha \models \varphi)$
2. *If e-implication is regular, then* $(\forall \beta \in dQ_1)(\forall \varphi \in \mathsf{Pres}Q)(\Gamma \cup \beta \models \varphi)$.

Evocation and erotetic implication

Both types of inferential structures can appear together and we are going to investigate their interaction.

As shown in the next example, e-implication does not preserve evocation. If we know $\Gamma \models Q_1$ and $Q_1 \models Q_2$, it does not mean that it must be $\Gamma \models Q_2$.

Example 3.17 (in CPL)

- $\{(\alpha \vee \beta)\} \models ?|\alpha, \beta|$ *and* $?|\alpha, \beta| \models ?(\alpha \vee \beta)$*, but* $\{(\alpha \vee \beta)\} \not\models ?(\alpha \vee \beta)$.
- $\{\alpha\} \models ?|\alpha, \beta|$ *and* $\{\alpha\}, ?|\alpha, \beta| \models ?\alpha$*, but there is an answer to* $?\alpha$ *in* $\{\alpha\}$.

Of course, we do not see anything pathological in this example. Knowing $(\alpha \vee \beta)$, resp. α, it is superfluous to ask $?(\alpha \vee \beta)$, resp. $?\alpha$.

Generally speaking, this brought us back to the role of an auxiliary set of declaratives in e-implication. Due to the admissibility of weakening in declaratives (Fact 3.2) we can arrive at structures of e-implications with Γ containing (direct) answers to some of the two questions. On the other hand, there are some solutions of this problem proposed by erotetic logicians.[13)]

In contrast to the previous example, we can prove that evocation carries over through a regular e-implication.

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13) See, for example, the definition of strong e-implication given by Wiśniewski in Wiśniewski [1995]. Fact 3.9 was the inspiration for the definition of strong e-implication. The definition is the same as that of e-implication, but $\Gamma \not\models\!\!\models \Delta$ is added into the second clause.

Lemma 3.18 *If* $\Gamma \models Q_1$ *and* $Q_1 \vdash Q_2$*, then* $\Gamma \models Q_2$.

Proof of Lemma 3.18. The first condition requires $\mathcal{M}^\Gamma \subseteq \bigcup_{\beta \in dQ_2} \mathcal{M}^\beta$. It is valid because of the same semantic range of both questions.

Let us suppose that there is $\beta \in dQ_2$ entailed by Γ. Then $\mathcal{M}^\Gamma \subseteq \mathcal{M}^\beta$ and, thanks to regularity of e-implication, $\mathcal{M}^\Gamma \subseteq \mathcal{M}^\alpha$, for some $\alpha \in dQ_1$. But it is in contradiction with $\Gamma \models Q_1$. qed

Lemma 3.18 can be formulated not only in the version of pure regular e-implication.

Theorem 3.19 *If* $\Gamma \models Q_1$ *and* $\Gamma, Q_1 \vdash Q_2$*, then* $\Gamma \models Q_2$.

Proof of Theorem 3.19. First, we prove $\Gamma \mid\models dQ_2$. Supposing it is not true, then there is a model $\mathbf{M}_0$ of Γ such that $\mathbf{M}_0 \not\models \beta$, for each $\beta \in dQ_2$. Because of $\Gamma \mid\models dQ_1$, there is $\alpha_0 \in dQ_1$ and $\mathbf{M}_0 \models \alpha_0$. From $\Gamma \cup \alpha \mid\models dQ_2$, for each $\alpha \in dQ_1$, there must be some $\beta_0 \in dQ_2$ such that $\mathbf{M}_0 \models \beta_0$ and that is a contradiction.

Secondly, let us suppose that there is $\beta_0 \in dQ_2$ and $\Gamma \models \beta_0$. Regularity and second condition of e-implication give $\Gamma \cup \beta_0 \models \alpha$ and it follows $\Gamma \models \alpha$ that is in contradiction with $\Gamma \models Q_1$. qed

Since regularity was used only in the second part of the proof, we get an expected fact that $\Gamma \models Q_1$ and $\Gamma, Q_1 \vdash Q_2$ gives soundness of an implied question Q_2 relative to Γ.[14)]

At the first sight, it need not be $\Gamma, Q_1 \models Q_2$ or $\Gamma, Q_2 \models Q_1$ if we only know that $\Gamma \models Q_1$ as well as $\Gamma \models Q_2$.[15)] Generally, neither evocation nor e-implication says something new about structures of engaged questions. Nevertheless, we can expect that some clearing up of the structure of sets of direct answers could be helpful for the study of inferences. This will be discussed in the next section.

Comparing questions—relations of questions based on direct answers

So far we have introduced inferences that can produce certain relations between questions. However, it would be useful to be able to compare

14) The second part of the proof of Theorem 3.19 could be slightly changed and we obtain that strong e-implication carries over as well.

15) Let us take as an example (in CPL) the case $\{\varphi\} \models ?\alpha$ and $\{\varphi\} \models ?\beta$. Then neither $\{\varphi\}, ?\alpha \models ?\beta$ nor $\{\varphi\}, ?\beta \models ?\alpha$.

questions with respect to their "answerhood power". The chosen set-of-answers methodology brings us to a natural approach.

Let us start with relations among questions based on pure comparison of sets of direct answers.[16)]

Definition 3.20

- *Two questions are* equal *($Q_1 = Q_2$) iff they have the same set of direct answers ($dQ_1 = dQ_2$).*
- *A question Q_1 is* included *in a question Q_2 ($Q_1 \subset Q_2$) iff $dQ_1 \subset dQ_2$.*

This approach could be extended in a semantic way. We say that (an answer) α gives an answer to a question Q iff there is $\beta \in dQ$ such that $\alpha \models \beta$. Having two questions Q_1 and Q_2 we can define a relationship of "giving answers":

Definition 3.21 *A question Q_1* gives *a (direct) answer to Q_2 iff $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\alpha \models \beta)$.*

In this definition the first question is considered as to be (semantically) "stronger" than the second one. For this we use the symbol $\geq$ ($Q_1 \geq Q_2$).

If $Q_1 = Q_2$ or $Q_1 \subset Q_2$, then $Q_1 \geq Q_2$ and, moreover, each direct answer to Q_1 not only gives an answer to Q_2 but also is a (direct) answer to Q_2, i.e., $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\alpha \equiv \beta)$.

The relation $\geq$ has a slightly non-intuitive consequence: a completely contradictory question is the strongest one. However, the class of evoked questions is not affected by this problem.

Let us note an expected fact—stronger questions presuppose more than weaker ones.

Fact 3.22 *If $Q_1 \geq Q_2$, then $\mathsf{Pres}Q_2 \subseteq \mathsf{Pres}Q_1$.*

This fact is not too useful. It is better to notice the relationship among maximal presuppositions. We have $\mathcal{M}^{\mathsf{Pres}Q_1} \subseteq \mathcal{M}^{\mathsf{Pres}Q_2}$. Each maximal presupposition of a stronger question entails a maximal presupposition of a weaker one, respectively, a prospective presupposition of a stronger question entails a prospective presupposition of a weaker question. The semantic range of a stronger question is included in the semantic range of a weaker question.

[16)] The original definition refers to equivalent questions instead of equal (cf. Wiśniewski [1995]), but we use the first term for erotetically equivalent or semantically equivalent.

Fact 3.23 *If* $Q_1 \geq Q_2$*, then* $\bigcup_{\alpha \in dQ_1} \mathcal{M}^{\alpha} \subseteq \bigcup_{\beta \in dQ_2} \mathcal{M}^{\beta}$.

It follows that the set of safe questions is closed under weaker questions.

Fact 3.24 *If* Q_1 *is safe and* $Q_1 \geq Q_2$*, then* Q_2 *is safe.*

The next example shows that safeness of weaker questions is not transferred to stronger ones.

Example 3.25 (in CPL) $?\{\beta \wedge \alpha, \neg\beta\} \geq ?\beta$

Answerhood, evocation, and erotetic implication

We can show some results of evocation and e-implication based on properties of the $\geq$-relation. The first one is an obvious fact that an implied stronger question is regularly implied.

Lemma 3.26 *If* $\Gamma, Q_1 \models Q_2$ *and* $Q_2 \geq Q_1$*, then* $\Gamma, Q_1 \vdash Q_2$.

Recall what is required of the regular e-implication: $(\forall\beta \in dQ_2)(\exists\alpha \in dQ_1)(\Gamma, \beta \models \alpha)$. Then the lemma follows.

If the relation is converted, i.e., Q_1 gives an answer to Q_2, then whenever Q_1 implies Q_2, Q_2 regularly implies Q_1 (both with respect to Γ). Moreover, both questions are erotetically equivalent relative to Γ.

Theorem 3.27 *If* $\Gamma, Q_1 \models Q_2$ *and* $Q_1 \geq Q_2$*, then* $\Gamma, Q_2 \vdash Q_1$.

Proof of Theorem 3.27. First, we need to show that $\Gamma \cup \beta \mid\models dQ_1$, for each $\beta \in dQ_2$. But it is an easy conclusion from $\Gamma, Q_1 \models Q_2$ because there is a subset $\Delta \subseteq dQ_1$ for each $\beta \in dQ_2$ such that $\Gamma \cup \beta \mid\models \Delta$.

The second condition of regular e-implication is justified by the same way as it is in Lemma 3.26. qed

Now, as it was stated before, we are going to study the influence of "giving answers" on the relationship of evocation and e-implication. We know that, generally, if Γ evokes Q_1 and Q_2, it need not be that either Q_1 implies Q_2 or Q_2 implies Q_1 (with respect to Γ). If a stronger question is evoked by Γ, then every weaker question regularly implies this stronger one with respect to Γ.

Theorem 3.28 *If* $\Gamma \models Q_1$ *and* $Q_1 \geq Q_2$*, then* $\Gamma, Q_2 \vdash Q_1$.

Proof of Theorem 3.28. First, $\Gamma \cup \beta \mid\models dQ_1$ is required for each $\beta \in dQ_2$. We get $\Gamma \mid\models dQ_1$ from $\Gamma \models Q_1$.

Second, from $Q_1 \geq Q_2$ we have $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\alpha \models \beta)$ and it gives the second condition of regular e-implication $(\forall\alpha \in dQ_1)(\exists\beta \in dQ_2)(\Gamma \cup \alpha \models \beta)$. qed

To digress for a moment, this repeated connection of $\geq$ and regular e-implication is not an accident. The definition of regular e-implication says that if $Q_1 \vdash Q_2$, then Q_2 gives an answer to Q_1, i.e., $Q_2 \geq Q_1$. However, "giving an answer" does not produce e-implication, see the next example.

Example 3.29 (in CPL) $?\{(\alpha \wedge \varphi), (\beta \wedge \psi)\} \geq ?\{\alpha, \beta\}$, *but neither* $?\{(\alpha \wedge \varphi), (\beta \wedge \psi)\} \models ?\{\alpha, \beta\}$ *nor* $?\{\alpha, \beta\} \models ?\{(\alpha \wedge \varphi), (\beta \wedge \psi)\}$.

To go back to evocation, it is clear that two equal questions are both evoked by a set of declaratives if one of them is evoked by this set. Generally, it is not sufficient to know $\Gamma \models Q_1$ and $Q_1 \geq Q_2$ to conclude $\Gamma \models Q_2$. An evoked stronger question only implies the soundness of weaker questions relative to Γ. Let us illustrate it in the case that the first question is included in the second one ($Q_1 \subset Q_2$); there could be a direct answer to Q_2 which is entailed by Γ. This reminds us of the non-monotonic behavior of evocation. Notice that $Q_2 \subset Q_1$ will not help us either. In the connection with the relation $\geq$ we have to require a version of an equality.[17)]

Fact 3.30 *Let* $Q_1 \geq Q_2$ *and* $Q_2 \geq Q_1$. *Then*

- $\Gamma \models Q_1$ *iff* $\Gamma \models Q_2$,
- $Q_1 \vdash Q_2$ *as well as* $Q_2 \vdash Q_1$.

Controlling the cardinality of sets of direct answers

The set of direct answers of a weaker question can be much larger than that of a stronger question. The book Wiśniewski [1995] introduces two relations originated from Tadeusz Kubiński that prevent this uncontrolled cardinality.

Definition 3.31 *A question* Q_1 *is stronger then* Q_2 ($Q_1 \succeq Q_2$) *iff there is a surjection* $j : dQ_1 \rightarrow dQ_2$ *such that for each* $\alpha \in dQ_1$, $\alpha \models j(\alpha)$.

17) We are not going to introduce a special name for this relationship; it is included in the erotetic equivalence.

The number of direct answers of the weaker question Q_2 does not exceed the cardinality of dQ_1, i.e., $|dQ_1| \geq |dQ_2|$. From the surjection, additionally, we know that each direct answer of a weaker question is given by some direct answer to a stronger question. We have used the term stronger in a bit informal way for questions that "give an answer" to weaker ones. It is clear that if $Q_1 \succeq Q_2$, then $Q_1 \geq Q_2$. But, unfortunately, we cannot provide any special improvement of previous results for $\succeq$-relation. In particular, Examples 3.25 and 3.29 are valid for $\succeq$-relation as well.

The other definition corresponds to a both-way relationship of "being stronger".

Definition 3.32 *A question Q_1 is* equipollent *to a question Q_2 ($Q_1 \equiv Q_2$) iff there is a bijection $i : dQ_1 \rightarrow dQ_2$ such that for each $\alpha \in dQ_1$, $\alpha \equiv i(\alpha)$.*

In this case, both sets of direct answers have the same cardinality ($|dQ_1| = |dQ_2|$). Let us add expected results gained from equipollency.

Fact 3.33 *If $Q_1 \equiv Q_2$, then*

- *both $Q_1 \succeq Q_2$ and $Q_2 \succeq Q_1$,*
- *$\Gamma \models Q_1$ iff $\Gamma \models Q_2$,*
- *$Q_1 \vdash Q_2$ as well as $Q_2 \vdash Q_1$.*

Of course, two equal questions are equipollent.

Partial answerhood

We declared that the study of various types of answers (generally speaking, answerhood) is not the central point of this paper. However, we can utilize the idea evoked by the second clause of Definition 3.1. Narrowing down the set of direct answers of an implying question seems to be a good base for the term partial answer.

Definition 3.34 *A declarative φ* gives a partial answer *to a question Q iff there is a non-empty proper subset $\Delta \subset dQ$ such that $\varphi \mid\models \Delta$.*

This definition allows us to cover many terms from the concept of the answerhood. Every direct answer gives a partial answer. Whenever ψ gives a (direct) answer, then ψ gives a partial answer. As a useful conclusion we get a weaker version of Theorem 3.28:

Fact 3.35 *If* $\Gamma \models Q_1$ *and each* $\alpha \in dQ_1$ *gives a partial answer to* Q_2*, then* $\Gamma, Q_2 \models Q_1$.

Questions and sets of questions

Working in the classical logic, let us imagine we would like to know whether it is the case that α or it is the case that β. The question $?\{\alpha, \beta\}$ is posed. But there could be a problem when a device, to which we are going to address this question, is not able to accept it. (This can be caused, e.g., by a restricted language-acceptability.) However, assume that there exist two devices such that: the first one can be asked by the question $?\alpha$, and the other one is able to work with the question $?\beta$. From both machines, independently, we can get the following pairs of answers: $\{\alpha, \beta\}$, $\{\neg\alpha, \beta\}$, $\{\alpha, \neg\beta\}$ or $\{\neg\alpha, \neg\beta\}$.

Posing the question $Q = ?\{\alpha, \beta\}$ we expect that if an answer to Q is true, then there must be a true answer to each question from the set $\{?\alpha, ?\beta\}$. Thus, we require soundness condition "from an initial question to a set of questions".

Generally speaking, let us suppose that there are a question Q and a set of questions $\phi = \{Q_1, Q_2, \ldots\}$. For each model of a direct answer to Q there must be a direct answer in each Q_i valid in this model.

$$(\forall \alpha \in dQ)(\forall Q_i \in \phi)(\alpha \mid\models dQ_i)$$

Possible states (of the world) given by answers to questions in the set ϕ must be in a similar relation to the initial question. Whenever we keep a model of the choice of direct answers from each question in ϕ, then there must be a direct answer to Q true in this model. For this, let us introduce a choice function ξ such that $\xi(Q_i)$ chooses exactly one direct answer from dQ_i. For each set of questions ϕ and a choice function ξ there is a choice set $A^{\phi}_{\xi} = \{\xi(Q_i) \mid Q_i \in \phi\}$.[18] The soundness condition in the other direction ("from a set to initial question") will be expressed, generally, by $(\forall A^{\phi}_{\xi})(A^{\phi}_{\xi} \mid\models dQ)$.

Back to our example, there are four choice sets:

$$\begin{aligned} A_{\xi_1} &= \{\alpha, \beta\} \\ A_{\xi_2} &= \{\neg\alpha, \beta\} \\ A_{\xi_3} &= \{\alpha, \neg\beta\} \\ A_{\xi_4} &= \{\neg\alpha, \neg\beta\} \end{aligned}$$

18) If the set ϕ is clear from the context, we will write only A_{ξ}.

