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A Lifting Argument for the Generalized Grigorieff Forcing

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Abstract In this short paper, we describe another class of forcing notions which preserve measurability of a large cardinal κ from the optimal hypothesis, while adding new unbounded subsets to κ. In some ways these forcings are closer to the Cohen-type forcings — e.g. we show that they are not minimal — however, they share some properties with tree-like forcings. We show that they admit fusion-type arguments which allow for a uniform lifting argument.

1 Introduction

In this short paper, we describe another class of forcing notions which preserve measurability of a large cardinal κ from the optimal hypothesis, while adding new unbounded subsets to κ. A typical application is to force the failure of GCH at a measurable cardinal from the assumption

Assumption 1.1 There exists $j : V \to M$ with critical point κ and
(i) ${}^{\kappa}M \subseteq M$;
(ii) there is $f : \kappa \to \kappa$ such that $j(f)(\kappa) = \kappa^{++}$.

Woodin was first to force the failure of GCH at a measurable from these assumptions (which are optimal); he used the iteration of the Cohen forcing to achieve this. At the crucial step, when a suitable generic is needed for the Cohen forcing, he solved the problem by modifying an existing generic to fit a certain condition; this is sometimes called "a surgery argument" (see [3]).

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There is an alternative approach, which is more uniform in that the required generic is obtained directly in the current universe. This approach is based on tree-like forcings. The first such construction ([8]) used the generalized Sacks forcing and the accompanying "tuning fork" argument. With the introduction of perfect trees splitting only at certain cofinalities, as in [7] or [4], it was possible to avoid the necessity to choose branches splitting at κ in order to define the desired generic filter. More applications of the tree-like forcings are now available – for instance generalizations of Miller forcing in [9], or the abstract treatment in [6].

We propose here another class of forcing notions which allow equally uniform constructions, and yet do not have a tree-like structure. These forcings are obtained by generalizing the forcing notions defined with respect to ideals on ω, as introduced by Grigorieff in [11]. Variants of the perfect-tree forcing and of Grigorieff forcing for uncountable cardinals have been studied extensively, see for instance [2] and [1]. Although Grigorieff forcing can be generalized to iterations, and successor cardinals as well, for the sake of brevity we treat here only the case of products at inaccessibles (see Remarks 3.9 and 3.10 for information on generalizations). In defining the generalized Grigorieff forcing, we introduce the notion of a *lifting-friendly* normal ideal on a large cardinal κ – as it turns out, uniform lifting is determined by this property.

We show the lifting argument for generalized Grigorieff forcing on the test case (1.1). Many other results in literature can be reproved using Grigorieff forcing, such as obtaining a tree property at a regular $\kappa > \omega_1$, by collapsing a weakly compact cardinal. After a more detailed analysis of the combinatorial properties of Grigorieff forcing, we think one can obtain new results concerning cardinal invariants at a regular $\kappa > \omega$; see open questions at the end of the paper. As a new result, we obtain that the uniform lifting argument does not depend on the minimality properties of the tree-like forcings (see Section 3.3 for definitions). Indeed, we prove that the generalized Grigorieff forcing admits a uniform lifting argument and yet is not minimal.

The notation of the paper is standard. The paper is self-contained, though familiarity with [3] (lifting of embeddings) and [12] (Sacks forcing at an uncountable κ) is useful.

2 Definition of the forcing

2.1 Preliminaries

Notation 2.1 Assume κ is regular and $\mathrm{Club}(\kappa)$ is the closed unbounded filter on κ. Let S be stationary. Define:

$$\mathrm{Club}(\kappa)[S] = \{X \subseteq \kappa \mid \exists C \text{ closed unbounded in } \kappa \text{ and } X \supseteq S \cap C\}.$$

Observation 2.2 *For every stationary S, $\mathrm{Club}(\kappa)[S]$ is a normal (i.e. closed under diagonal intersections) proper filter extending $\mathrm{Club}(\kappa)$.*

Proof Properness and upwards closure are obvious from the definition. We show that $F = \mathrm{Club}(\kappa)[S]$ is closed under diagonal intersection. Let $\langle X_\alpha \mid \alpha < \kappa\rangle$ be a sequence of elements in F; for every α, let C_α be a closed unbounded set in κ such that $X_\alpha \supseteq S \cap C_\alpha$. Then $\Delta_\alpha X_\alpha \supseteq S \cap \Delta_\alpha C_\alpha$, where $\Delta_\alpha C_\alpha$ is closed unbounded and therefore $\Delta_\alpha X_\alpha$ is in F. □

2.2 Grigorieff forcing at an inaccessible cardinal Let κ be an inaccessible cardinal. Unless otherwise stated, all ideals on κ will be κ-complete and proper.

