Preserving measurability with Cohen iterations

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Abstract: We describe a weak version of Laver indestructibility starting just with an $H(\lambda)$-hypermeasurable κ for some $\lambda > \kappa^+$, where "weaker" means that the indestructibility refers only to the Cohen forcing at κ of a certain length. A succinct version of this result is as follows: if λ is equal to κ^{+n} for some $1 < n < \omega$, then one can get a model V^* where κ is measurable, and its measurability is indestructible by $\mathrm{Add}(\kappa, \alpha)$ for any $0 < \alpha < \kappa^{+n}$ (Theorem 2.8).

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1 Introduction

Assume κ is supercompact. In [9], Laver defined an iteration P of length κ such that in $V[P]$,[1] κ is still supercompact and every further κ-directed closed forcing preserves the supercompactness of κ (P is often called the *Laver preparation*). We also say that κ is Laver-indestructible in $V[P]$. The proof of this indestructibility result is essentially based on two useful properties of a supercompact cardinal κ in V: (i) for every $\lambda \geq \kappa$, one

[1] $V[P]$ indicates a P-generic extension of V whenever it is not important to distinguish specific P-generic filters. For instance the statement "φ holds in $V[P]$" means that φ holds in $V[G]$ for every P-generic filter G.

can choose an elementary embedding $j : V \to M$ with critical point κ such that M is closed under λ-sequences existing in V; this closure is then used to find a *master condition* in M and proceed with a lifting argument which ensures that supercompactness is preserved,[2] (ii) there is a single function $f : \kappa \to V_\kappa$ such that for every $x \in V$, one can choose an embedding j in (i) so that $j(f)(\kappa) = x$ (this f is often called the *Laver function*).

A typical example of a κ-directed closed forcing is the Cohen forcing at κ, which we will denote by $\mathrm{Add}(\kappa, \alpha)$,[3] where α is any ordinal larger than 0. The fact that over $V[P]$, $\mathrm{Add}(\kappa, \alpha)$ preserves the measurability of κ is very useful when one wishes to use some large cardinal properties of κ in $V[P][\mathrm{Add}(\kappa, \alpha)]$ (see for instance [4] where a model with the tree property at κ^{++}, κ strong limit singular with cofinality ω, is constructed starting with a supercompact κ).

A natural question is whether a "Laver-like" indestructibility is available also for smaller large cardinals. As it turns out, it is the property (i) above which is more important: it is known that for instance a strong cardinal[4] κ has the analogue of the Laver function, but it is not known whether it can be made indestructible under κ-directed closed forcings.[5]

In this short paper we use the idea of Woodin (as described in [2]) to argue that with a well-designed preparation it is possible to have a limited indestructibility of a $H(\lambda)$-hypermeasurable[6] κ, λ regular, $\lambda > \kappa^+$,

[2] Assume $j : V \to M$ is an elementary embedding, P is a forcing notion, G is P-generic over V, and H is $j(P)$-generic over M. Then a sufficient condition for j to *lift*, i.e. a sufficient condition for the existence of $j^+ : V[G] \to M[H]$ with $j^+ \restriction V = j$, is that we have $j``G \subseteq H$. With supercompactness, we can often argue that $j``G$ is a condition in M (a master condition), and H can then be built below this master condition. For more details, see [3].

[3] Formally speaking, conditions in $\mathrm{Add}(\kappa, \alpha)$ are partial functions of size $< \kappa$ from $\kappa \times \alpha$ to 2. The ordering is by reverse inclusion.

[4] An infinite cardinal κ is *strong* if for every $\lambda \geq \kappa$ there is $j : V \to M$ with $H(\lambda) \subseteq M$.

[5] A non-supercompact strong cardinal κ can be indestructible under κ-directed closed forcings by a method of [1], but κ needs to be supercompact in the ground model.

[6] κ is $H(\lambda)$-*hypermeasurable* (also $H(\lambda)$-*strong*) if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $H(\lambda) \subseteq M$, and M is

in the sense that we can successively extend $V \subseteq V^1 \subseteq V^*$ such that V^1 preserves the initial degree of hypermeasurability of κ, and forcing over V^* with $\mathrm{Add}(\kappa, \lambda)$ yields the measurability, in fact the hypermeasurability, of κ (Theorem 2.4). Under additional assumptions on λ, namely if $\lambda = \kappa^{+n}$ for some $1 < n < \omega$, forcing with $\mathrm{Add}(\kappa, \alpha)$ over V^* for $0 < \alpha \leq \lambda$ yields the measurability, in fact hypermeasurability, of κ (Theorem 2.6 and Theorem 2.8). Note that in V^*, κ may actually stop being measurable[7] depending on the iteration P_κ which gives $V^* = V^1[P_\kappa]$; compare the constructions in Theorem 2.6 and 2.8.