But the fourth one is not in compliance with the second soundness requirement, it is in contradiction with our (prospective) presupposition $(\alpha \vee \beta)$. If we admit the additional answer $(\neg\alpha \wedge \neg\beta)$ and a question in the form $?\{\alpha, \beta, (\neg\alpha \wedge \neg\beta)\}$, mutual soundness of this question and the set of questions $\{?\alpha, ?\beta\}$ will be valid. But this solution seems to be rather awkward. A questioner posing the question $?\{\alpha, \beta\}$ evidently presupposes $(\alpha \vee \beta)$. This will bring us to the definition of reducibility with respect to an auxiliary set of declaratives and the mutual soundness will be required in the following forms:

$$(\forall \alpha \in dQ)(\forall Q_i \in \phi)(\Gamma \cup \alpha \mid\models dQ_i)$$

and

$$(\forall A^{\phi}_{\xi})(\Gamma \cup A^{\phi}_{\xi} \mid\models dQ).$$

Our example produces more than soundness of Q relative to each A^{ϕ}_{ξ} (with respect to Γ), also efficacy of each A^{ϕ}_{ξ} with respect to a question Q is valid:

$$(\forall A^{\phi}_{\xi})(\exists \alpha \in dQ)(\Gamma \cup A^{\phi}_{\xi} \models \alpha).$$

It will be reasonable to keep this strengthening. We require to obtain at least one answer to an initial question from a choice set. Whenever Γ and A^{ϕ}_{ξ} describe the state of the world, there must be a direct answer to a question Q that does the same job.

Reducibility of questions to sets of questions

We can take advantage of the previous discussion for the direct definition of reducibility of a question to a set of questions. Now, we introduce pure reducibility that does not use any auxiliary set of declaratives.

Definition 3.36 *A question Q is* purely reducible *to a non-empty set of questions ϕ iff*

1. $(\forall \alpha \in dQ)(\forall Q_i \in \phi)(\alpha \mid\models dQ_i)$
2. $(\forall A^{\phi}_{\xi})(\exists \alpha \in dQ)(A^{\phi}_{\xi} \models \alpha)$
3. $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$

First two conditions express mutual soundness, the second one adds efficiacy, as it was discussed, and the last one requires relative simplicity. If Q is reducible to a set ϕ, we will write $Q \gg \phi$. The definition of pure reducibility was introduced by Andrzej Wiśniewski in Wiśniewski [1994].

Example 3.37 (in CPL)

- $?\{\alpha, \beta, (\neg\alpha \wedge \neg\beta)\} \gg \{?\alpha, ?\beta\}$
- $?|\alpha, \beta| \gg \{?\alpha, ?\beta\}$
- $?(\alpha \circ \beta) \gg \{?\alpha, ?\beta\}$, *where* $\circ$ *is any of the connectives:* $\wedge, \vee, \rightarrow$

In the first item, there is the "pure" version from the introductory discussion. All items display reducibilities between initial safe questions and sets of safe questions. The following theorem shows that it is not an accident.

Theorem 3.38 *If* $Q \gg \phi$*, then* Q *is safe iff each* $Q_i \in \phi$ *is safe,*

Proof of Theorem 3.38. The first condition of Definition 3.36 can be rewritten as $\bigcup_{\alpha\in dQ} \mathcal{M}^\alpha \subseteq \bigcup_{\beta\in dQ_i} \mathcal{M}^\beta$, for each $Q_i \in \phi$, and it gives the implication from left to right.

For the proof of the other implication, let us suppose $Q \gg \phi$ and that each $Q_i \in \phi$ is safe, but Q is not. It implies the existence of model $\mathbf{M}_0 \in \mathcal{M}$ such that $\mathbf{M}_0 \not\models \alpha$, for each $\alpha \in dQ$. The safeness of all Q_i gives $(\forall Q_i \in \phi)(\exists \beta \in dQ_i)(\mathbf{M}_0 \models \beta)$. Thus, there is A^ϕ_ξ made from these βs and $\mathbf{M}_0 \models A^\phi_\xi$. But it is in contradiction with the second condition of the definition of $Q \gg \phi$, which gives the existence of some $\alpha \in dQ$ such that $\mathbf{M}_0 \models \alpha$. qed

From this we know that if there is a risky question among questions in ϕ and $Q \gg \phi$, then Q must be risky too. [Wiśniewski, 1995, p. 197]

The rewritten first condition of Definition 3.36 is of the form

$$\bigcup_{\alpha\in dQ} \mathcal{M}^\alpha \subseteq \bigcap_i \bigcup_{\beta\in dQ_i} \mathcal{M}^\beta$$

and it brings out the relationship of semantic ranges. The semantic range of a reduced question is bounded by the intersection of all semantic ranges of Q_is.

The relation of pure reducibility is reflexive ($Q \gg \{Q\}$) and we can prove the following version of transitivity:

Theorem 3.39 *If* $Q \gg \phi$ *and each* $Q_i \in \phi$ *is reducible to some set of questions* ϕ_i*, then* $Q \gg \bigcup_i \phi_i$*.*

Proof of Theorem 3.39. The third condition of Definition 3.36 is clearly valid.

The first one is easy to prove. From $Q \gg \phi$ we get $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^{\beta}$, for each $Q_i \in \phi$, and from the reducibility of all Q_i in ϕ to an appropriate ϕ_i we have $\bigcup_{\beta \in dQ_i} \mathcal{M}^{\beta} \subseteq \bigcup_{\gamma \in dQ_j} \mathcal{M}^{\gamma}$, for each $dQ_j \in \phi_i$. It gives together $\bigcup_{\alpha \in dQ} \mathcal{M}^{\alpha} \subseteq \bigcup_{\gamma \in dQ_j} \mathcal{M}^{\gamma}$, for each $Q_j \in \bigcup_i \phi_i$.

For the second one we require the existence of $\alpha \in dQ$ such that $A_{\xi}^{\cup_i \phi_i} \models \alpha$, for each $A_{\xi}^{\cup_i \phi_i}$. From the reducibility of all Q_i in ϕ to an appropriate ϕ_i we have that each $A_{\xi'}^{\phi_i}$ is a subset of some $A_{\xi}^{\cup_i \phi_i}$. It implies that if there is any model $\mathbf{M}$ of $A_{\xi}^{\cup_i \phi_i}$, it must be a model of some $A_{\xi'}^{\phi_i}$. From $Q \gg \phi$ we know that there is $\alpha \in dQ$ for each choice set $A_{\xi''}^{\phi}$ on ϕ. This choice set is made by elements of all $dQ_i \in \phi$ which are valid in $\mathbf{M}$. It means that $A_{\xi''}^{\phi}$ is valid in $\mathbf{M}$ as well as $\alpha \in dQ$. qed

Now let us look at the relationship of pure reducibility and pure e-implication. The following example shows that it need not be that e-implication causes reducibility. Both definitions have the same first conditions, but the second condition of reducibility can fail.

Example 3.40 (in CPL) $?|\alpha, \beta| \models ?(\alpha \wedge \beta)$, *but* $?|\alpha, \beta| \not\gg \{?(\alpha \wedge \beta)\}$.

On the other hand, we can prove that regular e-implication implies reducibility.

Lemma 3.41 *Let* ϕ *be a set of questions such that* $Q \vdash Q_i$*, for each* $Q_i \in \phi$*. If* $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$*, then* $Q \gg \phi$*.*

Proof of Lemma 3.41. Let us prove the second condition of Definition 3.36 that requires existence of $\alpha \in dQ$ such that $\mathcal{M}^{A_{\xi}^{\phi}} \subseteq \mathcal{M}^{\alpha}$, for each A_{ξ}^{ϕ}. It is known that $\mathcal{M}^{A_{\xi}^{\phi}} \subseteq \mathcal{M}^{\beta}$, for each $\beta \in A_{\xi}^{\phi}$. If $Q \vdash Q_i$, then for each $\beta \in dQ_i$ there is $\alpha \in dQ$ such that $\mathcal{M}^{\beta} \subseteq \mathcal{M}^{\alpha}$. Thus, $\mathcal{M}^{A_{\xi}^{\phi}} \subseteq \mathcal{M}^{\alpha}$. qed

What about if we know $Q_i \models Q$ or, even, $Q_i \vdash Q$, for each $Q_i \in \phi$, and $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$, can we conclude that $Q \gg \phi$? Example 3.40 gives the negative answer to this question as well as $?(\alpha \wedge \beta) \vdash ?|\alpha, \beta|$.

Not even reducibility produces e-implication.

Example 3.42 (in CPL) $?(\alpha \wedge \beta) \gg \{?\alpha, ?\beta\}$*, but* $?(\alpha \wedge \beta)$ *is not implied neither by* $?\alpha$ *nor by* $?\beta$ *and* $?(\alpha \wedge \beta)$ *does not imply neither* $?\alpha$ *nor* $?\beta$*.*

In the next subsection we will study some special cases of links between reducibility and e-implication.

So far we have worked only with the pure reducibility. It could be useful to introduce the general term of reducibility as it was seen in the introductory example. The definition is almost the same as Definition 3.36, but the mutual soundness and efficacy conditions are supplemented by an auxiliary set of declaratives Γ (cf. Leśniewski and Wiśniewski [2001]). We will write $\Gamma, Q \gg \phi$.

Definition 3.43 *A question Q is reducible to a non-empty set of questions ϕ with respect to a set of declaratives Γ iff*

1. $(\forall \alpha \in dQ)(\forall Q_i \in \phi)(\Gamma \cup \alpha \mid\models dQ_i)$
2. $(\forall A_\xi^\phi)(\exists \alpha \in dQ)(\Gamma \cup A_\xi^\phi \models \alpha)$
3. $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$

The introductory discussion is displayed in this example:

Example 3.44 (in CPL) $(\alpha \vee \beta), ?\{\alpha, \beta\} \gg \{?\alpha, ?\beta\}$

As it is expected, the role of Γ is similar to the role of an auxiliary set of declaratives in e-implication:

Fact 3.45 *If $Q \gg \phi$, then $\Gamma, Q \gg \phi$, for each Γ.*

So we can speak of weakening in declaratives and it enables us to generalize Lemma 3.41.

Theorem 3.46 *If $\Gamma, Q \vdash Q_i$, for each $Q_i \in \phi$, and $(\forall Q_i \in \phi)(|dQ_i| \leq |dQ|)$, then $\Gamma, Q \gg \phi$.*

Proof of Theorem 3.46. The third and the first conditions of Definition 3.43 are obvious.

The second one requires that for each A_ξ^ϕ there is $\alpha \in dQ$ such that $\mathcal{M}^{\Gamma \cup A_\xi^\phi} \subseteq \mathcal{M}^\alpha$. From the construction of choice sets we know that for each A_ξ^ϕ and $Q_i \in \phi$ there is $\beta \in dQ_i$ (member of A_ξ^ϕ) such that $\mathcal{M}^{\Gamma \cup A_\xi^\phi} \subseteq \mathcal{M}^{\Gamma \cup \beta}$. The regular e-implication provides that there is $\alpha \in dQ$ for each $\beta \in dQ_i$ such that $\mathcal{M}^{\Gamma \cup \beta} \subseteq \mathcal{M}^\alpha$. qed

We close this subsection by reversing the "direction" of the reducibility relation. Let us suppose that we have generated a set of questions ϕ that are evoked by a set of declaratives Γ. Can we conclude that Γ evokes such a complex question which is reducible to the set ϕ? Generally, not. But if we know that the complex question gives an answer to some question from ϕ, the answer is positive.

Theorem 3.47 *If Γ evokes each question from a set ϕ, $Q \gg \phi$, and there is a question $Q_i \in \phi$ such that $Q \geq Q_i$, then $\Gamma \models Q$.*

Proof of Theorem 3.47. Soundness of Q relative to Γ requires the existence of an answer $\alpha \in dQ$ for each model $\mathbf{M} \models \Gamma$. From the evocation of each $Q_i \in \phi$ we have $(\forall \mathbf{M} \models \Gamma)(\forall Q_i \in \phi)(\exists \beta \in dQ_i)(\mathbf{M} \models \beta)$. So, each model of Γ produces some choice set such that $(\forall \mathbf{M} \models \Gamma)(\exists A^{\phi}_{\xi})(\mathbf{M} \models A^{\phi}_{\xi})$. Together with reducibility, where it is stated that $(\forall A^{\phi}_{\xi})(\exists \alpha \in dQ)(A^{\phi}_{\xi} \models \alpha)$, we get $\Gamma \mid\models dQ$.

Informativeness of Q with respect to Γ is justified by $\geq$-relation for some question $Q_i \in \phi$. If $\Gamma \models \alpha$, for some $\alpha \in dQ$, then it gives a contradiction with $\Gamma \models Q_i$. qed

Given the conditions of Theorem 3.47 are met, we obtain:

- $\Gamma, Q \models Q_i$, for each $Q_i \in \phi$, and
- $\Gamma, Q_i \vdash Q$, for $Q \geq Q_i$.

The first item is based on Fact 3.9 and the second one is given by the help of Theorem 3.28.

Reducibility and sets of yes-no questions

The concept of reducibility is primarily devoted to a transformation of a question to a set of "less complex" questions. The introductory discussion and its formalization in Example 3.44 evoke interesting questions:

- If we have an initial question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ with, at worse, a countable list of direct answers, is it possible to reduce it to a set of yes-no questions based only on direct answers of Q?
- Moreover, could we require the e-implication relationship between Q and questions in the set ϕ?

We can find an easy solution to these problems under condition that yes-no questions are safe and we have an appropriate set of declaratives. We will require Q to be sound with respect to Γ.

Lemma 3.48 *Let us suppose that yes-no questions are safe in the background logic. If a question* $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ *is sound with respect to a set* Γ*, then there is a set of yes-no questions* ϕ *such that* $\Gamma, Q \gg \phi$ *and* $\Gamma, Q \models Q_i$*, for each* $Q_i \in \phi$*.*

Proof of Lemma 3.48. First, we define the set of yes-no questions ϕ based on the initial question $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ such that

$$\phi = \{?\alpha_1, ?\alpha_2, \ldots\}.$$

Secondly, we prove the condition that is common for both reducibility and e-implication. The safeness of members of ϕ implies that $\mathcal{M}^\alpha \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^\beta$, for each $\alpha \in dQ$ and $Q_i \in \phi$. This gives $\mathcal{M}^{\Gamma \cup \alpha} \subseteq \bigcup_{\beta \in dQ_i} \mathcal{M}^\beta$.

To prove reducibility we have to justify the second condition of Definition 3.43. We need to find an $\alpha \in dQ$ for each A^ϕ_ξ such that $\Gamma \cup A^\phi_\xi \models \alpha$. Two cases will be distinguished.

1. If there is α from both A^ϕ_ξ and dQ, then choose this direct answer.
2. If there is no direct answer $\alpha \in dQ$ in A^ϕ_ξ, then $\mathcal{M}^{\Gamma \cup A^\phi_\xi} = \emptyset$ and we can take any α from dQ.

The final step is the proof of e-implication. We have to show that for each Q_i and each direct answer $\beta \in dQ_i$ there is a non-empty subset $\Delta \subset dQ$ such that $\Gamma \cup \beta \mid\models \Delta$. For this, we utilize the shape of questions in ϕ.

1. If $\beta \in dQ$, then Δ could be $\{\beta\}$ and $\Gamma \cup \beta \mid\models \{\beta\}$.
2. If $\beta \notin dQ$ and $\Gamma \cup \beta$ has at least one model, we recognize that $\mathcal{M}^{\Gamma \cup \beta} \subseteq \mathcal{M}^\Gamma$. Simultaneously, β must be of the form $\neg\alpha_j$ and Δ can be defined as $dQ \setminus \{\alpha_j\}$. Together with soundness of initial question Q with respect to Γ, which means $\mathcal{M}^\Gamma \subseteq \bigcup_{\alpha \in dQ} \mathcal{M}^\alpha$, we get $\mathcal{M}^{\Gamma \cup \beta} \subseteq \bigcup_{\alpha \in \Delta} \mathcal{M}^\alpha$.

qed

This lemma enables us to work with classes of questions which are known to be sound relative to sets of their presuppositions. (Normal and regular questions are the obvious example.) Whenever we know that the initial question is evoked by a set of declaratives, we get the following conclusion.

Fact 3.49 *Working in logics where yes-no questions are safe, if a question* $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ *is evoked by a set of declaratives* Γ*, then there is a set of yes-no questions* ϕ *such that* $\Gamma, Q \gg \phi$ *and* $\Gamma, Q \models Q_i$*, for each* $Q_i \in \phi$.

This fact corresponds to Lemma 1 in the paper Leśniewski and Wiśniewski [2001] where a bit different definition of reducibility is used, but the results are the same.

If dQ is finite or the entailment is compact, the set ϕ is finite set of yes-no questions. Simultaneously, it is useful to emphasize that the proof of Lemma 3.48 shows how to construct such a set.[19)]

In logics with risky yes-no questions, the first condition of reducibility as well as e-implication can fail. It need not be $\mathcal{M}^{\Gamma,\alpha} \subseteq (\mathcal{M}^{\beta} \cup \mathcal{M}^{\neg\beta})$, for each $\alpha \in dQ$ and each $?\beta \in \phi$. More generally, we can ask for a help the auxiliary set of declaratives again. Let us remind Fact 3.9 and put soundness of each $Q_i \in \phi$ with respect to Γ. Going through the proof of Lemma 3.48, the rest is valid independently of safeness of yes-no questions. As a conclusion we get

Fact 3.50 *If a question* $Q = ?\{\alpha_1, \alpha_2, \ldots\}$ *is sound relative to a set* Γ*, and there is a set of yes-no questions* $\phi = \{?\alpha_1, ?\alpha_2, \ldots\}$ *(based on* Q*) such that* $\Gamma \mid\models dQ_i$*, for each* $Q_i \in \phi$*, then* $\Gamma, Q \gg \phi$ *and* $\Gamma, Q \models Q_i$*, for each* $Q_i \in \phi$.