Definition 2.3 *Let κ be inaccessible and let I be a subset of $\mathscr{P}(\kappa)$. Let us define*

$$P_I = \{f \,\vdots\, \kappa \to 2 \mid \mathrm{dom}(f) \in I\},$$

where $f \,\vdots\, \kappa \to 2$ is a partial function from κ to 2. Ordering is by reverse inclusion: for p,q in P_I, $p \leq q \leftrightarrow p \supseteq q$.

Remark 2.4 If we let I be the ideal of bounded subsets of κ in the previous definition we obtain the usual Cohen forcing.

A generalization of the following definition will be important later on.

Definition 2.5 *For $\alpha < \kappa$ write*

$$p \leq_\alpha q \leftrightarrow p \leq q \ \& \ \mathrm{dom}(p) \cap (\alpha+1) = \mathrm{dom}(q) \cap (\alpha+1).$$

We say that $\langle p_\alpha \mid \alpha < \kappa\rangle$ is a fusion sequence *if for every α, $p_{\alpha+1} \leq_\alpha p_\alpha$ and for limit γ, $p_\gamma = \bigcup_{\alpha<\gamma} p_\alpha$.*

The following theorem is easy. We prove it for the convenience of the reader.

Theorem 2.6 *Assume GCH and let I be a κ-complete ideal extending the nonstationary ideal on κ. Then P_I preserves cofinalities if and only if I is a normal ideal.*

Proof Assume first that I is a normal. Then P_I is κ-closed. Under GCH it satisfies the κ^{++}-cc. So it suffices to show that P_I preserves κ^+.

Claim 2.7 *If $\langle p_\alpha : \alpha < \kappa\rangle$ is a fusion sequence, then the union $q = \bigcup_{\alpha<\kappa} p_\alpha$ is a condition in P_I which is the infimum of the sequence in P_I. Moreover $q \leq_\alpha p_\alpha$ for each $\alpha < \kappa$.*

Proof of Claim It is sufficient to show

$$\mathrm{Lim}(\kappa) \cap (\triangle_{\alpha<\kappa}(\kappa \setminus \mathrm{dom}(p_\alpha))) \subseteq \bigcap_{\alpha<\kappa} (\kappa \setminus \mathrm{dom}(p_\alpha)).$$

Let ξ be a limit ordinal in the diagonal intersection. Then for all $\zeta < \xi$, $\xi \notin \mathrm{dom}(p_\zeta)$. By continuity on the limit step of a fusion sequence, $\xi \notin \mathrm{dom}(p_\xi)$. By definition (2.5), $\xi \notin \mathrm{dom}(p_\alpha)$ for every $\alpha \geq \xi$. ⊣

To prove the theorem we will use fusion to show that, in the extension, every function $f : \kappa \to \kappa^+$ is bounded. So fix a name $\dot{f}$ for such a function and a condition p such that $p \Vdash \dot{f} : \kappa \to \kappa^+$.

We shall construct by induction a fusion sequence $\langle p_\alpha : \alpha < \kappa\rangle$ whose union will force that $\dot{f}$ is bounded. Let $p_0 = p$. If α is limit, let $p_\alpha = \bigcup_{\beta<\alpha} p_\beta$. Assume now that p_α is constructed. Enumerate all functions $q : (\alpha+1) \to 2$ which are compatible with p_α as $\{q^\alpha_\xi : \xi < 2^\alpha\}$ (possibly with repetitions). Now construct a $\leq_\alpha$-decreasing sequence $\langle p^\alpha_\xi : \xi < 2^\alpha\rangle$ of conditions with $p^\alpha_0 = p_\alpha$ and a sequence of values $\langle y^\alpha_\xi : \xi < 2^\alpha\rangle$ such that

$$p^\alpha_{\xi+1} \cup q^\alpha_\xi \Vdash \dot{f}(\alpha) = y^\alpha_\xi.$$

and finally let $p_{\alpha+1} = \bigcup_{\xi<2^\alpha} p^\alpha_\xi$. This can be done since P_I is κ-closed and $2^\alpha < \kappa$ and it completes the inductive construction.