Although much weaker than the indestructibility of a supercompact κ, it may still be a useful property: without giving the details, it is possible to re-do the argument in [4] starting with a $H(\lambda)$-hypermeasurable cardinals, where λ is the least weakly compact above κ.

Remark 1.1 We assume that the reader is familiar with the lifting arguments. The general reference is [3]; the more specific constructions used in the present paper are also given in [2].

2 Construction of the model

Assume GCH holds in the ground model V. We show that if κ is $H(\lambda)$-hypermeasurable for some regular $\lambda > \kappa^+$,[8] then in some cofinality-preserving generic extension V^* of V, Cohen forcing $\mathrm{Add}(\kappa, \lambda)$ yields the measurability of κ in $V^*[\mathrm{Add}(\kappa, \lambda)]$. We further show that under some additional assumptions on λ, we can say more (see Section 2.3).

The model V^* is equal to $V^1[G_\kappa]$, where V^1 prepares for the lifting with respect to the next extension to V^*, and moreover V^1 preserves the $H(\lambda)$-hypermeasurability of κ (Section 2.1). The model V^1 is then

closed under κ-sequences in V.

[7]If in V^*, κ is not measurable, and it is measurable again in $V^*[\mathrm{Add}(\kappa, \alpha)]$ (for a specific α), it is more appropriate to call this step a "resurrection" of the measurability of κ.

[8]We are interested in models where GCH fails at κ, so we focus on λ's such that $\lambda > \kappa^+$.

extended to $V^1[G_\kappa] = V^*$ by a standard reverse-Easton iteration of length κ (Section 2.2).

2.1 Stage 1: the model V^1

Let us fix $j : V \to M$ which witnesses the $H(\lambda)$-hypermeasurability of κ: in particular, the critical point of j is κ, M is closed under κ-sequences in V and $H(\lambda)$ is a subset of M. Let U be the normal measure derived from j, and let $i : V \to N$ be the ultrapower embedding generated by U.

Using a fast-function forcing of Woodin, we can assume that there is $f : \kappa \to \kappa$ in V such that $j(f)(\kappa) = \lambda$. Let us denote $f(\alpha)$ by λ_α; let $C(f)$ denote the closed unbounded set of the closure points of f: if $\alpha \in C(f)$, then for all $\beta < \alpha$, $f(\beta) < \alpha$.

We plan to extend V by forcing to some V^1 with the same cofinalities, so that the following hold in V^1:

(A) There is $j^1 : V^1 \to M^1$ with critical point κ such that $H(\lambda) \subseteq M^1$ and j^1 restricted to V is the original j.

(B) If U^1 is the normal measure derived from j^1, and $i^1 : V^1 \to N^1$ is the ultrapower embedding for U^1, then in V^1 there is g which is $i^1(P)$-generic over N^1, where $P = \mathrm{Add}(\kappa, \lambda)^{V^1}$. i^1 restricted to V is the original i.

The model V^1 is obtained as follows. Define P^1 in V as an Easton-supported iteration

$$\langle (P^1_\alpha, \dot{Q}_\alpha) \,|\, \alpha < \kappa, \alpha \text{ is measurable}, \alpha \in C(f) \rangle * \dot{Q}_\kappa,$$

where

- $\dot{Q}_\alpha$ is chosen generically[9] amongst all forcings R which satisfy the following: (a) there exists in $V[P^1_\alpha]$ a normal measure U_α on α such

[9] If the forcing is chosen "generically", the related construction is often called the "lottery sum of the relevant forcing notions"; see [7] for more information.

that the derived ultrapower embedding i_α satisfies

$$i_\alpha : V[P^1_\alpha] \to N_\alpha[i(P^1_\alpha)]$$

for some N_α, (b) R is the forcing

$$i_\alpha(\mathrm{Add}(\alpha, \lambda_\alpha)^{V[P^1_\alpha]}). \tag{2.1}$$

If the above is not possible, let $\dot{Q}_\alpha$ be $\{1\}$.

- Suppose i lifts in $V[P^1_\kappa]$ to

$$i_\kappa : V[P^1_\kappa] \to N[i_\kappa(P^1_\kappa)].$$

Define $\dot{Q}_\kappa$ to be equal to

$$i_\kappa(\mathrm{Add}(\kappa, \lambda)^{V[P^1_\kappa]}). \tag{2.2}$$

If i does not lift to i_κ in $V[P^1_\kappa]$, set $\dot{Q}_\kappa = \{1\}$.