The construction of yes-no questions provided by Lemma 3.48 does not prevent the high complexity of such yes-no questions. Observing the last item of Example 3.37, it seems worthwhile to enquire whether it is possible to follow this process and to reduce a question (with respect to an auxiliary set of declaratives) to a set of atomic yes-no questions based on subformulas of the initial question (cf. Wiśniewski [2006]). The first restriction is clear, yes-no questions must be safe. But it is not all, the second clause of pure reducibility (Definition 3.36) requires the truth-functional connection of subformulas. Then the answer is positive. We can use repeatedly a cautious extension of Theorem 3.39:

Fact 3.51 *If* $\Gamma, Q \gg \phi$ *and each* $Q_i \in \phi$ *is reducible to some set of questions* ϕ_i*, then* $\Gamma, Q \gg \bigcup_i \phi_i$.

19) The same result is provided by theorems 7.49–7.51 in Wiśniewski [1995].

There is a similar concept based on properties of the classical logic and e-implication: erotetic search scenarios (see Wiśniewski [2001, 2003, 2004]). If we recall Example 3.13, we can recognize the "truth-functional" role of the question $?|\alpha, \beta|$ as an interlink between $?(\alpha \wedge \beta)$ and $?\alpha$, resp. $?\beta$.

■ Five

A Quick Guide to Independence Results

Radek Honzík

1 Introduction.

Many people feel a bit suspicious as regards modern set theory since many of its results are stated in terms of consistency. Arguably, this might mean that set theory doesn't tell us what is true, or at least provable, but rather what is not provable. Be it as it may, if one wants to have basic understanding of modern set theory, he or she must have some familiarity with techniques used to obtain consistency results. This article should provide introduction to these techniques without assuming special training in set theory.

Consistency results are an integral part of set theory at least for the following two reasons. First, many questions about sets do not seem to have an intuitively acceptable unique solution; this suggests that the axioms (or basic "truths") which we accept today are not rich enough. Consistency results can thus be construed as an improvement of our understanding of sets in general. Examples of such questions include the Axiom of Choice, AC or the Continuum Hypothesis, CH which we discuss in some detail in this article. Second, there exists in set theory a very general technique for showing consistency results. Other areas of mathematics are not so "lucky" and hence – with some degree of

Supported by grant GA ČR no. 401/03/H047

exaggeration – we could claim that consistency results do not occur so frequently in other areas of mathematics because people do not know how to obtain such results (this in particular true of arithmetics). This technique for obtaining consistency results in set theory is called forcing, and it is the main focus of this article.

2 Naive set theory

Majority of introductory books into the set theory, such as Kunen [1980] and Balcar and Štěpánek [2000], start with the list of axioms, sometimes with motivation for each axiom. But even with motivations this may seem unattractive to an interested reader because he or she may (justly) argue that introductions to arithmetics or analysis don't proceed in this formal fashion. It may support the (mistaken) conviction that set theory is utterly artificial since it lacks a true background from which it could draw its intuitions.

Yielding some ground to these objections, we will do set theory in this section without formal restrictions, enjoying the "naive" approach. It will become increasingly obvious, however, that its strategy is not viable and will have to be changed.

Basic truths

Though not formulated as axioms in some formal system, we will still have to say what we consider true of sets. The very fact that we have to do this shows that we are less certain with respect to sets then, for instance, with respect to natural numbers. There will be no attempt to give some "minimal" list (neither "ultimate"), nor the most economical one.

Also – although it may be superfluous and redundant for most of the readers – it is prudent to clear one possible cause of misunderstanding just at the beginning. It is fairly common to think that there are two kinds of objects in set theory: sets and elements, and that an element is in some sense basic and connot contain other elements (the notorious example of a bag of apples). This picture is of course wrong. In fact, there is only one kind of objects, namely sets; "to be an element of" is a binary relation between two sets, and this fact is denoted as $x \in y$, where x, y are sets.

For better orientation we may divide the basic truths into two groups: (i) structural properties, and (ii) algebraic properties.

Definition 2.1 *The* basic truths *are the following statements.*

(i) Structural properties

– We completely ignore the question what sets are, both in the metaphysical and physical sense.

– Extensionality. Two sets will be identical iff[1] they have the same elements; i.e. we disregard any intensional properties of the elements.

– Infinity. The natural numbers, taken all together, are a set. In set theory, this set is customarily denoted as ω.

(ii) Algebraic properties

– Pairing. For any two sets x, y there is another set $\{x, y\}$ that contains exactly the sets x, y.

– Union. For any set x there is another set $\bigcup x$ that contains all elements of all the elements of x (i.e. y is in $\bigcup x$ iff there is another set z in x, and y is in z).

Comment. This operation has an obvious connection with the $\cup$ operation known from the basic (school-taught) set theory:

$$\bigcup\{x, y\} = x \cup y.$$

$\bigcup$ is obviously more general – unlike the $\cup$ operation which joins elements from two sets, $\bigcup$ can join the elements of arbitrarily many sets (their number is determined by the size of x in $\bigcup x$).

– Power set. For any set x there is another set $\mathscr{P}(x)$ which contains exactly all the subsets of x.

– Closure under arbitrary set-operations. For any operation F from sets to sets, the image of F from a set x is also a set, i.e. $F''x = \{y \mid \exists q \in x$ such that $y = F(q)\}$ for a set x is a set.

Comment. In formulating this property we have admittedly crossed the line of what is intuitively true. But a weakening of the above property is intuitive: if P is a property and x a set then there is

[1] "Iff" is shorthand for "in and only if".

a set y which contains exactly the elements of x satisfying property P. The stronger form is however necessary even for the most elementary proofs.[2)]

The reader may wonder what about other properties which are widely known, such as axiom of choice, continuum hypothesis and other. Well these are not intuitively true (or false) so we should try to show them true or false – but before we do this, we first have to know more about sets, based on the basic properties we have just given.

We close this paragraph with a technical note. The above assumptions postulate what objects are sets. For instance if x is a set, then $\mathscr{P}(x)$ is a set. But what about $X = \{x \,|\, x \notin x\}$? We know from the Russell's paradox, that X must not be a set, or else our system is inconsistent.[3)] The crucial point here is how new sets are formed – it is tempting, and this is what Frege, who much to his misfortune introduced the so called Russell's paradox into his system for arithmetics, effectively did, to be as general as possible: a set is an arbitrary collection of objects satisfying some property. If, however, we accept this general rule to form new sets, or technically speaking this form of axiom of comprehension, we get the above contradiction – $x \notin x$ is certainly a property and X should accordingly be a set. So we must be more restrictive and include some safeguards in our system. It turns out that the statements under the heading "Closure under arbitrary set-operations" provide these safeguards. In effect, they allow a new set to be created only from a set which already exists. For instance if x is a set, then $\{y \in x \,|\, y \notin y\}$ is a set; the reader may verify for himself or herself that this time we don't get a contradiction.

To complicate things a little, the object $X = \{x \,|\, x \notin x\}$ – which we have just now excluded from our universe of sets – is nonetheless too "intuitive" to be banned altogether. In fact, under the axiom of foundation (see Definition 3.2), the object X contains all sets which exist, i.e. $X = \{x \,|\, x = x\}$. We have agreed that this is not a set, but we still want to "refer" to it for practical purposes. To cut long story short, we will distinguish two kinds of objects: sets and classes. The latter objects may be too big to be sets and we will have to treat them with some

[2)] It is necessary for definition of a function by recursion. Recursion is for instance indispensable in the proof that axiom of choice is equivalent to the claim that all sets can be wellordered. For more details about recursion, see Fact 3.1.

[3)] Denote $X = \{x \,|\, x \notin x\}$. But then $X \in X$ implies $X \notin X$, and $X \notin X$ implies $X \in X$, so we have a contradiction.

caution.[4] Some classes can also be sets, but those which are not sets are called proper classes. Canonical examples of the proper classes are the class $X = \{x \mid x = x\}$, which is customarily denoted V, or the proper class of ordinal numbers, On, to be defined below. For more rigorous treatment, see discussion after Definition 3.2.

The notion of "size" or "magnitude"

There are many things which are true about sets, but it seems most natural to concentrate on the notion of "size" since set theory is, after all, about "big" sets.

The most obvious question is if we have "more" infinity than just the set of natural numbers. There can be several approaches which clarify the idea of "more infinity"; with some generalization, we can distinguish two main types: comparison via the 1-1 functions or via the types of wellordering.[5]

The first notion is in some sense stronger (but also coarser, as the reader shall see further on) so we will take it up first.

We say that x is smaller than y if there is a 1-1 function from x to y.[6] We shall denote this relation as $|x| \leq |y|$. If $|x| \leq |y|$, but $|y| \not\leq |x|$, we write $|x| < |y|$ and say that x is strictly smaller than y. Note that this definition is reasonable in the sense that the usual "sizes" of natural numbers satisfy this property, for instance $|n| < |n+1|$ is true for all n.[7]

Theorem 2.2 (Cantor) *For any set x,*

$$|x| < |\mathscr{P}(x)|.$$

In particular, if ω denotes the set of all natural numbers,

$$|\omega| < |\mathscr{P}(\omega)|,$$

i.e. we have more infinity than just natural numbers.

---

[4] This compares nicely with the situation in arithmetics: the set of all natural numbers is certainly not a natural number, but we still (indirectly) refer to it when we claim that something is true about all natural numbers.

[5] A function f is 1-1 if no two distinct x_1, x_2 in the domain of f have the same image, i.e. $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

[6] To avoid confusion, this function is itself a set, construed as a set of ordered pairs, and must be first shown to exist.

[7] Here we use the standard set-theoretical usage which identifies a natural numbers with the set of its predecessors: $n = \{0, \ldots, n-1\}$; in particular n has n elements.

Proof of Theorem 2.2. We only show that $|\omega| < |\mathscr{P}(\omega)|$ since the generalization to an arbitrary x is immediate. We first show that (i) $|\omega| \leq |\mathscr{P}(\omega)|$, and then we show (ii) $\mathscr{P}(\omega) \not\leq |\omega|$; this implies that $|\omega| < |\mathscr{P}(\omega)|$ as required. (i) is obvious: define g to map $n \in \omega$ to $\{n\} \in \mathscr{P}(\omega)$. To prove (ii) we argue by contradiction. For contradiction suppose there is 1-1 function f from $\mathscr{P}(\omega)$ into ω. Consider the set

$$A = \{n \in \omega \mid n \notin f^{-1}(n)\},$$

for all n in the range of f. As A is a subset of ω, A is in the domain of f; let a denote the image of A under f, i.e. $f(A) = a$. Then there are two possibilities: (a) $a \in A$, but then by definition of A, $a \notin f^{-1}(a) = A$, contradiction; (b) $a \notin A$, but then $a \notin f^{-1}(a)$, and thus satisfies the property defining A, hence $a \in A$, contradiction. As both possibilities lead to contradiction, we have proved that $|\mathscr{P}(\omega)| \not\leq |\omega|$. qed

It may be added that the structure of the proof is a "positive incorporation" of the Russell paradox into the set theory. See discussion after Definition 2.1.

At first glance, the set $\mathscr{P}(\omega)$ may seem somewhat artificial. We show that practically for all purposes, it can be identified with the real numbers, $\mathbb{R}$. We shall only show that $\mathscr{P}(\omega)$ has the same size as $\mathbb{R}$, but the analogy can be taken much further.[8)]

Lemma 2.3 $|\mathscr{P}(\omega)| = |\mathbb{R}|$, *i.e. there is 1-1 function from all subsets of natural numbers onto the set of all real numbers.*

Corollary 2.4 *There are more real numbers than natural numbers.*

Sketch of proof. Though it is possible to construct directly a function f which is 1-1 and onto, we will be more lenient and demand only that $|\mathscr{P}(\omega)| \leq |\mathbb{R}|$ and $|\mathbb{R}| \leq |\mathscr{P}(\omega)|$.[9)] First notice that the size of all real numbers is the same as the size of the unit interval $(0, 1)$. Without going into details, the basic idea is to write each $r \in (0, 1)$ in its (perhaps infinite) binary expansion (i.e. using only the numbers 0 and 1) and view this expansion as a characteristic function of a subset of the natural

[8)] For those familiar with topology, in set theory $\mathbb{R}$ is customarily identified with the product topology on 2^ω (the so called Cantor discontinuum), or on ω^ω (the so called Baire space).

[9)] It is a theorem in set theory, the so called Cantor-Bernstein theorem, which claims that we are perfectly entitled to do this, i.e. that $|A| \leq |B|$ and $|B| \leq |A|$ already implies $|A| = |B|$.

numbers (i.e. if there is 1 at place n of the binary expansion, then think of n as being a member of the corresponding set; if there is 0 at place n, then n is not in the set).[10)] qed

The other type of infinity – based on wellorderings – is in some sense finer than the comparisons utilizing 1-1 embeddings.

Definition 2.5 *A binary relation $<$ is a* partial ordering *on A if the following conditions hold:*[11)]

(i) $<$ is irreflexive, i.e. for all a in A, $a \not< a$;
(ii) $<$ is transitive, i.e. for all $a, b, c \in A$, $a < b$ and $b < c$ implies $a < c$.

Notice that partial ordering doesn't demand that all members of A are comparable, i.e. it doesn't have to be true that for all $a, b \in A$, either $a < b$ holds, or else $b < a$. If it does hold that for all $a, b \in A$, either $a < b$ holds, or else $b < a$, we call such ordering linear.

Definition 2.6 *We say that A is* wellordered *by $<$ if $<$ is a partial ordering on A and for every $X \subseteq A$ the ordering $<$ restricted to X has a least element.*

Note that this definition in particular implies that $<$ on A is linear. The most prominent example of a wellordered set is $\mathbb{N}$, or in set-theoretical notation, ω, i.e. the set of natural numbers (if $X \subseteq \omega$, pick any $x \in X$; then $\{y \leq x \,|\, y \in X\}$ is finite and certainly has a least element). Also note that the whole numbers, or integers, denoted $\mathbb{Z}$, are not wellordered – the set of negative numbers doesn't have a least element. Disregarding niceties, the assumptions given in Definition 2.1 allow us to prove that there is a set that contains one more element than ω and the ordering $\leq$ on ω can be so extended that the new element is bigger than all elements of ω (in simple terms, take all elements of ω and put one new element after them all). Let us denote this set suggestively as $\omega + 1$, and the new element ω.[12)]

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10) The more careful argument has to take into consideration that for instance the number $0,00\bar{1}$ is the same as $0,01\bar{0}$ though the corresponding sets are different; a hint to the reader: when constructing a function from $\mathscr{P}(\omega)$ into $(0,1)$, think of $(0,1)$ as a disjoint union of $(0,1/2)$ and $(1/2,1)$ and map the set $0,00\bar{1}$ into one half and the set $0,01\bar{0}$ into the other.

11) We must emphasize that $<$ is in fact a set consisting of ordered pairs; we write $x < y$ to denote $\langle x, y\rangle \in <$.

12) In fact, under the formal definition, $\omega + 1$ is identified with $\omega \cup \{\omega\}$, the ordering $\leq$ being the $\in$ relation.

We now show the simple observation that using the notion of infinity based on 1-1 functions, ω has the same size as $\omega+1$, but with respect to the wellordering $\leq$, the set $\omega+1$ is strictly longer (or "bigger").

Observation 2.7 $|\omega| = |\omega+1|$, *but* $(\omega, \leq)$ *and* $(\omega+1, \leq)$ *are not order isomorphic, i.e. there is no 1-1 function* i *from* ω *onto* $\omega+1$ *such that* $n \leq m$ *iff* $i(n) \leq i(m)$.

Proof of Observation 2.7. For the first claim, define $i(0) = \omega$ and $i(n) = n-1$, for $n > 0$. The second claim follows easily from the fact that whereas in ω all elements have finitely many predecessors, in $\omega+1$, ω has infinitely many predecessors. qed

Note also, that there are many sets that are not identical, but still are order isomorphic (such as the set of all even numbers with the inherited ordering and the set of all natural numbers) – hence our result is not automatic.

We can iterate the above process, and obtain a sequence of ever longer sets

$$\ldots \omega, \omega+1, \omega+2, \ldots, \omega+\omega \ldots,$$

where $\omega+\omega$ is intuitively a set consisting of two copies of ω put one after the other. But we needn't stop at that – writing $\omega \cdot 2$ for $\omega+\omega$, we obtain

$$\omega \cdot 2, \ldots, \omega \cdot 2 + 3, \ldots \omega^2,$$

where ω^2 is written for $\omega \cdot \omega$. But again, we may continue

$$\omega^2, \ldots, \omega^2 \cdot 3 + \omega \cdot 15, \ldots \omega^3, \ldots \omega^\omega.$$

And there is no need stop here, either.