We now show that the fusion limit r forces that $\dot{f}$ is bounded. Let $Y = \{y^\alpha_\xi \mid \alpha < \kappa, \xi < 2^\alpha\}$ and note that, by GCH, $|Y| \leq \kappa$. So it is enough to show that $r \Vdash \dot{f}(\alpha) \in Y$ for all $\alpha < \kappa$. Pick $\alpha < \kappa$ and let r_α be any extension of r deciding $\dot{f}(\alpha)$. By enlarging r_α, if necessary, we may assume that $\mathrm{dom}(r_\alpha)$ contains $\alpha+1$. Then $r_\alpha \restriction (\alpha+1)$ is compatible with r and hence with p_α ($r \leq_\alpha p_\alpha$) so it is equal to some q^α_ξ. Since $p^\alpha_{\xi+1} \cup q^\alpha_\xi$ forces $\dot{f}(\alpha) = y^\alpha_\xi \in Y$ it follows that so does r_α.

Assume now that I is not normal and that this is witnessed by a sequence $\langle A_\alpha : \alpha < \kappa\rangle$ (i.e. the diagonal union $D = \{\alpha : (\exists\beta < \alpha)(\alpha \in A_\beta)\}$ of the sequence is not in I while all A_α's are elements of I). Since I is κ-complete we may, without loss of generality, assume that $|A_\alpha| = \kappa$ and that the sequence is increasing and continuous (i.e. $A_\gamma = \bigcup_{\beta<\gamma} A_\beta$ for limit $\gamma < \kappa$).

Claim 2.8 *There is no $B \in I$ which almost covers all A_α's, i.e for which $|A_\alpha \setminus B| < \kappa$ for all $\alpha < \kappa$.*

Proof of Claim Fix $B \in I$ and let $g(\alpha) = \min\{\gamma : A_\alpha \setminus \gamma \subseteq B\}$. First notice that $\{\alpha < \kappa \,|\, g(\alpha) < \alpha\}$ must be nonstationary because otherwise, by Fodor's lemma, there is some $\gamma < \kappa$ and a stationary set S such that for all $\alpha \in S$, $A_\alpha \setminus \gamma \subseteq B$. Since we assume that the sequence of A_α's is increasing, it follows that for all α, $A_\alpha \setminus \gamma \subseteq B$. This contradicts our assumption that D is not in I since $D \setminus \gamma \subseteq B \in I$ and $\gamma \in I$. It follows that $\{\alpha < \kappa \,|\, \alpha \leq g(\alpha)\}$ contains a club. By continuity of the sequence $\langle A_\alpha \,|\, \alpha < \kappa \rangle$, there is a club C such that for all $\alpha \in C$, $g(\alpha) = \alpha$. It follows that $C \cap D$ is included in B. However $C \cap D$ is I-positive while $B \in I$ — a contradiction.

$\dashv$

We proceed by constructing a P_I name for an unbounded function $\dot{h} : \kappa \to 2^\kappa$. Enumerate all functions $f : A_\alpha \to 2$ as $\{f^\alpha_\beta \mid \beta < 2^\kappa\}$ and define $\dot{h}$ such that

$$f^\alpha_\beta \Vdash \dot{h}(\alpha) = \beta.$$

To show that P_I forces that $\dot{h}$ is unbounded, fix some $p \in P_I$ and $\beta < 2^\kappa$. Let $B = \mathrm{dom}(p) \in I$. By the previous claim, there is some $\alpha < \kappa$ such that $|A_\alpha \setminus B| = \kappa$. Then the set $H = \{q \in P_I \mid \mathrm{dom}(q) = A_\alpha,\ q||p\}$ of conditions with domain A_α which are compatible with p has size 2^κ. So there must be some $\gamma \geq \beta$ such that $f^\alpha_\gamma \in H$. This shows that p can be extended to q forcing $\dot{h}(\alpha) = \gamma \geq \beta$, finishing the proof of the theorem. □

As a preparation for the lifting construction, we will consider the following generalization of the definition of $\leq_\alpha$ and of the fusion construction. Let I be a normal ideal on κ and $S \in I^*$, where I^* is the dual of I. We will assume that S is composed of limit ordinals; this is without loss of generality because we can always shrink S by intersecting it with the class of limit ordinals, and still stay in I^*. Let P_I be the forcing defined above.