The following Lemma is implicit in [5]:

Lemma 2.1 *P^1 preserves cofinalities.*

PROOF. The following suffices: Suppose P^1_α, $\alpha \leq \kappa$, preserves regularity of cardinals and the only violation of GCH below α is that for every $\beta < \alpha$, where the forcing is non-trivial, we have $2^{\beta^+} = \lambda_\beta$ (this is without loss of generality). We show that this will hold for $P^1_{\alpha+1}$.

Suppose $\dot{Q}_\alpha$ is non-trivial and let G_α be P^1_α-generic. Then in $V[G_\alpha]$, $i_\alpha : V[G_\alpha] \to N_\alpha[i_\alpha(G_\alpha)]$ is a measure ultrapower embedding. It suffices to show that $R = (\dot{Q}_\alpha)^{G_\alpha}$ is α^+-closed and α^{++}-Knaster in $V[G_\alpha]$. Closure is obvious because $N_\alpha[i_\alpha(G_\alpha)]$ is closed under α-sequences in $V[G_\alpha]$ and R is α^+-closed in $N_\alpha[i_\alpha(G_\alpha)]$. Regarding the Knaster property, notice that every element of R can be identified with the equivalence class of some function $f : \alpha \to \mathrm{Add}(\alpha, \lambda_\alpha)$. For $f, g : \alpha \to \mathrm{Add}(\alpha, \lambda_\alpha)$, set $f < g$ if for all $i < \alpha$, $f(i) < g(i)$; it suffices to check that the ordering $<$ on these f's is α^{++}-Knaster. Let A be a maximal antichain in this ordering; take an elementary substructure $\bar{M}$ in some large enough $H(\theta)$ of $V[G_\alpha]$ which contains all relevant data, has size α^+ and is closed under α-sequences. Then it is not hard to check that $A \cap \bar{M}$ is maximal in the ordering (and therefore $A \subseteq \bar{M}$), and therefore has size at most α^+. $\square$

Lemma 2.2 *P^1 is non-trivial and forces (A) and (B) on page 4.*

Proof. Let $G_\kappa * g$ be $P^1_\kappa * \dot{Q}_\kappa$-generic over V; let R denote for the purposes of this item the forcing $(\dot{Q}_\kappa)^{V[G_\kappa]}$. It suffices to show that the stage κ is non-trivial, i.e. R is not $\{1\}$ (as the argument carries over to all $\alpha < \kappa$ which are non-trivial). This follows immediately by the standard lifting methods, as i lifts to $V[G_\kappa]$, witnessing the measurability κ in $V[P_\kappa]$.[10]

(A). We show that for the right choice of $j(G_\kappa * g)$, j lifts to j^1 as required. In $M[G_\kappa]$, work below a condition which ensures that at stage κ, $j(P^1_\kappa)$ chooses the forcing R.[11] Then the lifting is carried out using the standard methods.

(B). In $V[G_\kappa]$, R is a κ^+-closed forcing (and is an image of $\mathrm{Add}(\kappa, \lambda)$ under the embedding i lifted to $V[G_\kappa]$), and by a Woodin lemma (see below), g is actually R generic over $N[i^1(G * g)]$. □

Lemma 2.3 (Woodin) *Let κ be a measurable cardinal, U some measure on κ, and $i : V \to N$ the ultrapower embedding by U. Let P be a κ-closed partial ordering and let $Q = i(P)$. Let g be Q-generic over V and transfer it along i to get the lifting $i^1 : V[g] \to N[i^1(g)]$. Then g is in fact Q-generic over $N[i^1(g)]$.*

For proof, see [2], Fact 2, page 7.

2.2 Stage 2: the model V^*

Assume that κ, λ, f, and V^1, N^1, i^1, j^1 are as in Section 2.1.

We will prove the following theorem:

[10] This uses the fact that $i(P^1_\kappa)_\kappa = P_\kappa$ forces the tail iteration of $i(P^1_\kappa)$ in the interval $[\kappa, i(\kappa))$ to be κ^+-closed in $N[G_\kappa]$ (and hence also in $V[G_\kappa]$); moreover the number of antichains in the tail iteration is just κ^+ in $V[G_\kappa]$.

[11] Note that in $M[G_\kappa]$, κ is measurable and the measure corresponding to $i_\kappa : V[G_\kappa] \to N[i_\kappa(G_\kappa)]$ is also an element of $M[G_\kappa]$ since $H(\lambda)$ of $V[G_\kappa]$ is included in $M[G_\kappa]$; also note that in $M[G_\kappa]$, $\lambda_\kappa = j(f)(\kappa) = \lambda$, and therefore R is an element of $M[G_\kappa]$ and can legitimately be considered as $\dot{Q}_\kappa$ for the iteration $j(P^1_\kappa)$.