But what is perhaps surprising is that it is relatively simple to show that all these sets are still countable, i.e. there is a 1-1 function mapping them onto ω. The question presents itself whether we can reach something uncountable by this procedure.

All is about wellorderings ...

Yes, we can; and this fact follows from Theorem 2.2 and the fact that $\mathscr{P}(\omega)$ "should" be wellorderable.[13] This "should" deserves further clarification. Given a set x we could wonder whether there is a relation

[13] As a matter of fact, this assumption about wellordering of $\mathscr{P}(\omega)$ is not necessary; but it simplifies the argument and also motivates the notion of a wellordering.

$<$ such that x is wellordered by $<$. General agreement is that such $<$ should exist; the reasons for this agreement range from pragmatic issues (wellorderings add a powerful structuring to the universe and help to prove various theorems; see the discussion after Fact 2.11), to aesthetic considerations (some consider a universe without such wellordering as badly organized).

Without inquiring – at least for the time being – whether the existence of such wellorderings already follows from the other basic "truths" we have accepted earlier, we know add this principle into our stock of basic truths (it fits into the group (i) Structural properties). To be more specific, we now consider the statements WO and AC given below as true. It must be conceded, however, that the truth of WO and AC is less evident than of those given in Definition 2.1.

Definition 2.8 *The wellordering principle, WO, is the following statement: for any set x there exists a relation $<$ with domain x such that $(x, <)$ is a wellordered set.*

One of the earliest theorems in set theory, the one which has given rise to first axiomatization, see Zermelo [1904], is that WO is equivalent to famous AC, Axiom of Choice.

Definition 2.9 *The axiom of choice, AC, is the following statement: for any set x there is a function f with domain consisting of all non-empty elements of x such that if $y \in x$ and $y \neq \emptyset$, then $f(y) \in y$. Such f is sometimes called a choice function.*

In this abstract setting, AC doesn't look much appealing. However, it is known that many theorems in the usual mathematics are not provable without some form of AC; we will list just few of them:

(i) Every vector space has a basis;
(ii) Every field has a unique algebraic closure;
(iii) The Hahn-Banach Extension Theorem;
(iv) Tikhonov's Product Theorem for compact spaces.

Theorem 2.10 *WO is equivalent to AC.*

Sketch of proof. WO is easily seen to imply AC: given x let x' contain all elements of y, where $y \in x$; formally $x' = \bigcup x$. WO applied to x' yields some wellordering $<_{x'}$. Define $f(y)$, where $y \in x, y \neq \emptyset$, as the $<_{x'}$ least element of y.

The converse direction uses a transfinite recursion, which we have not defined rigorously.[14] Intuitively, apply AC to obtain an f on $\mathscr{P}(x)$. Then successively build the wellordering $<_x$ by choosing as the next element in $<_x$ the value of f applied to the subset of x which has not yet been included in the domain of $<_x$. qed

Cardinals vs. ordinals

The most intriguing property of set theory is its ability to unify seemingly diverse concepts. Above, we have introduced two different concepts of size. We now show that we may identify the sizes obtained via the 1-1 functions with some of the wellordered sets. The basic property of the wellordered sets is captured in the following result we state without a proof.[15]

Fact 2.11 *Suppose* $(A, \leq_A)$ *and* $(B, \leq_B)$ *are two wellordered sets. Then there are three possibilities,*

(i) $(A, \leq_A)$ and $(B, \leq_B)$ are order isomorphic;
(ii) $(A, \leq_A)$ can be embedded into $(B, \leq_B)$ as a proper initial segment; or
(iii) $(B, \leq_B)$ can be embedded into $(A, \leq_A)$ as a proper initial segment.

If we abstract from the concrete sets, then the above fact claims that well-ordered sets can be nicely put one after the other, and that they effectively form a stick which can be used to "measure" the length of sets (endowed with some ordering). Their role in set theory is omnipresent and cannot be overstated – they are the true generalization of the natural numbers. It turns out that if we define the equivalence on wellordered sets (two sets will be equivalent iff they are order isomorphic), then in each equivalence class we can find one special set which we declare to be the "representative" of the class.[16] This representative is simplest possible in the sense that it is of the form $(X, \in)$, i.e. the ordering is given by $\in$ (i.e. the binary relation $\in$ satisfies Definition 2.5 on the domain X).

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[14] See Fact 3.1.

[15] The proof is straightforward, but requires some technical apparatus.

[16] This is common mathematical practice. Recall that the same happens with rational numbers: 1/2 is considered the same number as 2/4, 3/6 . . ., and the representative is defined as the one with non-divisible constituents.

Definition 2.12 *These representatives are called* ordinals, *or* ordinal numbers.[17] *They are two types of ordinals:*

(i) Successor ordinals; i.e. ordinals which have an immediate predecessor. For instance all natural numbers are successor ordinals, but also the set $\omega + 1$ – which is technically speaking identical with $\omega \cup \{\omega\}$ – is a successor ordinal;
(ii) Limit ordinals; i.e. ordinals which don't have an immediate predecessor. For instance ω, i.e. the set of all natural numbers, or $\omega + \omega$ are limit ordinals.

Ordinal numbers, for short "ordinals", will be denoted by small Greek letters from the beginning of the alphabet ($\alpha, \beta, \gamma \dots$). The class of all ordinal numbers will be denoted On. Some of the members of the class On were introduced – without the term "ordinal" – at the end of Section 2.

Now, we may apply the property of being of the same size (using the 1-1 functions) to the structure of wellordered sets. We have seen that for instance ω and $\omega + 1$ are different ordinal numbers, but modulo 1-1 functions are equivalent (see Observation 2.7). Formally,

Definition 2.13 *An ordinal number κ is called a* cardinal number *if there is no 1-1 function f and an ordinal number $\alpha < \kappa$ such that f maps 1-1 the ordinal κ onto the ordinal α.*

To draw an intuitive picture: there is a wellordered class of all ordinal numbers (we may picture it as a line), representing the types of wellordered sets (in the sense of Fact 2.11), and some ordinals on this line have the additional property that there are the least in the segment of ordinal numbers with the same size with respect to 1-1 functions.

We have seen that the cardinal numbers are very rare between the ordinal numbers: recall that $\omega + 1, \omega^2, \omega^\omega, \omega^{(\omega^\omega)}$ etc. are still countable ordinals, i.e. there is a 1-1 function mapping them onto ω. But there is some cardinal above $\omega + 1, \omega^2, \omega^\omega, \omega^{(\omega^\omega)}$, as the following corollary claims:

Corollary 2.14 *Assume AC (or, equivalently, WO) for simplicity. There is an ordinal number α such that $(\alpha, \in)$ is isomorphic to $(\mathscr{P}(\omega), <)$, where $<$ is some wellordering ensured by AC, and there is no 1-1 function from α onto ω. In other words there is a cardinal above ω.*

[17] The real definition must be more careful, and is consequently more obtuse: x is ordinal number iff it is transitive and wellordered by $\in$.

Proof of Corollary 2.14. Due to Fact 2.11, the wellordering $(\mathscr{P}(\omega), <)$ implies that there is some ordinal $(\alpha, \in)$ order-isomorphic with $(\mathscr{P}(\omega))$.[18] The second part of the claim follows from Cantor's theorem 2.2. qed

The above result easily generalizes to all cardinals by induction (for instance the size of $\mathscr{P}(\mathscr{P}(\omega))$ is clearly bigger than $\mathscr{P}(\omega)$).

Corollary 2.15 *Assume AC (or, equivalently, WO) for simplicity. Then the class of all cardinal numbers is unbounded and continuous among the ordinal numbers; in particular there is a proper class of cardinal numbers.*

ALEPHS: $\aleph_\alpha$

We are now in a position to define the notion of an $\aleph$, arguably the most popularized symbol of set theory.

We will start with the following fact.

Fact 2.16 *Whenever $(X, <)$ is a proper class wellordered by $<$ with the additional property that for each $x \in X$, $\{y \in X \mid y < x\}$ is a set (not a proper class), then $(X, <)$ is isomorphic to the class of all ordinal numbers ordered by $\in$.*

Due to Fact 2.16 and Corollary 2.15, there is a function i from ordinal numbers onto the cardinal numbers which is 1-1. In other words, this function enumerates the cardinal numbers in the sense that $i(\alpha)$ is the αth cardinal number. Since i is identity on ω (i.e. $i(n) = n$ for all $n \in \omega$), it is customary to start the enumeration of the cardinals with ω, the least infinite cardinal. The function which enumerates the infinite cardinals is denoted $\aleph$, where for notational reasons $\aleph_\alpha$ is written for $\aleph(\alpha)$. For illustration, $\aleph_0 = \omega$, $\aleph_1 =$ the first cardinal greater then ω, and so on.[19]

[18] We do a little cheating here: we use the additional (unmentioned) fact that the totality of all ordinal numbers is a class – something bigger than any set; since $\mathscr{P}(\omega)$ is certainly a set (see the statement of power set property in Definition 2.1), there must be some ordinal number – an element of the class of all ordinal numbers – corresponding to $(\mathscr{P}(\omega), <)$. The point is that it could conceivably happen that what corresponds to $(\mathscr{P}(\omega), <)$ is the wellordered totality of all ordinal numbers.

[19] To avoid future cause of confusion, it is customary in set theory to use two kinds of notation for alephs: $\aleph_\alpha$, vs. ω_α. Strictly speaking, it is always true for all $\alpha \in \mathrm{On}$ that $\aleph_\alpha = \omega_\alpha$; however, the notation ω_α is used in the situations where we look at cardinal number $\aleph_\alpha$ in terms of its being an ordinal. A typical example of this convention is the notation $\aleph_{\omega_1}$, where we stress the point that we talk about a cardinality which is indexed by the first uncountable cardinal $\aleph_1 = \omega_1$.

Due to Fact 2.16, the enumeration by $\aleph$ is defined on all ordinal numbers – for any ordinal α there is the αth cardinal number, denoted $\aleph_\alpha$. By way of illustration, the following are cardinal numbers:

$$\aleph_0, \aleph_1, \cdots, \aleph_\omega = \aleph_{\aleph_0}, \cdots, \aleph_{\aleph_1} = \aleph_{\omega_1}, \cdots .$$

Note that we can now give a more precise meaning to the notation $|x| \leq |y|$ which we have introduced earlier. Under AC, every x can be wellordered, and so the size of x can be defined as the unique cardinal $\aleph_\alpha$ such that x can be mapped 1-1 in an order-preserving fashion onto $\aleph_\alpha$ (see Fact 2.11). The relation $|x| \leq |y|$ thus translates to saying that the cardinal corresponding to x is less than or equal to the cardinal corresponding to y.

Though the statement of Fact 2.16 may seem very innocuous and plausible at the first glance it should be remembered that it implies that the number of ordinal numbers and the number of cardinal numbers is the same; so even if there are great "gaps" between cardinal numbers, there is still the same number of them as of all ordinal numbers. This is paradigmatic example of results which clearly run contrary to intuition formed when dealing with finite objects.

How big is $\mathbb{R}$

The natural question now is what is the place of $\mathbb{R}$ on the $\aleph_\alpha$ scale. Theorem 2.2 and Lemma 2.3 together with AC imply that $\aleph_1 \leq |\mathbb{R}|$, but can we say something more? Surprisingly, even if the question may seem very trivial, Cantor and other set-theoreticians worked for more than 30 years and obtained only partial results.[20)]

Definition 2.17 *Continuum hypothesis, CH, claims that the size of* $\mathbb{R}$*, or* continuum*, is the least possible, i.e.* $|\mathbb{R}| = \aleph_1$*.*

Now we – as the set-theoreticians in the early 30s – have arrived at an important crossroads – if we wish to be faithful to our "naive" set-theoretical framework (i.e. no axioms, just "truths"), then all left to us would be to keep trying to decide whether the CH is true or not. With the benefit of hindsight, we know that such efforts would be hopeless.

20) The above question about the size of $\mathbb{R}$ may be reformulated as follows: does there exist an infinite set $X \subseteq \mathbb{R}$ that is neither of size ω, nor of the maximum size possible, i.e. $|\mathbb{R}|$? If there is no such set X, then indeed $|\mathbb{R}| = \aleph_1$. Cantor and others defined subsets X of $\mathbb{R}$ of ever increasing complexity and always succeeded in proving that it is either of size ω, or has the maximum size. For instance all closed subsets of $\mathbb{R}$ have this property. But they never succeeded in showing this for all subsets X.

3 Axiomatic set theory

If there is a persistent failure to decide a given statement (such as CH in the 30s), it is reasonable to ask whether it is in principle possible to decide this statement. However, in the framework of our "naive" set theory – centered on truth of statements – any inquiry about the principal feasibility of such a task is a priory meaningless. Truth is an absolute concept, every statement is true or false, regardless of our ability to determine it.

For practical reasons we may therefore decide to replace the concept of truth by something weaker. In principle, there may be innumerable ways how to do it. In practice, this weaker concept is almost exclusively taken as that of a proof inside some formal calculus. In the case of set theory, this formal calculus is almost universally based on the first-order predicate calculus.[21)]

The replacement of the notion of "truth" by the concept of "proof" has some benefits, as well as deficiencies. By restricting the domain of our inquiries to proofs[22)] we are suddenly able to show that a given statement is not decidable inside our system, i.e. is not provable from the chosen assumptions; on the other hand we implicitly exclude some truths from our formal system.[23)]

In devising the formal system, i.e. deciding upon the axioms we choose, we obviously aim at capturing as big a portion of the interesting and intuitively true statements as possible. Accordingly, if we show that a given interesting statement φ is independent of our system, i.e. is not provable nor refutable, we will construe this positively as an opportunity to increase our understanding of the given area of mathematics.We will give examples of such statements in set theory in the next paragraphs. A famous example from another part of mathematics is the 5th Euclid postulate – the realization that it is independent of the first four postulates opened doors to non-standard geometry and hence better understanding of the subject.

[21)] Arguably, the main reason for the adoption of the first order predicate calculus seems to be the ease of use of the calculus, in particular the completeness theorem and Löwenheim-Skolem theorems; in other systems some of these benefits are always lost.

[22)] For obvious reasons, the formal framework is always designed so that what is provable is also true, so that we may fail to capture every true statement, but we only capture true statements and no false ones.

[23)] In case of reasonable theories, such as set theory or even arithmetics, it can be shown that we always exclude some truths; this is of course the consequence of Gödel's incompleteness theorem.

Zermelo-Fraenkel axiomatization

The following list of axioms (formulated in the 1st order predicate calculus) is now considered as standard.

Before we give the list of axioms, we will formulate in more detail the concept of recursion, or equivalently induction. We have mentioned recursion in passing above, but from this spot on, this concept deserves more rigorous treatment. We shall not give a formal proof of the properties of the construction, but rather explain what is going on.

Fact 3.1 *For each class function G from V to V there exists a unique class function F such that for all ordinal numbers α,*

$$F(\alpha) = G(F \restriction \alpha),$$

where $F \restriction \alpha$ is the function F restricted to the domain α.

Comment. The transfinite recursion given in Fact 3.1 above is a simple generalization of the classical definition by recursion along the natural numbers – the function G can be viewed as procedure which will calculate the value of F at ordinal α based on the values F takes on $\beta < \alpha$ (we may see G as the simpler function of the two defining the more complex function F).[24)] The recursion along On must be defined at limit ordinals as well, but the principle is the same.

Now we may give the list of axioms.

Definition 3.2 *The Zermelo-Fraenkel, ZF set theory is comprised of the following axioms; cf. with the list in Definition 2.1.*[25)]

- Extensionality.

  $\forall x, y\, [x = y \leftrightarrow (\forall q\, q \in x \leftrightarrow q \in y)]$.

- Pairing.

  $\forall x, y\, \exists z\, [\forall q\, (q \in z \leftrightarrow q = x \vee q = y)]$

  Comment. We shall write $z = \{x, y\}$.

[24)] Consider the following easy example: $F(0) = 1$ and $F(n + 1) = 2^{F(n)}$. The existence of such a function follows from Fact 3.1. Note that we only need to know the value of $F(n)$ to calculate $F(n + 1)$; and hence the relevant G for this F is a function which maps n to 2^n. (This particular G is in fact obtained by recursion itself: this time the relevant G' takes n to $2n$.)

[25)] For technical reasons the listed order of the axioms is different than in Definition 2.1; for instance, it is convenient to have the pairing function defined before the axiom of infinity is stated. For more detailed and rigorous treatment of the axioms, see Balcar and Štěpánek [2000], Kunen [1980], and Jech [2003].

– Union.

$\forall x\, \exists z\, [\forall q\, (q \in z \leftrightarrow \exists y\, y \in x \wedge q \in y)]$

Comment. We shall write $z = \bigcup x$.

– Powerset.

$\forall x\, \exists z\, [\forall q\, (q \in z \leftrightarrow q \subseteq x)]$

Comment. The symbol $q \subseteq x$ is shorthand for $\forall q'\, (q' \in q \rightarrow q' \in x)$. We shall write $z = \mathscr{P}(x)$.

– Schema of replacement; closure under arbitrary set-operations.