Definition 2.9 *Define the relation $\leq^S_\alpha$ as follows.*

(i) if α is in S:

$$p \leq^S_\alpha q \leftrightarrow p \leq q \;\&\; \mathrm{dom}(p) \cap (\alpha + 1) = \mathrm{dom}(q) \cap (\alpha + 1)$$

(ii) if α is in $\kappa \setminus S$:

$$p \leq^S_\alpha q \leftrightarrow p \leq q \;\&\; \mathrm{dom}(p) \cap \alpha = \mathrm{dom}(q) \cap \alpha.$$

We say that $\langle p_\alpha \,|\, \alpha < \kappa\rangle$ *is an* S*-fusion sequence if* $p_{\alpha+1} \leq^S_\alpha p_\alpha$ *for every* α *and* $p_\gamma = \bigcup_{\alpha<\gamma}$ *for limit* α.

Notice that $S = \kappa$ gives the original definition of $\leq_\alpha$ and fusion.

Lemma 2.10 *Assume* I *is a normal ideal on* κ*, and* S *is a set in* I^* *which contains only limit ordinals. Then* P_I *is closed under limits of* S*-fusion sequences.*

Proof Let $\langle p_\alpha \,|\, \alpha < \kappa\rangle$ be an S-fusion sequence. Then

$$S \cap (\triangle_{\alpha<\kappa}(\kappa \setminus \mathrm{dom}(p_\alpha))) \subseteq \bigcap_{\alpha<\kappa} (\kappa \setminus \mathrm{dom}(p_\alpha)).$$

To see this, let ξ be a limit ordinal in the set on the left hand side. By the properties of the diagonal intersection, $\xi \notin \mathrm{dom}(p_\zeta)$ for every $\zeta < \xi$; by continuity of the fusion sequence, $\xi \notin \mathrm{dom}(p_\xi)$; by (i) of definition (2.9), and the fact that ξ is in S, $\xi \notin \mathrm{dom}(p_{\xi+1})$ and therefore by (ii) of definition (2.9), $\xi \notin \bigcap_{\alpha<\kappa} \mathrm{dom}(p_\alpha)$. □

Notice that to be a fusion sequence or an S-fusion sequence for $S \in I^*$ in P_I are properties of certain sequences of conditions in the same underlying forcing notion $(P_I, \leq)$.

3 Lifting

3.1 Elementary facts about lifting We now provide a quick review of the results relevant to lifting of embeddings.

Definition 3.1 *Assume GCH. We say that* $j : V \to M$ *with critical point* κ *is a* (κ,λ)-extender ultrapower embedding *if*

$$M = \{ j(f)(\alpha) \,|\, f : \kappa \to V \;\&\; \alpha < \lambda\}$$

for some regular λ *with* $\kappa \leq \lambda < j(\kappa)$.

For more details and more general definitions, see [3].

Fact 3.2 *Let* $\mathbb{P}$ *be a forcing notion,* G *a* P*-generic filter over* V*, and* $j : V \to M$ *an embedding with critical point* κ*. Then the following hold:*

- *(i) (Silver) Assume* H *is* $j(\mathbb{P})$*-generic over* M *such that* $j[G] \subseteq H$*. Then there exists an elementary embedding* $j^* : V[G] \to M[H]$ *such that* $j^* \restriction V = j$*, and* $H = j^*(G)$*. We say that* j *lifts to* $V[G]$*.*
- *(ii) If* j *is moreover a* (κ,λ)*-extender ultrapower embedding and* $\mathbb{P}$ *is a* κ^+*-distributive forcing notion, then the filter* G^* *in* $j(\mathbb{P})$ *defined as*

  $$G^* = \{q \,|\, \exists p \in G, j(p) \leq q\}$$

  is $j(\mathbb{P})$*-generic over* M*.*

(iii) If $j : V \to M$ is a (κ,λ)-extender ultrapower embedding then so is $j^* : V[G] \to M[H]$ (with the same κ and λ).

Proof For proofs, see [3]. □

3.2 Preserving measurability

Definition 3.3 *Let $j : V \to M$ be an elementary embedding with critical point κ. We say that a normal ideal I on κ is* lifting-friendly *if*

$$\kappa \notin j(A), \quad \text{for some } A \in I^*,$$

where I^* is the dual of I.

Examples. The nonstationary ideal on κ is not lifting friendly because κ is an element of $j(C)$ for every closed unbounded subset C of κ. For any regular $\mu < \kappa$, let E^μ_κ denote the set of all limit ordinals with cofinality μ. If I is dual to $\mathrm{Club}(\kappa)[E^\mu_\kappa]$ (see (2.1) for notation), then I is lifting-friendly.

Definition 3.4 *Let P be a forcing notion and let κ be a regular cardinal. Assume that every decreasing sequence of conditions in P of length $\leq \kappa$ has an infimum in P and let $X \subseteq P$ be given. Then*

$$\mathrm{Cl}_{\leq\kappa}X = \{p \in P \mid (\exists \text{ decreasing } \langle p_\alpha \mid \alpha < \kappa\rangle \subseteq X)(\inf(\langle p_\alpha \mid \alpha < \kappa\rangle) \leq p)\}$$

*is called the κ-*closure of X.

It is easy to see that that if X is a directed family (for every x, y in X there exists z in X such that $z \leq x$ & $z \leq y$) closed under limits of sequences of length less than κ, then $\mathrm{Cl}_{\leq\kappa}X$ is a filter in P.

Notation For an inaccessible cardinal α, an ordinal $\beta \geq 1$, and a normal ideal I_α on α, let $P_{I_\alpha}(\alpha,\beta)$ denote the product of β-copies of P_{I_α} with support $\leq \alpha$, where P_{I_α} is defined as in Definition 2.3.

We now introduce fusion sequences in the context of product forcings.

Definition 3.5 *Let $p, q \in P_{I_\alpha}(\alpha,\beta)$. Given $S \in I^*_\alpha$, $F \subseteq \beta$ with $|F| < \alpha$ and $\delta < \alpha$ we define*

$$p \leq^S_{F,\delta} q \quad \leftrightarrow \quad p \leq q \text{ and } p(\xi) \leq^S_\delta q(\xi) \text{ for all } \xi \in F.$$

Moreover, we say that a sequence

$$(\langle p_\delta \mid \delta < \alpha\rangle, \langle F_\delta \mid \delta < \alpha\rangle)$$

*is an S-*fusion sequence *if it satisfies the following conditions:*

(i) $|F_\delta| < \alpha$, $F_\delta \subseteq F_{\delta+1}$ *for every* $\delta < \alpha$,

(ii) $F_\gamma = \bigcup_{\delta<\gamma} F_\delta$ *for every limit* $\gamma < \alpha$ *and* $\bigcup_{\delta<\alpha} F_\delta = \bigcup_{\delta<\alpha} \mathrm{supp}(p_\delta)$

(iii) $\mathrm{supp}(p_\gamma) = \bigcup_{\delta<\gamma} \mathrm{supp}(p_\delta)$ *and* $p_\gamma(\xi) = \bigcup_{\delta<\gamma} p_\delta(\xi)$ *for limit* $\gamma < \alpha$, ξ *in the support of* p_γ *and*

(iv) $p_{\delta+1} \leq^S_{F_\delta,\delta} p_\delta$ *for every* $\delta < \alpha$.

The limit of such a sequence is a condition q *with*

$$\mathrm{supp}(q) = \bigcup_{\delta<\alpha} \mathrm{supp}(p_\delta) \quad \text{and} \quad q(\xi) = \bigcup_{\delta<\alpha} p_\delta(\xi) \quad \text{for } \xi \in \mathrm{supp}(q).$$

We now state and prove the main result of this paper. In the theorem we use an apparently stronger assumption on the strength of j than the one given in (1.1). However, it can be shown, possibly with some collapsing, that the condition (1.1) is sufficient (see [10] for details).

Theorem 3.6 *Assume GCH and let* κ *be a critical point of a* (κ,κ^{++})*-extender ultrapower embedding* $j : V \to M$ *such that*

(i) ${}^{\kappa}M \subseteq M$ *and*

(ii) $\kappa^{++M} = \kappa^{++}$.

Fix some regular cardinal μ *below the first inaccessible and let* $\mathbb{P} = \mathbb{P}_{\kappa+1}$ *be the reverse-Easton iteration of length* $\leq \kappa$ *which forces at each inaccessible* $\alpha \leq \kappa$ *with* $P_{I_\alpha}(\alpha,\alpha^{++})$ *(where* I_α *denotes the dual ideal to* $\mathrm{Club}(\alpha)[E^\mu_\alpha]$ *in* $V^{\mathbb{P}_\alpha}$*).*

If $G * g$ *is* $\mathbb{P}_{\kappa+1} = \mathbb{P}_\kappa * P_{I_\kappa}(\kappa,\kappa^{++})$*-generic over* V*, then one can lift* j *to* $V[G*g]$ *inside* $V[G*g]$*, thus showing that* κ *remains measurable in* $V[G*g]$.