We say that a formula $\varphi(x, y)$ determines a function if

$\forall x, y, y'\, (\varphi(x, y) \wedge \varphi(x, y') \rightarrow y = y')$

then the following is taken as an axiom:

$\varphi(x, y)$ determines a function $\rightarrow [\forall x\, \exists z\, [\forall q\, (q \in z \leftrightarrow \exists y \in x\, \varphi(y, q))]]$

Comment. The rather awkward formulation can be translated as follows: given some arbitrary function φ which doesn't even have to be a set (note it is given by a formula), the image of any set x under this function φ is also a set (z in the axiom above). Also note that there is such axiom for any formula φ; since there is infinitely many of such formulas, the axiomatization ZF is infinite.

– Infinity.

$\exists y\, [\emptyset \in y \wedge (\forall q\, (q \in y \rightarrow q \cup \{q\} \in y))]$

Comment. At this stage it would be technically cumbersome to define first the notion of natural numbers and then claim that there is the set of all natural numbers (though it is perfectly possible to do so). So for simplicity, we just claim that there is a set with a specific property, i.e. being inductive (for $q \in y$, $q \cup \{q\} \in y$); there may be more such sets, we just claim that there exists at least one. Once we define the notion of a natural number, the existence of an inductive set will imply the existence of the set of all natural numbers.

– Foundation.

Assume we have build ordinal numbers and showed Fact 3.1 based on the axioms above. The axiom of foundation claims the following:[26)]

26) Axiom of foundation can also be stated directly without using ordinal numbers.

Define class WF by recursion along the ordinal numbers On as follows:

$$\begin{aligned} WF_0 &= \emptyset \\ WF_{\alpha+1} &= \mathscr{P}(WF_\alpha) \\ WF_\lambda &= \bigcup_{\alpha<\lambda} WF_\alpha, \text{ for } \lambda \text{ limit ordinal} \\ WF &= \bigcup_{\alpha\in \mathrm{On}} WF_\alpha \end{aligned}$$

The axiom of foundation claims that every x in the universe V is present in some level WF_α, i.e. $V = WF$.

Comment. This axiom was not included in the list of truths in Definition 2.1 because its intuitive underpinning is not so obvious. Amongst its consequences is for instance the exclusion from the set-theoretic universe V of the sets x such that $x \in x$. It can be argued that the existence of sets with the property $x \in x$ is simply of no consequence for the usual arguments in set theory. Last but not least, the foundation axiom adds a very convenient structuring into the universe and consequently simplifies some arguments.

Remark. We owe the reader some rigorous comments as regards the notion of a class. If $\varphi(x, \vec{p})$ is a first order formula with parameters $\vec{p}$, then the collection $\{x \mid \varphi(x, \vec{p})\}$ is a class; we may denote this class by some letter, for instance P, and use the notation $x \in P$. But we need to remember that $x \in P$ is only a shorthand for $\varphi(x, \vec{p})$. Recalling the case of the class $X = \{x \mid x \notin x\}$, we now see that the expression $X \in X$ is meaningless since it would mean substitution of a formula for a set variable. Also note that some classes are sets;[27] classes which are not sets are called proper classes.

For the time being, we consider ZF as the complete list of assumptions which we consider true of sets. By Gödel theorem we cannot hope that ZF decides every statement in set theory, but we may still cherish hopes that it decides all interesting statements we might consider. We already have two candidates to decide: the Axiom of Choice (AC) and the Continuum Hypothesis (CH). In the next paragraph, we will introduce a set-theoretical technique known as forcing powerful enough to answer the question of decidability not only for AC or CH but for virtually any set-theoretical statement.

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[27] If x is a set, then $P = \{q \mid q \in x\}$ is a class in the sense of the above comment and is equal to x, so it denotes a set.

For the sake of completeness we shall give the exact formulation of AC, WO and CH in the first order calculus.

Definition 3.3 *The following is the reformulation of AC, WO, CH in the first-order predicate calculus.*

(i) Axiom of Choice is the following formula:

$\forall x \, \exists f \, \forall y[(y \in x \wedge y \neq \emptyset) \rightarrow y \in \mathrm{dom}(f) \wedge f(y) \in y]$,

and in view of Theorem 2.10, this is equivalent to the following statement in item (ii):

(ii) WO, the wellordering principle, is the following formula:

$\forall x \, \exists < \; [(\forall a, b, c \in x \, a \not< a \wedge (a < b \wedge b < c) \rightarrow a < c) \wedge (\forall y \subseteq x \, \exists x_0 \in y \, \forall z \in y \, (z = x_0 \vee x_0 < z))]$

(iii) Continuum Hypothesis is the following formula (where we write $2^{\aleph_0}$ for the size of $\mathscr{P}(\omega)$):

$2^{\aleph_0} = \aleph_1$.

4 Independence

In the previous paragraph, we have promised to inquire whether the statements AC and CH are decidable in ZF or not. The question of decidability is obviously composed of two parts: (i) whether AC and CH are consistent with respect to ZF, and (ii) whether ¬AC and ¬CH are consistent with respect to ZF. Formally speaking, a statement φ is consistent with respect to ZF if

$$ZF \nvdash \neg\varphi,$$

i.e. there is no proof of the negation of the statement derivable from the axioms of ZF.[28)]

Now, it is generally easier to show that some object exists, rather than verifying that no object from the infinitely many possibilities satisfies some property (for instance, being a proof of a statement). To

[28)] The meaningfulness of the following discussion of course presupposes that ZF itself is consistent, i.e. that it does not prove a contradiction; again due to Gödel, this time his second theorem, we cannot show that ZF is consistent; rather, we decide to take the consistency of ZF on belief.

transform our task to this more convenient form, we take advantage of the following basic equality between syntactical and semantical side in the predicate calculus:

$$ZF \nvdash \neg\varphi \Leftrightarrow \text{ exists a model } M \models ZF + \varphi.$$

From now on, then, to show that a statement φ is consistent amounts to finding a model for ZF + φ.[29] Note however that since we can find no model for the theory ZF due to Gödel theorem (unless we use something stronger than a set theory), we will technically speaking assume the existence of a model for ZF and from this model we shall derive a model for ZF + φ; this is known as relative consistency.

Transitive structures

All structures, or models, we will consider will be of the form $(M, \in)$ where $M \subseteq V$ is a class. In particular, the binary relation "to be an element of" will always be realized over the domain M by the predicate $\in$ (and the same goes for $=$). As is common in set theory, we "translate" the provability relation $ZF \vdash \varphi^M$ into a model-theoretic language and say that φ holds in M.[30]

Definition 4.1 *We say that a class X is* transitive *if for all $x \in X$ we have $x \subseteq X$, or in other words if $y \in x \in X$, then also $y \in X$.*

All models we will consider will have a transitive domain. We might even say, with a degree of exaggeration, that there are no other interesting models but transitive ones. The main reason for this is that a lot of basic properties is absolute for transitive models; this is a technical term, but its intuitive import as follows. When inquiring about properties holding in a model, say $(M, \in)$, we implicitly work in the universe V (everything we do in set theory takes place in V). It is of great advantage if some properties hold in M iff they hold in V – we have some understanding of V and it can help us understand the model M. We shall give a couple of specific examples.

[29] We need to excercise some care as regards the exact meaning of the word "model"; see discussion at the beginning of Section 4 for more information.

[30] φ^M is obtained from φ by restricting all quantifiers to M: $\exists x$ becomes $\exists x \in M$ and similarly for $\forall$. If M is a set, then $ZF \vdash \varphi^M$ is equivalent to $ZF \vdash (M \models \overline{\varphi})$, where $\overline{\varphi}$ is a formal translation of the metamathematical formula φ. If M is a proper class, then the relation $\models$ may not be definable, and so the notation $ZF \vdash \varphi^M$ is preferable.

Since the axiom of extensionality holds in V, we may conclude that any transitive model $(M, \in)$ satisfies the axiom of extensionality:

Lemma 4.2 *Assume M is a transitive class, then $(M, \in)$ satisfies the axiom of extensionality, i.e. $ZF \vdash (extensionality)^M$.*

Proof of Lemma 4.2. We need to verify the following formula for all $x, y \in M$: $x = y \leftrightarrow \forall q \in M\,(q \in x \leftrightarrow q \in y)$. The direction $(\rightarrow)$ is trivial. So let us take up the converse direction $(\leftarrow)$. Assume for contradiction, that there is $q_0 \in x$ and $q_0 \notin y$; however, because M is transitive, $q_0 \in M$, and by the assumption q_0 must be in y, contradiction. Notice however, that if M fails to be transitive, then existence of such q_0 cannot be ruled out. qed

As regards the absoluteness, look for instance at the notion of a function. In the set-theoretical use, a function is a set of ordered pairs, where an ordered pair $\langle x, y\rangle$ is defined as $\{\{x\}, \{x, y\}\}$. Without transitivity of M, if $f \in M$ and even $f \subseteq M$, we still cannot conclude that f is a function in M ($\langle x, y\rangle \in M$ doesn't imply that x or y is in M unless M is transitive). On the other hand if M is transitive and $f \in M$, then f is a function in V whenever f is a function in M.

Amongst the properties which are absolute for transitive models belong: the ordered pair, function, relation, ordinal numbers, the set ω, and other.

But even nicer is that some properties are not absolute for transitive models $(M, \in)$; indeed, we must remember why we want to construct such models in the first place. In constructing a model $(M, \in)$, we want to make sure that some desired property holds in M, a property which is generally much more difficult – or outright impossible – to verify in $(V, \in)$. If AC, for instance, were absolute, then $ZF \vdash AC^M$ is equivalent to $ZF \vdash AC$, and this would tell us nothing new. To emphasize: the whole point of constructing models is that some properties are not absolute.

In a nutshell, we have an almost optimal situation with transitive models $(M, \in)$: the basic properties are absolute – so we can "see" into the transitive models – but this absoluteness stops conveniently at properties which we need to show consistent.[31)]

Importantly, being a wellordering or the concept of cardinality – i.e. concepts relevant to AC or CH – are not absolute.

[31)] Formally, if $(M, \in)$ is any transitive structure, then all properties given by bounded formulas, i.e. Δ_0 formulas, are absolute; if $(M, \in)$ is a model of ZF, then all Δ_1^{ZF} formulas are absolute.

Inner models

Historically, the breakthrough on the AC came at the beginning of 30's when Gödel showed that AC is consistent with respect to ZF; this result was supplemented in 1938, see Gödel [1938], also by a result by Gödel that CH is consistent with respect to ZF. On both occasions he used a nice transitive model $L \subseteq V$ where both AC and CH hold.

First we define a concept of an inner model.

Definition 4.3 *A transitive class model* $(M, \in)$ *is an* inner model *if it contains all ordinal numbers, i.e.* $\mathrm{On} \subseteq M$*, and satisfies axioms of ZF, i.e.* $ZF \vdash \varphi^M$ *for every axiom* φ *of ZF.*

Now we will define an inner model L which will satisfy all axioms of ZF plus AC (and a lot of other things, such as CH).

Reviewing the axiom of foundation in Definition 3.2, we see that V is constructed from an empty set $\emptyset$ by iterating two simple operations: powerset $\mathscr{P}(x)$, and union $\bigcup x$. Gödel realized that the powerset operation is perhaps to generous – for the model to satisfy the axioms of ZF, it only needs to contain the definable subsets present in the universe. We will shortly make it more rigorous, but the basic idea behind the construction of L is simple enough: instead of taking the whole powerset of the earlier stage of construction (as in $\mathscr{P}(WF_\alpha)$, see the axiom of foundation), we just take the definable subsets.

Definition 4.4 *A subset* y *of the set* x *is* definable *in the model* $(x, \in)$*, to be denoted* $\mathrm{Def}_x(y)$*, if there is a formula* φ *and parameters* $\vec{p}$ *in* x *such that* y *is the set of all* q *in* x *which satisfy in* $(x, \in)$ *the property* $\varphi(q, \vec{p})$*, i.e.*

$$y = \{q \in x \mid (x, \in) \models \varphi(q, \vec{p})\}.$$

Definition 4.5 *The Gödel constructible universe L is defined as follows:*[32)]

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \{y \subseteq L_\alpha \mid \mathrm{Def}_{L_\alpha}(y)\} \\ L_\lambda &= \textstyle\bigcup_{\alpha<\lambda} L_\alpha, \ \textit{for } \lambda \textit{ limit ordinal} \\ L &= \textstyle\bigcup_{\alpha\in\mathrm{On}} L_\alpha \end{aligned}$$

Note that L is referred to as a constructible universe since the amorphous character of the powerset operation in the definition of the class $V = WF$ is restricted to subsets which are constructible from the already existing elements using a defining property.

Lemma 4.6 *L is a transitive class.*

Proof of Lemma 4.6. We will in fact show that for every ordinal α, L_α is transitive. This already implies that L is transitive: if $x \in L$, then $x \in L_\alpha$ for some α by the definition of L, and hence $x \subseteq L_\alpha \subseteq L$.

We will proceed by induction. Assume that $x \in L_\alpha$ and all L_β for $\beta < \alpha$ are transitive. If α is a limit ordinal, then $x \in L_\beta$ for some $\beta < \alpha$, and by the induction assumption $x \subseteq L_\beta \subseteq L_\alpha$. If α is a successor ordinal, say $\alpha = \beta + 1$, then by the definition of $L_{\beta+1}$, x is a subset of L_β. It follows that it is enough to show that $L_\beta \subseteq L_{\beta+1}$ because then $x \subseteq L_\beta \subseteq L_{\beta+1}$. Assume that $y \in L_\beta$, then

$$y = \{q \in L_\beta \mid L_\beta \models q \in y\} \in L_{\beta+1}$$

because by the induction assumption $y \subseteq L_\beta$. qed

We will not prove all steps necessary to show that L is an inner model, i.e. that it satisfies all axioms of ZF. But we shall prove some crucial points to give the reader some flavour of what is going on.

Theorem 4.7 *L satisfies all axioms of ZF, i.e. $ZF \vdash \varphi^L$ for every axiom φ of ZF.*

32) For an attentive reader we should add that the formulas used in the definition of the predicate Def_{L_α} cannot be the usual “metamathematical” formulas – a cursory review of the predicate Def_{L_α} shows that it contains the existential quantification over formulas; to be able to quantify over formulas, we must first build formal formulas (formal formulas are sets, and hence objects of our theory) inside our set-theoretical universe, much as the logical syntax is built inside arithmetics in the proof of Gödel incompleteness theorem.

Sketch of proof. We will just show that L satisfies the axiom of extensionality, infinity, and the powerset axiom.

(i) L satisfies the axiom of extensionality. This is a direct consequence of Lemma 4.2 and 4.6.

(ii) L satisfies the axiom of infinity. Since all finite sets are definable, $\omega \in L$ and consequently the axiom of infinity holds in L.

(iii) The powerset axiom is true in L.

We need to show the following:

$$ZF \vdash \forall x \in L \, \exists z \in L \, z = \{q \subseteq x \,|\, q \in L\}.$$

Working in V we define a function F that given a $q \subseteq x$, $q \in L$, finds the least α such that $x \in L_\alpha$. Let γ be the supremum of $\{F(q) \,|\, q \in L, q \subseteq x\}$. In L_γ we have all relevant subsets and also the set x. We now show that the powerset of x in L is a member of $L_{\gamma+1}$. To this effect we need to find a defining formula, but this is easy:

$$z = \{q \in L_\gamma \,|\, (L_\gamma, \in) \models q \subseteq x\}.$$

qed

Incidentally, this proof of powerset axiom in L again shows the benefits of transitivity. For instance, we tacitly used the fact that the property "being a subset of x" is the same in L and in V. If it were different, then the enumerating function F defined in V would be useless.[33] Notice however that although the property "to be a subset" is absolute, the collection of all such subsets is not absolute – some subsets of x existing in V can be missing in L.

Before we give arguments for the consistency of AC and CH, we need to focus on the absoluteness of the construction of L. In fact, we need to show that L constructed inside L is again L. Reasons why we need this property are mostly technical: for instance when showing that AC holds in L we need to know that a particular construction of a wellordering of L is the same in V and L. We state this result without a proof.

[33] A reader might object that we could redefine F to some F' which would be defined on subsets of x in the sense of L. After verifying that there are only set-many of these (which is not automatic, notice that in our case we used the fact that if $q \subseteq x$ in L then $q \subseteq x$ in V, and $\mathscr{P}(x)$ in V is certainly a set), we would encounter another obstacle. Defining z at stage γ, how would we know that being a subset in L is the same as being a subset in L_γ? Here again transitivity is used.

Theorem 4.8 *L satisfies the sentence $V = L$, i.e. $ZF \vdash (V = L)^L$. It follows that if ZF is consistent, so $ZF + V = L$.*

Finally we may turn to the proof of consistency of AC and CH with respect to ZF. It should be emphasized however that the theorem leaves open the question whether in fact ZF already proves AC or CH (see Section 4).

Theorem 4.9 *L satisfies AC, i.e. $ZF \vdash (AC)^L$. This implies that if ZF is consistent, then $ZF \nvdash \neg AC$.*

Sketch of proof. We first show that $ZF \vdash (AC)^L$ implies $ZF \nvdash \neg AC$. We reason by contradiction: if $ZF \vdash \neg AC$, then as L satisfies all axioms of ZF, it also needs to satisfy all consequences of ZF – in particular $ZF \vdash (\neg AC)^L = \neg(AC)^L$; but as also $ZF \vdash (AC)^L$, and this contradicts the consistency of ZF.

We have stated above that AC is equivalent to the statement that all sets can be wellordered (WO). Instead of finding a distinct wellordering for every set, we will find a single (class) wellordering $<_L$ definable in L which wellorders all sets in L at once.

Notice that if x and y are in L and there are $\alpha < \beta$ such that x first appears in L_α and y first appears in L_β, then we can postulate that $x <_L y$ (here we use the fact that ordinals themselves are wellordered).

So it remains to define $<_L$ on the individual levels L_α (if x, y first appear at the same L_α, then we have to say which is the smaller one in the desired $<_L$). As the limit stages are just unions of the previous stages, it is enough to say how to extend $<_L$ from L_α to the next level $L_{\alpha+1}$. Let $\langle \varphi_i \,|\, i < \omega \rangle$ be some fixed enumeration of all formulas (there are only countably many of these); by induction assumption $<_L$ already wellorders L_α. Assume x, y first appear in $L_{\alpha+1}$ and x is defined by a formula φ_x using a single parameter p and y by a formula φ_y using a parameter r,[34] i.e.

$$x = \{q \in L_\alpha \,|\, L_\alpha \models \varphi_x(q, p)\} \text{ and } y = \{q \in L_\alpha \,|\, L_\alpha \models \varphi_y(q, r)\}.$$

We set $x <_L y$ iff the formula φ_x comes before the formula φ_y in the enumeration $\langle \varphi_i \,|\, i < \omega \rangle$ or if $\varphi_x = \varphi_y$, then p comes before r in the enumeration $<_L$ on L_α.

Now we are done, or rather almost done. We have constructed in V an ordering $<_L$ which wellorders L. But we need more: $ZF \vdash (<_L$ is

[34] The case with more parameters is essentially the same.

a wellordering of the universe)^L, i.e. $<_L$ should be definable in L and L should think that it is indeed a wellordering. Fortunately, we have Theorem 4.8 which is enough to argue that this is really the case. qed

Now we turn to CH.

Theorem 4.10 *L satisfies CH, i.e. $ZF \vdash (CH)^L$. This implies that if ZF is consistent, then $ZF \nvdash \neg CH$.*

Sketch of proof. The technical apparatus needed to show that CH holds in L goes well beyond the scope of this article. But some intuitive hints are readily available. Recall that CH says that there are as few subsets of ω as possible. The definition of $V = WF$ leaves a great leeway as regards the potential subsets of ω, not so L however. Notice that the subsets of ω in L are only constructed by using countably many formulas plus parameters from earlier levels of L. It is not all that easy so as to claim that all subsets of ω are in $L_{\omega+1}$ – this is not true, but it can be shown[35] that the levels of L where a new subset of ω can be created is bounded by $\omega_1 = \aleph_1$. Since at each L_α, $\alpha < \omega_1$, we have only countably many parameters to choose from and also have only countably many formulas to construct new sets, there can be at maximum $\aleph_1$ many of these.

The proof of CH in L easily generalizes to other cardinalities as well: for all $\alpha \in \mathrm{On}$,

$$ZF \vdash (2^{\aleph_\alpha} = \aleph_{\alpha+1})^L.$$

This statement is known as the generalized continuum hypothesis, GCH. qed

As ZF + AC is consistent, we can enlarge our theory and include AC.

Definition 4.11 *ZFC will denote the theory ZF + AC.*

Forcing

By the above-mentioned results of Gödel, the consistency of AC and CH was decided in the 30s; but we have already mentioned that this result

[35] The proof to show this uses a rather remarkable property of L, called condensation: if $(X, \in)$ is an elementary substructure of $(L_\lambda, \in)$, where λ is a limit ordinal, then in fact $(X, \in)$ is isomorphic to some $(L_\alpha, \in)$, for $\alpha \leq \lambda$. So if $x \subseteq \omega$ is in L_γ, we take the Skolem hull inside L_γ containing x, obtaining a countable structure which is due to the condensation property isomorphic to some $(L_\alpha, \in)$ for $\alpha < \omega_1$. After making sure that $x \in L_\alpha$, we are done.

answers just half the question – the other half being the consistency of ¬AC or ¬CH. An obvious strategy would be to use the technique successful in the case of AC and CH and define a suitable inner model $M \subseteq V$ where ¬AC or ¬CH would hold. However, it turns out that we cannot do this:

Theorem 4.12 *Assume that ZF is consistent. There is no inner model M such that M satisfies $\neg CH$ or $\neg AC$ in ZF, or formally, there is no inner model M such that $ZF \vdash (\neg CH)^M$ or $ZF \vdash (\neg AC)^M$.*

Proof of Theorem 4.12. We shall use the following fact (note that this Fact implies Theorem 4.8: L is an inner model, and hence $L^L = L$ by this Fact):

Fact 4.13 *Let M be an inner model. If the definition of L is repeated*[36] *inside M instead of V – the result of this construction is denoted L^M –, then we obtain the same L as when the construction is carried out in V, i.e. $L = L^M$. In particular, this implies that $L \subseteq M$.*

Assume for contradiction that $ZF \vdash (\neg CH)^M$ for some inner model M (the case of $(AC)^M$ is analogous). By the above Fact, $ZF \vdash L \subseteq M$; but it cannot be true that $L = M$ as otherwise we would have $ZF \vdash (CH)^M$ by Theorem 4.10 and $ZF \vdash (\neg CH)^M$ by our assumption, and this is a contradiction. It follows that $ZF \vdash L \subsetneq M \subseteq V$, i.e. $ZF \vdash L \neq V$, but this contradicts Theorem 4.8. qed

The impossibility to use inner models to show the consistency of ¬CH and ¬AC stalled the progress on this problem for the next 30 years. It was only in the 60s when Paul Cohen managed to develop a new technique which finally succeeded in showing the consistency of ¬CH and ¬AC, see Cohen [1963].

The idea of Cohen is both simple and ingenious: if we cannot use a transitive model M such that $M \subseteq V$, then what about going outside the universe V? In other words, we may wish to construct a transitive class M, containing all ordinal numbers and satisfying the axioms of ZF such that $M \supseteq V$. Taken literally, this idea is obviously not workable; by

[36] Recall that as M is an inner model, it satisfies all axioms of ZF; in particular all apparatus used in V when L is constructed is available inside M as well. But it is conceivable that the same apparatus used in V yields another result when applied in M. For instance, in both models we can form the power set of ω, but the actual sets representing the power set in M and in V can be different – depending on the subsets available in V, or in M.

definition, there can be nothing outside V – after all, V is the universe containing all sets.

But set theory is very flexible in its means: though apparently contradictory, the idea of an outer model can be formalized inside ZF with surprising ease. First we realize that if we assume there is a transitive set model M of ZFC,[37] then it suffices to extend the "smaller" universe M, instead of V. In fact, what Lemma 4.12 says is that if M is a transitive model (set-like, or a proper class), then there is no M-definable transitive model $N \subseteq M$ satisfying sentences contradictory with $V = L$ – however, the situation of $M \subseteq N$, where N is defined in V, is permissible.[38]

Consequently, instead of starting with V, we can start with some transitive set model of ZF, called M, extend M by some transitive $N \supseteq M$ and plan this construction in such a clever way to ensure that N also satisfies ZF.

Extending the universe

In the following sections, we shall briefly describe how a universe can be extended into a larger universe, also satisfying ZF. The emphasis will be on connections and background, whereas the technicalities will be suppressed. An interested reader can find an excellent rigorous treatment in Kunen [1980].

Recalling the discussion from the previous section, we fix a transitive countable model[39] of ZF, or ZFC.[40] We will from now on forget about the universe V, and will "live" in M instead. For instance, an ordinal number for us will be an ordinal number in M – we act as if we cannot

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[37] Due to Gödel theorem, we cannot show the existence of any set model of ZFC (inside ZFC), the less so of a transitive one; but much as with the existence of a model of arithmetics $\mathbb{N}$, we may decide that we believe in existence of such a model. Moreover, as will be apparent later, see page 130, the assumption of the existence of a transitive set model of ZFC is convenient, but by no means necessary for the development of the forcing.

[38] The contradiction in Lemma 4.12 centers around an existence of a hypothetical element $x \in M$ which is not constructible, i.e. $x \notin L$. But here is the snag: if M contains all ordinal numbers – as we assumed in Theorem 4.12, then "being constructible" and "being constructible in the sense of M" is indeed the same thing, i.e. $L = L^M$. But if M were a set, for instance, then this is no longer true: all we have is that $L^M \subseteq L$; in particular, it is perfectly possible that $L^M \subseteq M$ and $L^M \neq M$, with the offending $x \in M \setminus L^M$ existing in L!

[39] The countability conditions shall be used later, see Theorem 4.25.

[40] It is convenient to have AC in our model; by theorem 4.9 we may assume that much. In fact, very often our M will be a model of $V = L$ because this assumption greatly simplifies the cardinal arithmetics which is essential for rigorous development of forcing.

see ordinal numbers outside M. By absoluteness properties discussed in Section 4, we know that ordinal numbers are absolute; it follows that ordinal numbers we see in M are the real ordinal numbers. Also, by a simple argument, the transitivity of M implies that the ordinal numbers in M form an initial segment of the ordinal numbers; in other words, there is an ordinal number o_M such that $o_M = \{\alpha \mid \alpha \in \mathrm{On}^M\}$, i.e. o_M is the least ordinal α such that $\alpha \notin M$. The ordinal o_M is called the height of M, and if M is countable, then $o_M < \omega_1$.

It turns out that a reasonable concept of an extension of M should preserve the ordinal height o_M, i.e. that for the desired extension $N \supseteq M$, it should hold that $o_N = o_M$. An intuitive argument for this requirement might run as follows: we need to have some control over the properties of N, and introducing new ordinal numbers into N makes this almost impossible – we need some correspondence between constructions in M and N; and since constructions in set theory heavily use iteration over the ordinal numbers, a new ordinal number in N would be literally "inaccessible" from the point of view of M.

We have so far decided that we need to extend M, a transitive countable model of ZF, into a model $N \supseteq M$, which is also to be a model of ZF. The following lemma shows that in extending the model M we must be very careful about the use of any "extra" information available in V, but not in M. Consequently, we cannot hope to argue recklessly in V and pick some arbitrary $x \in V \setminus M$ and build the extension N around this x. Indeed, the correct selection of such x lies at the heart of the forcing construction.

Lemma 4.14 *Assume M satisfies ZF. Let $\alpha < \omega_1$ be greater then o_M, i.e. $o_M < \alpha$. Let $(\omega, <_R)$ be isomorphic to $(\alpha, \in)$. Then for all transitive extensions $N \supseteq M$ such that $o_N = o_M$,*

if the relation $<_R$ is in N, then N is not a model of ZF.

Proof of Lemma 4.14. Because $\alpha < \omega_1$, there is a function f mapping α 1-1 onto ω. If we define a binary relation $<_R$ such that $n <_R m$ iff $f^{-1}(n) \in f^{-1}(m)$, then $(\omega, <_R)$ is isomorphic to $(\alpha, \in)$; sometimes we say that $<_R$ "codes" α. Now it is a theorem of ZF that each wellordered set corresponds to exactly one ordinal, to which it is isomorphic (see Fact 2.11). If N satisfies ZF and contains $<_R$, then inside N there must be an ordinal $\alpha' \in N$ such that $(\omega, <_R)$ is isomorphic to $(\alpha', \in)$. But this isomorphism would be in V as well (since $N \subseteq V$), and consequently there would be two distinct ordinals in V, α and α', corresponding to $(\omega, <_R)$, and this is a contradiction. qed

It follows that in extending the model M we have to avoid adding any binary relation $<_R$ which happens to code an ordinal α between o_M and ω_1 – since they are $\aleph_1$ many of these, this is quite a lot of sets that we must avoid.

The structure of names

Our horizon being limited by the structure M, where we "live", we cannot point to a set which is outside M, but we may give it a name inside our model M. Introducing names for the postulated objects outside our universe M has the benefit that we may exercise some level of control over the desired model N; assuming of course that we define the names in such a clever way that we achieve sufficient degree of correspondence between the names and the elements in the desired N.

Obviously, for this approach to work, we must have a sufficient number of names inside our model M; in a way – because our N should be bigger than M – we need more names inside M than is the totality of all the elements of M. But having infinity as our ally, this is easy. Though it is not our official definition of a name, consider the following example.

Example 4.15 *For $x \in M$, we will view the pair $\langle x, 0\rangle$ as the canonical name for x.*[41] *We will also define other names: for $x \in M$, and $n \in \omega$, a pair $\langle x, n\rangle$ will also be a name. Thus, pairs of the type $\langle x, 0\rangle$ will represent the elements of M, while the names $\langle x, n\rangle$ will denote* some other objects *– ideally, the elements in the desired extension $N \supseteq M$. However, notice that the set of all names is a proper subset of M. Indeed, while each $\langle x, n\rangle$ is a member of M, there are certainly sets $y \in M$ which are not ordered pairs, and consequently are not names.*

Our official definition of names will be more technical, but the basic idea of the previous example will be preserved, i.e. we will have some special names denoting the elements of M, and in addition we will have other names which will denote the elements of the extension N.

Before plunging into definitions, we mention one more ingredient which will be used in defining the names. Since forcing is a very general technique, it cannot do to have a single class of names that would work equally well for all possible extensions; it is reasonable to expect that the model where $\neg$CH holds may be quite different to the model where $\neg$AC holds. Accordingly, our names will be defined with respect to a given set

[41] Note that since M satisfies ZF, it is in particular closed under $\langle , \rangle$; hence for $x, y \in M$, also $\langle x, y\rangle \in M$.

of parameters $\mathbb{P}$, with $\mathbb{P} \in M$. From the technical point of view, $\mathbb{P}$ will be a partially ordered set with a greatest element $1_{\mathbb{P}}$, while the elements $p \in \mathbb{P}$ will be called conditions.

Remark. Though the following exposition works equally well with arbitrary partially ordered sets $\mathbb{P}$, we will fix one concrete example $\mathrm{Add}(\omega, 1)$ of such a set of condition $\mathbb{P}$ to make the exposition more transparent.

Definition 4.16 *The domain of the partially ordered set* $\mathrm{Add}(\omega, 1)$ *will contain all finite sequences of* 0 *and* 1*; for instance* $p = \langle 1, 0, 1, 1\rangle$ *is an example of such a sequence. The conditions in* $\mathrm{Add}(\omega, 1)$ *will be ordered by end-extension; if* p *end-extends* q*,*[42] *we write* $p \leq q$*.*[43] *Note that the greatest element of* $\mathrm{Add}(\omega, 1)$ *is the emptyset* $\emptyset$*.*

To anticipate a little, the name "Add" suggests that the corresponding extension N will "add" (at least) one new subset of ω.

We will now define the class of names $M^{\mathrm{Add}(\omega,1)} \subseteq M$ with respect to the partially ordered set $\mathrm{Add}(\omega, 1)$. The reader will find it helpful to think about the conditions $p \in \mathrm{Add}(\omega, 1)$ as parameters which say to what extent the given name has the right to be in the desired extension N.

Definition 4.17 *Adapting the definition in Foundation axiom, see Definition 3.2, we will define (inside* M*) the class of names* $M^{\mathrm{Add}(\omega,1)}$ *as follows:*

$$\begin{array}{rcl} M_0^{\mathrm{Add}(\omega,1)} & = & \emptyset \\ M_{\alpha+1}^{\mathrm{Add}(\omega,1)} & = & \mathscr{P}(M_\alpha^{\mathrm{Add}(\omega,1)} \times \mathrm{Add}(\omega,1)) \\ M_\lambda^{\mathrm{Add}(\omega,1)} & = & \bigcup_\alpha M_{\alpha<\lambda}^{\mathrm{Add}(\omega,1)},\ \textit{for } \lambda \textit{ limit} \\ M^{\mathrm{Add}(\omega,1)} & = & \bigcup_{\alpha\in\mathrm{On}} M_\alpha^{\mathrm{Add}(\omega,1)}. \end{array}$$

General names will be denoted by letters with a dot above it, as in $\dot{x} \in M^{\mathrm{Add}(\omega,1)}$.

[42] For instance $\langle 1, 0, 1, 1\rangle$ end-extends $\langle 1, 0, 1\rangle$.

[43] The reader should notice that p is "less than" q if it is "stronger" in the sense that it contains more information. This usage may seem confusing at first; it can be defended by the argument that the "stronger" a condition is, the more things it prohibits, and consequently restricts the number of suitable models. Compare this with Theorem 4.22.