Proof Using standard arguments, one can lift in $V[G*g]$ to

$$j : V[G] \to M^* = M[G*g*H].$$

To see this, realize that $j(\mathbb{P}_\kappa)$ restricted to $\kappa+1$ is identical to $\mathbb{P}_{\kappa+1}$. By the extender representation of j, and by Fact 3.2(iii) each relevant dense open subset of the the iteration $j(\mathbb{P}_\kappa)$ in the interval $(\kappa+1, j(\kappa))$ is of the form $j(f)(\alpha)$ for some $\alpha < \kappa^{++}$ and $f : \kappa \to DO(\mathbb{P}_\kappa)$, where $DO(\mathbb{P}_\kappa)$ is the set of dense open subsets of $\mathbb{P}_\kappa$. Since the iteration is $(\kappa^{+3})^M$-distributive over $M[G*g]$ in the interval the following sets

$$D_f = \bigcap_{\alpha<\kappa^{++}} j(f)(\alpha)$$

are all open dense. Since GCH holds in V there are only κ^+ many functions $f : \kappa \to DO(\mathbb{P}_\kappa)$, so we can build our generic H by induction of length κ^+.

By Silver's theorem (Fact 3.2(i)) it will be sufficient to prove the following claim.

Claim 3.7 *Let* P *denote* $P_{I_\kappa}(\kappa,\kappa^{++})$*. We claim that*

$$h = \mathrm{Cl}_{\leq\kappa}\, j[g] \text{ is a } j(P)\text{-generic filter over } M^*.$$

Proof of Claim 3.7 It is easy to see that h is a filter and is well-defined because by standard arguments M^* is closed under κ-sequences in $V[G*g]$, and $j(P)$ is κ^+-closed in M^*.

By Fact 3.2 (iii), every dense open set in $j(P)$ is of the form $j(f)(\alpha)$ for some f in $V[G]$ and $\alpha < \kappa^{++}$. Moreover, we can assume that $\langle f(\alpha) \mid \alpha < \kappa \rangle$ is a sequence of dense open sets in P in $V[G]$ for every such f.

Fix a dense open set D in $j(P)$, represented as $j(f)(\alpha_0)$ for some f as in the preceding paragraph, and $\alpha_0 < \kappa^{++}$. We will show that $h \cap D$ is non-empty.

Now work in $V[G]$. Choose some function $e : \kappa \to \kappa$ such that $j(e)(\kappa) \geq \kappa^{++}$ and $e(\xi) \geq \xi$ for each $\xi < \kappa$; for instance $e(\alpha) = |\alpha|^{++}$. We say that $\alpha < \kappa$ is a *closure point* of e if $e(\beta) < \alpha$ for every $\beta < \alpha$. Let S denote the stationary set $E^{\mu}_{\kappa} = \{\alpha < \kappa : \operatorname{cf} \alpha = \mu\}$.

Given $p \in P$, we will construct an S-fusion sequence

$$(\langle p_\alpha \mid \alpha < \kappa \rangle, \langle F_\alpha \mid \alpha < \kappa \rangle)$$

with limit q. Let $p_0 = p$. At limit stage $\alpha < \kappa$ for ξ in the domain of p_α, let

$$F_\alpha = \bigcup_{\delta<\alpha} F_\delta \quad \text{and} \quad p_\alpha(\xi) = \bigcup_{\delta<\alpha} p_\delta(\xi),$$

so that conditions (ii,iii) of definition (3.5) are satisfied.

At successor stage $\alpha+1$ where α is not a regular closure point of e greater than μ, do nothing, i.e. $F_{\alpha+1} = F_\alpha$ and $p_{\alpha+1} = p_\alpha$. Note that all elements of S are in this category.

At successor stage $\alpha+1$, where α is a regular closure point of e greater than μ, do the following. When defining $F_{\alpha+1}$ all that is required is some bookkeeping device so that in the end $\bigcup_{\alpha<\kappa} F_\alpha$ is equal to the support of the fusion limit of p_α's. So it remains to describe the construction of $p_{\alpha+1}$. We first fix some $\lambda < \kappa$ and an enumeration $\langle x^{\xi}_{\alpha} \mid \xi < \lambda \rangle$ of all functions f with domain F_α such that for each $\zeta \in F_\alpha$, $x^{\xi}_{\alpha}(\zeta)$ is a function with domain α which is compatible with $p_\alpha(\zeta)$. (This can be done since κ is inaccessible.)