Also, we will single out canonical *names for elements in* M*. For* $x \in M$*, the canonical name* $\check{x}$ *is defined by recursion as follows:*

$$\check{x} \quad = \quad \{\langle \check{y}, \emptyset \rangle \mid y \in x\},$$

where $\check{y}$ *is defined in the earlier stage of construction.*

Recall that $\emptyset$ *is the greatest element of* $\mathrm{Add}(\omega, 1)$ *in the ordering* $\leq$ *for* $\mathrm{Add}(\omega, 1)$*. Thus, if* $\mathbb{P}$ *is an arbitrary partially ordered set with a greatest element* $1_{\mathbb{P}}$*, this element would be in place of* $\emptyset$ *in the definition of a canonical name.*

Example 4.18 *The following are examples of names.* $\emptyset$ *is a name as it is in* $M_1^{\mathrm{Add}(\omega,1)}$*. Notice that names are some relations* R*, where if* $\langle x, y \rangle \in R$*,* x *is a name defined in the previous stages of the construction, and* y *is an element of* $\mathrm{Add}(\omega, 1)$*. Accordingly,* $\{\langle \emptyset, \langle 0, 1 \rangle \rangle\}$*,* $\{\langle \emptyset, \langle 0, 1, 1, 1, 0 \rangle \rangle, \langle \emptyset, \langle 0, 1 \rangle \rangle\}$*, and* $\{\langle \{\langle \emptyset, \langle 0, 1, 1, 1, 0 \rangle \rangle\}, \langle 0, 1, 1 \rangle \rangle\}$ *are all names.*

As for the canonical names, a canonical name for $\emptyset$ *is just* $\emptyset$*, for* $1 = \{\emptyset\}$*, the canonical name is* $\{\langle \emptyset, \emptyset \rangle\}$*, and so on.*

Notice that the basic idea of the definition of both the general and the canonical names does correspond to the simple example in 4.15; the important difference is that instead of the parameters in ω*, as in 4.15, we use more complicated parameters in* $\mathrm{Add}(\omega, 1)$*; also to ensure transitivity of the desired model* N*, we require that members of the names are themselves names – in example 4.15 we ignored the issue whether the* x *in* $\langle x, n \rangle$ *is itself a name or not.*

Making names into objects

We have shown in Lemma 4.14 that we must be very careful about the elements we will add to our extended universe N. We also mentioned that the crux of the forcing technique is to build the desired extension N around a carefully chosen element x – an element which is new, i.e. lies outside M, but simultaneously avoids to contain "unwanted" information, such as the relation $<_R$ in Lemma 4.14.

In the standing terminology, this new element is called a generic object, to be denoted G, and the extension N will be denoted as $M[G]$, i.e. the least model $N \supseteq M$, such that $G \in N$. G will always be some subset of the set of conditions $\mathbb{P}$. Considering the set of conditions $\mathrm{Add}(\omega, 1)$ in our example, the generic object G will determine a new subset of ω – i.e. G will be composed of sequences $p \in \mathrm{Add}(\omega, 1)$ such that for all $p, q \in \mathrm{Add}(\omega, 1)$, either $p \leq q$ or $q \leq p$; in particular, if the conditions in G are put one after the other, they will form a characteristic function of a subset of ω. If $p \in G$, we think about p as a finite approximation of the generic object. It must be emphasized that while $G \subseteq \mathrm{Add}(\omega, 1) \subseteq M$, G itself is required to be outside the model M.[44] Figuratively, while M contains all letters (i.e. finite approximations $p \in \mathrm{Add}(\omega, 1)$) necessary to determine the ideal object, the infinite word which is composed of these letters (the object G) lies outside the scope of people living in M.

We will describe how to select a generic object G later in the text. For now, assume we have chosen some $G \subseteq \mathrm{Add}(\omega, 1)$ which determines a new subset G of ω; for technical reasons, assume furthermore that $\emptyset \in G$. We will show how the universe $M[G]$ can be described.

Definition 4.19 *An interpretation $\dot{x}_G$ of a name $\dot{x} \in M^{\mathrm{Add}(\omega,1)}$ is defined by recursion as follows.*

$$\dot{x}_G = \{\dot{y}_G \mid \exists p \in G\, \langle \dot{y}, p\rangle \in \dot{x}\},$$

where $\dot{y}_G$ is defined in the earlier stages of the construction.

We set

$$M[G] = \{\dot{x}_G \mid \dot{x} \in M^{\mathrm{Add}(\omega,1)}\}.$$

Remark. Notice that the definition of $\dot{x}_G$ makes specific the hint given before Definition 4.17, namely that the conditions $p \in \mathrm{Add}(\omega, 1)$ determine how much "right" has the specific $\dot{x}$ to be in $M[G]$ – if it has some $p \in G$ next to itself, it can get into the universe $M[G]$, if it has no such $p \in G$, it may not. This rather vague comment is illustrated by some examples:

Example 4.20 *Assume that our G contains (among other sequences) the sequences $\emptyset$, $\langle 0, 1\rangle$, and doesn't contain $\langle 0, 0\rangle$.*

We will show some examples how the names are interpreted. Let $x \in M$ be an arbitrary element in M, $\dot{y}_0 = \{\langle\emptyset, \langle 0, 1\rangle\rangle\}$, $\dot{y}_1 = \{\langle\emptyset, \langle 0, 1\rangle\rangle, \langle\emptyset, \langle 0, 0\rangle\rangle\}$, $\dot{y}_2 = \{\langle\emptyset, \langle 0, 0\rangle\}$, and finally $\dot{g} = \{\langle\check{p}, p\rangle \mid p \in \mathrm{Add}(\omega, 1)\}$. Then the interpretation is as follows:

$$\begin{aligned} \check{x}_G &= x \\ (\dot{y}_0)_G &= \{\emptyset\} \\ (\dot{y}_1)_G &= \{\emptyset\} \\ (\dot{y}_2)_G &= \emptyset \\ \dot{g}_G &= G. \end{aligned}$$

[44] Since M is countable, it contains at most countably many subsets of ω available in V; due to Cantor theorem 2.2, there are $2^\omega > \omega$ subsets of ω in V. It follows that we have a lot of candidates to choose from when determining the generic object $G \subseteq \mathrm{Add}(\omega, 1)$.

Some comments are in order here. The fact that the canonical names $\check{x}$ are always realized by x follows (by induction) from the requirement that $\emptyset \in G$. The interpretation of $\dot{y}$'s is determined by the presence, or absence of $\langle 0, 1\rangle \in G$. As for $\dot{g}_G$, we reason as follows:

$$\dot{g}_G = \{\check{p}_G \mid \exists p \in G\, \langle \check{p}, p\rangle \in G\} = \{p \mid p \in G\} = G.$$

The rationale behind the definition of the names $M^{\mathrm{Add}(\omega,1)}$ was to ensure we can "talk" about the elements of $M[G]$ inside our model M; in this connection it is instructive to realize that a single name $\dot{g}$ always denotes the generic object G, irrespective of what the object G in fact is.[45)]

In the following sections we shall show that this uniformity between the model M and the extension $M[G]$ can be taken much further.

Choosing the generic object

The selection of the generic object $G \subseteq \mathrm{Add}(\omega, 1)$ is determined by the requirement that the model $M[G]$ should satisfy all the axioms of ZF. Taking into account that each $p \in G$ is in M and should function as a finite approximation of G, the ideal situation would be the following:

Let $\varphi(v_0, \ldots)$ be an arbitrary formula and let it be true in $M[G]$ under the interpretations $(\dot{x}_0)_G, \ldots$, i.e.

$$M[G] \models \varphi[(\dot{x}_0)_G, \ldots].$$

Then there is some $p \in G$, a finite approximation of G, such that

$$p \text{ "decides" } \varphi(\dot{x}_0, \ldots) \text{ inside } M.$$

The nature of the "deciding" needs more clarification, but the intuitive idea is clear: not only we can refer to particular objects, such as different G's, with a single name, i.e. $\dot{g}$, we can also decide each property holding in $M[G]$ by an element which exists in M, i.e. some $p \in G \subseteq \mathrm{Add}(\omega, 1)$. Notice that this is a priori not contradictory: although we can decide each property with an element in M, there are infinitely many properties to decide – but this would require to know which infinitely many objects $p \in \mathrm{Add}(\omega, 1)$ do the deciding; in other words it would require that we know (inside M) the whole $G \subseteq \mathrm{Add}(\omega, 1)$. This leads up to the following central definition and a theorem.

45) Without giving details, there exist (in V) 2^ω many generic objects in total for any "interesting" $\mathbb{P}$ – and $\mathrm{Add}(\omega, 1)$ is interesting in this sense, i.e. there is a sequence $\langle G_\alpha \mid \alpha \in 2^\omega\rangle$ of generic objects existing in V; $\dot{g}$ denotes in each extension $M[G_\alpha]$ exactly the object G_α, i.e. $\dot{g}_{G_\alpha} = G_\alpha$ for every α.

Definition 4.21 *We say that* $p \in \mathrm{Add}(\omega, 1)$ ***forces***[46] $\varphi(\dot{x}_0, \ldots)$*, in symbols*

$$p \Vdash \varphi(\dot{x}_0, \ldots),$$

if for every generic object $G \subseteq \mathrm{Add}(\omega, 1)$*, if* $p \in G$*, then*

$$M[G] \models \varphi[(\dot{x}_0)_G, \ldots].$$

Theorem 4.22 (Correspondence theorem) *If* G *denotes a generic object for* $\mathrm{Add}(\omega, 1) \in M$ *and* $\dot{x}_0 \ldots$ *are arbitrary names, then the following is true:*

$$M[G] \models \varphi[(\dot{x}_0)_G, \ldots] \text{ iff } \exists p \in G\, p \Vdash \varphi(\dot{x}_0, \ldots).$$

A proof of this theorem is well outside the scope of this article. But some intuitive arguments will be given in the rest of this section.

The name "forcing" in Definition 4.21 signifies that p forces some formula, or property, to hold in the generic extension $M[G]$, irrespective of what the object G in fact is. However, notice that Definition 4.21 is really just a definition: the main import of the forcing relation $\Vdash$, in correspondence with the motivation given at the beginning of this section, should be that it is expressible inside M; this is similar to the name $\dot{g}$ for the generic object – recall that the name $\dot{g}$ exists inside M. Accordingly, the hardest task in verifying the properties of forcing is to show that the relation $\Vdash$ in Definition 4.21 is in fact definable inside M. In Definition 4.25, we show how the relation $\Vdash$ is defined for the existential quantifier.

As suggested earlier, the members $p \in \mathrm{Add}(\omega, 1)$ can be viewed as finite approximations of the extension $M[G]$; Theorem 4.22 in fact claims that these finite approximations decide everything about the extension $M[G]$. Given $p \in G$, there are some properties φ which p decides, but there are some other which are left undecided by p. If $q \le p$, i.e. if q is stronger than p, then q should intuitively decide all properties which p does, and possibly some additional properties as well.[47]

Example 4.23 *If* q *is stronger than* p*, then* q *should "know" more than* p *does, and moreover if* $p \Vdash \exists x\, \varphi$*, then it is reasonable to expect that*

[46] This relation has given its name to the entire technique, called "forcing".

[47] To a reader familiar with models for intuitionistic logic, this property of the forcing relation may seem very similar to a Kripke frame; there are some important analogies, but some important distinctions as well. For such a reader, to construe members of $\mathbb{P}$ as possible worlds may help to grasp intuitively what is going on. The book Fitting [1969] studies in detail this analogy.

there is some witness $\dot{x}$ *for* φ*, such that* $q \Vdash \varphi(\dot{x})$.[48] *It turns out that it would be too strong to demand that such a witness exists for every* $q \leq p$*; however,* eventually*, each* $q \leq p$ *should have such a witness. This leads to the following definition.*

Definition 4.24 *Let* $D \in M$*; we say that* $D \subseteq \mathrm{Add}(\omega, 1)$ *is* dense *if*

$$\forall p \in \mathrm{Add}(\omega, 1)\, \exists d \in D \text{ such that } d \leq p.$$

For $p \in \mathrm{Add}(\omega, 1)$*, we say that* D *is* dense below p *iff*

$$\forall p' \leq p\, \exists d \in D \text{ such that } d \leq p'.$$

Definition 4.25 *The inductive definition of* $\Vdash$ *for an existential quantifier is as follows:*

$$p \Vdash \exists x\, \varphi \text{ iff the set } \{q \leq p \mid \exists \dot{x}\, q \Vdash \varphi(\dot{x})\} \text{ is dense below } p.$$

It turns out that the concept of "being dense below a condition" is the correct apparatus which is used in defining the forcing relation $\Vdash$ for an arbitrary formula φ. If σ is a sentence in the forcing language, we actually obtain the following nice property: the set of all p such that p decides σ, i.e. $p \Vdash \sigma$ or $p \Vdash \neg\sigma$ is dense in $\mathrm{Add}(\omega, 1)$.

We will now turn to the definition of a generic object G where the concept of denseness is also crucial.

Definition 4.26 *A subset* $G \subseteq \mathrm{Add}(\omega, 1)$ *is a generic object if it satisfies the following conditions:*

(i) The greatest element of $\mathrm{Add}(\omega, 1)$, i.e. $\emptyset$, is in G;
(ii) If $p \in G$ and $p \leq q$, then $q \in G$;
(iii) If $p, q \in G$, then there is some $r \in G$ such that $r \leq p$, and $r \leq q$;
(iv) If $D \in M$ is a dense subset of $\mathrm{Add}(\omega, 1)$, then there is some $d \in G$ such that $d \in D$.

Some comments are in order here. Technically speaking, the first three conditions imply that G is a filter – it is easy to show that there are many G's satisfying these three conditions which exist in M. It is the fourth condition of denseness which makes sure that G cannot exist in M; see Lemma 4.29.

[48] Recall that $\Vdash$ is about semantics, not syntax; this should be understood as follows, if an arbitrary theory T proves the formula $\exists x\, \varphi$, then it doesn't have to be the case that there is a term t_x such that $T \vdash \varphi(t_x)$; on the other hand, if M is a model of T, then $M \models \exists x\, \varphi$ does imply that there is a member $m_x \in M$ such that $M \models \varphi(m_x)$. The definition of $\Vdash$ should therefore take this into account.

Theorem 4.27 (Existence of a generic object) *If M is countable transitive set with $\mathrm{Add}(\omega, 1) \in M$, then there is a set $G \in V$ such that G is a generic object for $\mathrm{Add}(\omega, 1)$. In general, the same applies to an arbitrary partial order $\mathbb{P} \in M$.*

Proof of Theorem 4.27. Since M is countable, there are only countably many dense sets $D \subseteq \mathbb{P}$ existing in M. Let $\langle D_n \mid n < \omega \rangle$ be their enumeration. Let $p_0 \in \mathbb{P}$ be an arbitrary element. By induction construct a decreasing sequence $p_{n+1} \leq p_n$ such that $p_{n+1} \in D_n$. Set

$$G = \{p \in \mathbb{P} \mid \exists p_n \, p_n \leq p\}.$$

It is not hard to verify that G satisfies all the required properties.

Notice the importance of the countability of M in the proof. If M were not countable, then the construction could fail at a limit step: for the decreasing sequence of p_n for $n \in \omega$, there might exist no p_ω below all of them. qed

Example 4.28 *We say that $\mathbb{P}$ is* non-trivial *if for all $p \in \mathbb{P}$ there are $q_0, q_1 \in \mathbb{P}$ such that $q_0 \leq p$ and $q_1 \leq p$ and q_0, q_1 don't have a common stronger condition; i.e. there is no r such that $r \leq q_0$ and $r \leq q_1$. This is denoted as $q_0 \perp q_1$ and we say that q_0, q_1 are incompatible; this name clearly refers to the fact that there can be no generic object G which will contain both q_0 and q_1. Notice that $\mathrm{Add}(\omega, 1)$ is non-trivial in this sense.*

Lemma 4.29 *If $\mathbb{P}$ is non-trivial, then no generic object is a member of M.*

Proof of Lemma 4.29. *By contradiction. If $G \in M$, then in view of Definition 4.26, item (iv), it is enough to show that the set $D_G = \{p \in \mathbb{P} \mid p \notin G\}$ is dense in $\mathbb{P}$ (since G is in M by our assumption, D_G is in M as well; if D_G were dense, then there is some $p \in G$ in D_G, contradicting the definition of D_G). But this is easy: assume $p \in \mathbb{P}$ is arbitrary and q_0, q_1 are two incompatible conditions below p; since q_0, q_1 cannot be both in G, one of them is in D_G.* qed

We will now illustrate the importance of the condition (iv) in Definition 4.26 which demands that G meets all dense subsets of $\mathrm{Add}(\omega, 1)$ (or $\mathbb{P}$ in general).

Corollary 4.30 *If G is a generic filter for $\mathrm{Add}(\omega, 1)$, then $M[G]$ contains a new subset of ω.*

Proof of Corollary 4.30. Work in M and define for each $n \in \omega$ a set $D_n = \{p \in \mathrm{Add}(\omega, 1) \,|\, n \in \mathrm{dom}(p)\}$. Obviously, each D_n is dense: if $q \in \mathrm{Add}(\omega, 1)$ is arbitrary, we can always find some $p \leq q$ such that n is in the domain of p.

Let G be a generic filter for $\mathrm{Add}(\omega, 1)$. Define $g(n) = 1$ if there is some $p \in G$ such that $p(n) = 1$; define $g(n) = 0$ otherwise. As D_n is dense for each $n \in \omega$, g is defined on all elements of ω. Note that if $p(n) = 1$ for some $p \in G$, then there can be no other $q \in G$ such that $q(n) = 0$, by the condition (iii) in Definition 4.26, and so g is correctly defined.

We claim that g is a new subset of ω. By Lemma 4.29, G cannot be in M; if g were in M, so would be G as G is definable from g: $G = \{p \in \mathrm{Add}(\omega, 1) \,|\, p$ is compatible with $g\}$.[49] It follows that $g \in M[G] \setminus M$. qed

We shall end this section with a general theorem which says that for every forcing $\mathbb{P}$ and every generic filter $G \subseteq \mathbb{P}$, the resulting model $M[G]$ satisfies ZF(C).

Theorem 4.31 *For any partially ordered set $\mathbb{P} \in M$ and any generic object $G \subseteq \mathbb{P}$, $M[G]$ satisfies all axioms of ZF; if M satisfies also the axiom of choice, AC, so does $M[G]$. In particular, if φ is an axiom of ZF(C), then $1_{\mathbb{P}} \Vdash \varphi$, where $1_{\mathbb{P}}$ is the greatest element in $\mathbb{P}$.*

The proof of the theorem requires a rigorous treatment of the forcing relation. It must suffice to say that the main tool used in the proof is the fact that the forcing relation $\Vdash$ is definable in M and consequently to make sure φ holds in $M[G]$ we can use that some formulas relevant to φ hold in M, as M is a model of ZF(C).

To prevent misunderstanding, however, it must be emphasized that while the truth of AC in M does imply that AC holds in $M[G]$, this is by no means true for an arbitrary formula φ. In particular, by nature of the forcing construction, $M[G]$ can never satisfy $V = L$ if G is not in M,[50] while M can satisfy $V = L$, and in practice it often does.

The combination of Theorem 4.31 and Corollary 4.30 gives the following theorem:

Theorem 4.32 *If ZF is consistent, so is ZF + $V \neq L$.*

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[49] This does require some more detailed argument which we omit.

[50] Since the ordinal height of M and $M[G]$ is the same, it holds that $L^M = L^{M[G]}$. It follows that $L^M \subseteq M \subsetneq M[G]$, and this contradicts $V = L$ in $M[G]$. See also Theorem 4.32.

Proof of Theorem 4.32. Let M be a ground model satisfying ZF. Let $M[G]$ be a generic extension for $G \subseteq \mathrm{Add}(\omega, 1)$. By Theorem 4.31, $M[G]$ satisfies ZF. By Corollary 4.30, $M[G]$ cannot satisfy $V = L$ as the the new set $g \subseteq \omega$ derived from G (see Corollary 4.30) is certainly not in L (recall that $L = L^M = L^{M[G]} \subseteq M \subsetneq M[G]$). qed

Notice that Theorem 4.32 is already non-trivial. By Theorem 4.12, we cannot construct an inner model to show consistency of $V \neq L$ (just replace $\neg CH$ in the proof of Theorem 4.12 by $V \neq L$).

Making CH false

By Corollary 4.30, forcing with $\mathrm{Add}(\omega, 1)$ adds a new subset g of ω. In fact, since $M[G]$ satisfies ZF, many more subsets of ω will be added into $M[G]$; for instance for any $n \in \omega$, g above n is a new subset. But to make sure CH fails in $M[G]$, we need to add at least $\aleph_2$ many new subsets. It can be shown, however, that $\mathrm{Add}(\omega, 1)$ will not add as many new sets.

An obvious strategy to add at least $\aleph_2$ many new subsets of ω is to apply $\mathrm{Add}(\omega, 1)$ $\aleph_2$-many times.

Theorem 4.33 *There exists a partial order* $\mathrm{Add}(\omega, \aleph_2)$*, namely, roughly speaking,* $\aleph_2$*-many copies of* $\mathrm{Add}(\omega, 1)$ *put one after the other, such that if G is generic for* $\mathrm{Add}(\omega, \aleph_2) \in M$*, then $M[G]$ satisfies* $2^{\aleph_0} = \aleph_2$*, i.e.* $\neg CH$*. In particular, in view of Theorem 4.10, CH is independent on the axioms of ZF, and ZFC.*

We will not give a rigorous proof, but will address instead some finer technical points which are important for the practical applications of the forcing technique, and in particular for the failure of CH. Notice that the development of forcing described above works for an arbitrary partial order $\mathbb{P}$ (we have used the example of $\mathrm{Add}(\omega, 1)$ but in fact we used none of its specific properties so far). It is clear, however, that some properties of the forcing extension must depend on the specific properties of the given partial order. In Section 4, we mentioned that the concept of a cardinal number is not absolute for transitive models of ZF. In other words, it may happen that some cardinal numbers in M are destroyed, or are collapsed in the standing terminology, in $M[G]$ after forcing with some partial order $\mathbb{P}$ (if $\kappa < \lambda$ are cardinals in M, a forcing $\mathbb{P}$ may add a 1-1 function from κ onto λ, collapsing λ). Although we may sometimes wish to collapse cardinals, it is not difficult to see that in the case of Theorem 4.33 and the partial order $\mathrm{Add}(\omega, \aleph_2)$ we had better avoid collapsing.

The reason is that $\mathrm{Add}(\omega, \aleph_2)$ adds $\aleph_2$-many new subsets of ω, where $\aleph_2$ (i.e. the second uncountable cardinal) is calculated in M. If for instance $\aleph_1$ is collapsed in $M[G]$ (and $\aleph_2$ is not collapsed), then $\aleph_2$ of M becomes in fact the first uncountable cardinal in $M[G]$, i.e. $\aleph_1$ of $M[G]$, and this would mean that CH again holds in $M[G]$!

Here we come to the crucial task which concerns practical application of forcing: one needs to carefully verify various properties of the partial order $\mathbb{P}$ to make sure that $\mathbb{P}$ achieves the right thing. By way of illustration, we show how to ensure that the cardinal $\aleph_1$ of M is not collapsed.

Lemma 4.34 $\mathrm{Add}(\omega, 1)$ *doesn't collapse* $\aleph_1$ *of* M.

Sketch of proof. It is enough to show that if G is a generic filter for $\mathrm{Add}(\omega, 1)$, then in $M[G]$ there is no countable subset X cofinal in the ordinal $\aleph_1$. In detail, if we denote $\gamma = \aleph_1$ then in $M[G]$ there can be no increasing sequence of ordinals $X = \langle x_n \,|\, n \in \omega \rangle$ such that the limit of $\langle x_n \,|\, n \in \omega \rangle$ is γ (if there were such a sequence in $M[G]$, then it easily follows that γ cannot be the first uncountable cardinal in $M[G]$).

We will proceed by contradiction. Assume that there is in $M[G]$ some such sequence $X = \langle x_n \,|\, n \in \omega \rangle$ cofinal in $\gamma = \aleph_1$ of M. By Correspondence theorem 4.22, there is some $p \in \mathrm{Add}(\omega, 1)$ such that $p \Vdash (\dot{X}$ is a cofinal sequence in $\check{\gamma})$, where $\dot{X}$ is a name for the sequence $X = \langle x_n \,|\, n \in \omega \rangle$. The first element of the sequence X, i.e. x_0, has some name $\dot{x}_0$ pertaining to it. As it is a name, it may be interpreted by different ordinal numbers under different generic filters G; but whichever ordinal it is (in a given generic extension), it has to be forced by some condition to be this ordinal, again by Correspondence theorem 4.22. However, $\mathrm{Add}(\omega, 1)$ has only size ω, and consequently $\dot{x}_0$ can represent only countably many ordinals below $\aleph_1$. The same applies to $\dot{x}_n$ for every $n \in \omega$. This means that we can find inside M some countable family of ordinals below $\aleph_1$ which contains all the possible ordinals which can be represented by the names $\dot{x}_n$. But as $\aleph_1$ is really uncountable in M, no such family existing in M can be cofinal in it. This is a contradiction. qed

Notice that the argument in the previous lemma used the size of the forcing notion to find in M some countable family of ordinals which may by interpreted by the names $\dot{x}_n$. The forcing notion $\mathrm{Add}(\omega, \aleph_2)$ in Theorem 4.33 (which we have defined very vaguely) is certainly bigger than ω. We will show that this obstacle can be overcome by a more detailed analysis of the forcing relation.

Definition 4.35 *Let $\mathbb{P}$ be a partial order. A subset A of $\mathbb{P}$ is called an* antichain *if all elements of A are pairwise incompatible, i.e. for all $p, q \in A$, there is no $r \in P$ such that $r \leq p, r \leq q$.*

Recall that if $G \subseteq \mathbb{P}$ is a generic filter, than all elements of G must be pairwise compatible. It follows that G can contain at most one element of A.

The partial order $\mathrm{Add}(\omega, \aleph_2)$ can be formally defined in such a way as to satisfy the condition that all antichains in $\mathrm{Add}(\omega, \aleph_2)$ are at most countable. Naively, one would put in $\mathrm{Add}(\omega, \aleph_2)$ all functions p with domain $\omega \times \aleph_2$ and range included in $\{0, 1\}$ such that p restricted to $\alpha < \aleph_2$ is some finite sequence of zeros and ones – just like a condition in $\mathrm{Add}(\omega, 1)$. The idea being that p approximates $\aleph_2$-many new subsets of ω. However, $\mathrm{Add}(\omega, \aleph_2)$ defined in this way contains antichains of uncountable size. It turns out that the right solution is to require that p be non-trivial only at finitely many $\alpha < \aleph_2$, i.e. except for finitely many α, p restricted to a coordinate $\alpha' < \aleph_2$ must in fact be an empty set (empty set is regarded as a trivial finite sequence, and hence the name "non-trivial"). Such p are said to have a finite support.

Definition 4.36 *The forcing $\mathrm{Add}(\omega, \aleph_2)$ contains all conditions p with domain $\omega \times \aleph_2$ and range included in $\{0, 1\}$ such that the projection of p to a coordinate $\alpha < \aleph_2$ is a finite sequence of 0's and 1's and p has finite support. The relation $p \leq q$ is the reverse inclusion: $p \leq q$ iff $p \supseteq q$.*

The following fact will be given without a proof.

Fact 4.37 *All antichains in $\mathrm{Add}(\omega, \aleph_2)$ are at most countable.*

This Fact allows us to show:

Lemma 4.38 *$\mathrm{Add}(\omega, \aleph_2)$ doesn't collapse $\aleph_1$ of M.*

Sketch of proof. The fact that all antichains in $\mathrm{Add}(\omega, \aleph_2)$ are at most countable is enough to infer that $\aleph_1$ is not collapsed in a generic extension $M[G]$ by $\mathrm{Add}(\omega, \aleph_2)$, and consequently CH is false in $M[G]$. Realize that the argument in Lemma 4.34 applies here: if p and q force distinct ordinals for the interpretation of $\dot{x}_n$, then p and q must be incompatible; it follows that they form an antichain, and consequently the number of such p and q can be at most countable – even if $\mathrm{Add}(\omega, \aleph_2)$ itself is bigger. Thus, there is again in M a countable family of ordinals which contains all the possible interpretations of the names $\dot{x}_n$.[51] qed

Making AC false

The model for the negation of AC is technically more demanding, so we won't be able to give an intuitive outline of the proof. It must suffice to say that it is possible to construct a forcing extension $M[G]$ and an inner model $N \subseteq M[G]$, where the axiom of choice fails.

Forcing: Frequently asked questions

Models or syntax? We have mentioned above that the assumption about the existence of a transitive countable model of ZF is not necessary for the development of forcing. The key properties of forcing which enable us to completely avoid the use of models are the following:

(i) The definability of the relation $\Vdash$ inside M without any recourse to G – and consequently in V as well if we do the definition in V in place of M;
(ii) Preservation of the forcing relation under the provability, see Fact 4.39 below;
(iii) The fact that the definition of the forcing relation rules out the possibility that there exists $p \in \mathbb{P}$ forcing both φ and $\neg\varphi$, where φ is an arbitrary formula.

Fact 4.39 *Assume that* $\varphi_0 \wedge \ldots \wedge \varphi_n \vdash \psi$*; then if* $p \Vdash \varphi_0 \wedge \ldots \wedge \varphi_n$*, then it is also true that* $p \Vdash \psi$*.*

The following lemma shows how to avoid the use of models.

Lemma 4.40 *Let* $\mathbb{P}$ *be an arbitrary forcing notion. Assume that there is* $p \in \mathbb{P}$ *such that* $p \Vdash \varphi$*, then* $ZF \nvdash \neg\varphi$*.*

Proof of Lemma 4.40. First we have to emphasize that the property $p \Vdash \varphi$ is completely expressible in V, without any recourse to $M[G]$. Recalling that the purpose of forcing is to derive consistency results, the above lemma is entirely sufficient for our needs.

[51] Technically speaking, we need to work below a condition p_X that forces that $\dot{X}$ is a sequence of ordinals. If we set $A = \{p \leq p_X \mid \exists \alpha_p \, p \Vdash \dot{X}(n) = \check{\alpha}_p\}$, then the set of distinct α_p's is at most countable as the relevant p's form an antichain.

Assume for contradiction that $ZF \vdash \neg\varphi$; then there exist a finite list of axioms of ZF $\varphi_0, \ldots, \varphi_n$ such that

$$\varphi_0 \wedge \ldots \wedge \varphi_n \vdash \neg\varphi.$$

By Theorem 4.31, $1_{\mathbb{P}} \Vdash ZF$; by property in Fact 4.39, we also have $1_{\mathbb{P}} \Vdash \neg\varphi$. We have mentioned above that if $p \leq q$, then p forces at least the same formulas as q does; $p \leq 1_{\mathbb{P}}$ hence implies that $p \Vdash \neg\varphi$. This is a contradiction with the item (iii) above. qed

For readers with more familiarity with logic, we can add that the true benefit of the syntactical approach is that we can formulate the consistency results on the level of arithmetics, instead of relying on set theory and its consistency.

Partial orders or Boolean algebras? In many books, in Balcar and Štěpánek [2000] for one, the development of forcing seems intrinsically dependent on (complete) Boolean algebras. This may be a deterring feature for students inadequately familiar with the theory of Boolean algebras; fortunately, the use of Boolean algebras can be completely avoided. Indeed, the incorporation of Boolean algebras into the theory of forcing came only later by work of Vopěnka and Solovay; Boolean algebras add more understanding into the way forcing works, and also provide nice connection to logic. In what follows, we describe very briefly – and in a zig-zag way, in what respect the Boolean algebras are perhaps more natural to use than general partial orders, and also give some argument to show that both approaches are for the most purposes identical. Due to lack of space, however, some familiarity with Boolean algebras must be assumed.

Some simplifications brought in by Boolean algebras appear already in the definition of names. Recall that in the definition of the names in $M^{\mathrm{Add}(\omega,1)}$, see Definition 4.17, if $\dot{x}$ is a name then there can be many elements $p \in \mathrm{Add}(\omega, 1)$ such that $\langle \dot{x}, p\rangle$ is an element of some other name $\dot{y}$. Let us fix a complete Boolean algebra $\mathbb{B}$ with its canonical ordering $<_{\mathbb{B}}$ (and for technical reasons, remove the least element in the ordering $<_{\mathbb{B}}$); then $(\mathbb{B}, <_{\mathbb{B}})$ is a partially ordered set and all the development of forcing can be applied to it (recall that it works for any partially ordered set). Assume that there is a name $\dot{y} = \{\langle \dot{x}, p\rangle \,|\, p \in I\}$, where $I \subseteq \mathbb{B}$ is some set. By the standard definition of a name in Definition 4.17, this is a regular name and generally it may not be replaceable by some "simpler" name; now, since $\mathbb{B}$ is a complete Boolean algebra, there exist a supremum $\hat{p} = \bigwedge p_{i \in I}$. It turns out that the name $\dot{y}$ can be equivalently replaced by a name $\dot{y}' = \{\langle \dot{x}, \hat{p} \rangle\}$, i.e. in the sense of forcing relation, all the names p_i can be "approximated" by the greatest weaker element of $\mathbb{B}$, namely the supremum $\hat{p}$. Note that such replacement needs to take place in all names, so unless the Boolean algebra is complete, such replacement cannot be carried out.

Similarly, in the definition of the forcing relation we can use the analogy between the connectives in logic and the operations in the Boolean algebra $\mathbb{B}$ – they are even sometimes denoted by the same symbols: $\wedge, \vee, \neg = {}'$, $\exists =$ infimum, and $\forall =$ supremum. The most important consequence of this analogy is that we may inductively calculate a Boolean value, i.e. an element of $\mathbb{B}$, for each formula $\varphi(\dot{x}_0, \ldots)$. In fact this value may be taken as a generalized truth value attributed to a formula – recall that the usual truth value in logic takes either the value 0 or 1, where $\{0, 1\}$ is a (trivial) complete Boolean algebra. These analogies can be taken much further, but they cannot make the forcing technique more efficient than it already is, as the following – rather vaguely formulated – theorem shows.

Theorem 4.41 *For every partially ordered set $\mathbb{P}$, there is a unique complete Boolean algebra $\mathbb{B}_{\mathbb{P}}$, called the* completion *of $\mathbb{P}$, such that the forcing with $\mathbb{P}$ and $\mathbb{B}_{\mathbb{P}}$ achieves the same thing.*

In practice, it is a matter of personal preferences of the individual mathematicians whether they use the partial order approach, or the Boolean algebra one.[52)]

5 Conclusion

In the present article, we have argued that it is in the nature of set theory to be "open-ended" in the sense that many interesting properties, such as AC or CH, can be added into our system both in their positive and negative form.

There are many other topics in set theory which shed more light on the issues discussed in this article. We have, for instance, completely ignored the question of the existence of the so called large cardinals which, at least in the hopes articulated by Gödel, see for instance Gödel [1999], might have had some impact on the intuitive validity or falsity of CH. These issues will be hopefully brought to the reader's attention in the projected second part of this article.

52) In some special cases, the use of Boolean algebras does lead to new insights, but such examples require a more detailed knowledge of forcing.

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