We let $p_{\alpha+1}$ be the limit of a $\leq^{S}_{F_\alpha,\alpha}$-decreasing sequence $\langle p^{\xi}_{\alpha} \mid \xi < \lambda \rangle$ below p_α constructed as follows. We let $p^0_\alpha = p_\alpha$ and, since $\alpha \notin S$, we can also ensure that

$$(\star) \qquad \alpha \in \operatorname{dom}(p^1_\alpha(\zeta)) \text{ for each } \zeta \in F_\alpha.$$

At limit stages we take the infima and at successor stages we make sure that $p^{\xi+1}_{\alpha}$ strengthened by x^{ξ}_{α} coordinate-wise is in $D_{e(\alpha)} = \bigcap_{\beta<e(\alpha)} f(\beta)$,

i.e. if p is defined as

$$p(\zeta) = \begin{cases} x^{\xi}_{\alpha}(\zeta) \cup p^{\xi+1}_{\alpha}(\zeta) & \text{for } \zeta \in F_\alpha \\ p(\zeta) = p^{\xi+1}_{\alpha}(\zeta) & \text{for } \zeta \in \mathrm{dom}(p^{\xi+1}) \setminus F_\alpha \end{cases}$$

then p is in $D_{e(\alpha)}$.

By construction, $(\langle p_\alpha \,|\, \alpha < \kappa\rangle, \langle F_\alpha \,|\, \alpha < \kappa\rangle)$ is an S-fusion sequence. Let q be its limit. Since we worked below an arbitrary p, we can assume that q is in g.

Observe that by ($\star$), if α is a regular closure point of e greater than μ, then

(†) $$\alpha \in \mathrm{dom}(q(\zeta)) \text{ for } \zeta \in F_\alpha.$$

Moreover, for each regular closure point α of e greater than μ and every $r \leq q$, it holds:

(‡) If $[0,\alpha] \subseteq \mathrm{dom}(r(\xi))$ for every $\xi \in F_\alpha$, then $r \in D_{e(\alpha)}$.

Denote $F = \bigcup_{\alpha<\kappa} F_\alpha = \mathrm{supp}(q)$. Note that $F = F^*_\kappa$, where

$$\langle F^*_\alpha \,|\, \alpha < j(\kappa)\rangle = j(\langle F_\alpha \,|\, \alpha < \kappa\rangle).$$

Choose below $j(q)$ a $\leq$-decreasing sequence $\langle j(r_\alpha) \,|\, \alpha < \kappa\rangle$ of conditions in $j[g]$ such that $r_0 = q$, each r_α is in g, $\mathrm{supp}(r_\alpha) = \mathrm{supp}(q)$, and satisfies that $[0,\alpha]$ is included in the domain of $r_\alpha(\xi)$ for each $\xi \in \mathrm{supp}(q)$. Such a sequence exists by a density argument. Let r be the limit of $\langle j(r_\alpha) \,|\, \alpha < \kappa\rangle$ in M^*; r exists because ${}^\kappa M^* \subseteq M^*$ in $V[G*g]$ and $j(P)$ is κ^+-closed in M^*. We claim:

Claim 3.8 *Condition r is in $h \cap D$.*

Proof of Claim 3.8 The condition r is clearly in h, and so it suffices to check that it hits $D = j(f)(\alpha_0)$ as well. Notice that κ is a regular closure point of $j(e)$ greater than $j(\mu) = \mu$. By (‡), the inequality $r \leq j(q)$, and elementarity, it suffices to show that $[0,\kappa]$ is included in the domain of $r(\zeta)$ for each $\zeta \in F = F^*_\kappa$ – then r meets $\bigcap_{\beta<j(e)(\kappa)} j(f)(\beta) \subseteq D$ as desired. However, this is easy: The cardinal κ is in the domain of $r(\zeta)$ as an element for each $\zeta \in F$ because this already holds for $j(q)$ by (†) and by elementarity, and κ is included in the domain of $r(\zeta)$ for $\zeta \in F$ as a subset because r is the limit of $j(r_\alpha)$'s. ⊣

This finishes the proof of Claim (3.7) and hence the proof of the theorem. □

Note that as a corollary of the proof of the theorem (with $\alpha = \kappa$), we obtain that $P_{I_\alpha}(\alpha,\beta)$ preserves α^+ for α inaccessible. It follows that under GCH, $P_{I_\alpha}(\alpha,\beta)$ preserves cofinalities.

Remark 3.9 By incorporating ideas of [12], the above argument carries over to iterations of the forcing P_I. Essentially, since we deal with names here, one needs to "determine" the proper initial segments of the conditions to carry out the fusion argument.

Remark 3.10 By incorporating ideas from [12] and [5] and a $\diamondsuit'$-based fusion construction, one can use the Grigorieff forcing at successor cardinals. This is useful in the context of supercompact cardinals, or generic elementary embeddings (which can have a critical point a successor cardinal in the larger universe). Without going into much details, note that the key point of the constructions in [12] and [5] is an appropriate version of the fusion argument for trees which is easily generalizable to the fusion properties of the Grigorieff forcing introduced in this paper.

Remark 3.11 (With the same notation as in Theorem 3.6.) If I_κ is not lifting-friendly, then the closure $\mathrm{Cl}_{\leq\kappa} j[g]$ does not give rise to a generic filter: Fix $\xi < \kappa^{++}$. If p is in g, then $\mathrm{dom}(p(\xi))$ is in I_κ, and therefore $j(p)$ at $j(\xi)$ is not defined on $A_p = j(\kappa \setminus \mathrm{dom}(p(\xi)))$. By the assumption of not being lifting-friendly, every A_p contains κ as an element and therefore

$$\bigcup_{p\in g} \mathrm{dom}(j(p)(j(\xi))) = \bigcup \{\mathrm{dom}(r(j(\xi))) \mid r \in \mathrm{Cl}_{\leq\kappa} j[g]\}$$

does not contain κ as an element. This implies that $\mathrm{Cl}_{\leq\kappa} j[g]$ is not generic over M^* because by a density argument, such a generic must be defined on κ on every $\xi < j(\kappa)^{++}$.

3.3 Lifting and minimality We say that a forcing P is *minimal* if for every P-generic filter G and every subset Y of ordinals in $V[G]$, either $Y \in V$, or $G \in V[Y]$. In other words there is no inner model strictly included between V and $V[G]$.

Fact 3.12 *Let κ be an inaccessible cardinal. Then the Sacks forcing at κ is minimal.*

For a proof of this fact and more information on the topic of minimality, see [2].

By an easy generalization of Proposition 3.3 in [11], the Grigorieff forcing we have used is not minimal:

Observation 3.13 *Let κ be regular and I a normal non-prime ideal on κ. Then P_I is not minimal over the ground model.*

Proof Let S be a subset of κ such that neither S nor $S' = \kappa \setminus S$ is in I; this is possible because I is not prime. Let $I|S$ denote the set $\{X \cap S \,|\, X \in I\}$, and let $P_{I|S}$ denote the following set of forcing conditions:

$$P_{I|S} = \{f \colon \kappa \to 2 \,|\, \mathrm{dom}(f) \in I|S\},$$

and similarly for $P_{I|S'}$. Then clearly

$$P_I \cong P_{I|S} \times P_{I|S'}.$$

By our assumption on S, both forcings $P_{I|S}$ and $P_{I|S'}$ are nontrivial and therefore P_I is not minimal: if G is P_I-generic, then $V[G] = V[G_1][G_2]$, where $G_1 \times G_2$ is $P_{I|S} \times P_{I|S'}$-generic. □

Corollary 3.14 *Assume κ is regular and let $S = E^{\mu}_{\kappa}$ for some regular $\mu < \kappa$. Let I be the dual ideal to $\mathrm{Club}(\kappa)[S]$. Then P_I is not minimal.*

By a more complicated argument it can be shown that Grigorieff forcing at uncountable cardinals (even when defined for co-ideals) is never minimal, see [1] for details.

4 Questions

Question. Is there a combinatorial property related to cardinal invariants at a regular $\kappa > \omega$ which distinguishes the generic extension by the Grigorieff forcing and by the Sacks forcing (product and iteration)? The techniques of this paper would be useful to obtain a model with this property with a measurable cardinal κ.

Question. In [13] it was shown that the classical Grigorieff forcing on ω either collapses cardinals or can be decomposed into an iteration of an ω_1-closed notion of forcing followed by a ccc notion of forcing. Does a similar decomposition work for the general case?

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