Theorem 2.4 *There is a forcing iteration P_κ defined in V^1 such that*

$$V^1[P_\kappa][\mathrm{Add}(\kappa,\lambda)] \models \text{ } \kappa \textit{ is measurable},$$

where $\mathrm{Add}(\kappa,\lambda)$ *is defined in* $V[P_\kappa]$*. Hence the model* $V^1[P_\kappa] = V^*$ *is as desired: the measurability of* κ *is obtained by* $\mathrm{Add}(\kappa,\lambda)$*.*

Proof. Define P_κ to be the following Easton-supported iteration:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \,|\, \alpha < \kappa \text{ is measurable}, \alpha \in C(f) \rangle,$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathrm{Add}(\alpha, \lambda_\alpha)$, and λ_α equals $f(\alpha)$.

The proof uses the usual surgery argument (see [3]) with the following lemma which allows us to use the generic filter g added in V^1 (for the i^1-image of $\mathrm{Add}(\kappa,\lambda)^{V^1}$) in the model $V^1[P_\kappa]$.

We first show a general lemma (for more details, see Fact 2 in [2]).

Lemma 2.5 *Let S be a κ-cc forcing notion of cardinality κ, $\kappa^{<\kappa} = \kappa$. Then for any λ, the term forcing $Q_\lambda = \mathrm{Add}(\kappa,\lambda)^{V[S]}/S$ is isomorphic to* $\mathrm{Add}(\kappa,\lambda)$*.*

Proof. Recall that Q_λ is the term forcing defined as follows: the elements of Q_λ are names τ such that τ is an S-name and it is forced by 1_S to be in $\mathrm{Add}(\kappa,\lambda)$ of $V[S]$. The ordering is $\tau \leq \sigma \leftrightarrow 1_S \Vdash \tau \leq \sigma$.

If β is a cardinal, let T_β denote the partial order with a top element and β incomparable extensions (which we identify with the elements in β). Recall first that $\mathrm{Add}(\kappa,\lambda)$ is isomorphic to the product of λ copies of T_2 with support $< \kappa$. Using the κ-cc of S, one can show that Q_λ is the product of λ copies of T_2/S with $< \kappa$-support. Further, T_2/S is isomorphic to T_κ since the size of S is κ: thus Q_λ is isomorphic to a product of λ copies of T_κ with $< \kappa$-support, and therefore isomorphic to $\mathrm{Add}(\kappa,\lambda)$. □

Let $G_\kappa * h$ be $P_\kappa * \mathrm{Add}(\kappa,\lambda)^{V^1[P_\kappa]}$-generic.

By Lemma 2.5 applied to $S = P_\kappa$, $\mathrm{Add}(\kappa,\lambda)^{V^1[P_\kappa]}/P_\kappa$ is isomorphic in V^1 to $\mathrm{Add}(\kappa,\lambda)$ of V^1. It follows that after lifting to

$$i^1 : V^1[G_\kappa] \to N^1[i(G_\kappa)]$$

in $V^1[G_\kappa * h]$, the generic filter g constructed at stage 1 is (or more precisely, yields a generic which is)

$$i^1(\mathrm{Add}(\kappa,\lambda))\text{-generic over } N^1[i(G_\kappa)]. \tag{2.3}$$

This suffices to proceed with the usual surgery argument (for details see [2]) and lift the embedding j^1 to $V^1[G_\kappa * h]$, showing that κ is measurable in $V^1[G_\kappa * h]$. □

2.3 Some generalizations

It seems natural to extend Theorem 2.4 and have that the measurability of κ ensured by $\mathrm{Add}(\kappa,\alpha)$ for any ordinal α, $0 < \alpha \le \lambda$. We will show that this can be achieved with some additional assumptions on λ. For concreteness, we will focus on the example where $\lambda = \kappa^{+n}$ for some $1 < n < \omega$.

First, in Theorem 2.6, we provide a standard construction which actually forces κ to stop being measurable in V^*; the measurability of κ is then resurrected by $\mathrm{Add}(\kappa,\alpha)$ for any $\kappa^+ \le \alpha \le \kappa^{+n}$.

Theorem 2.6 *(GCH) Let $1 < n < \omega$ be fixed and assume κ is $H(\kappa^{+n})$-hypermeasurable. Then there is an iteration P^1 such that in $V[P^1] = V^1$, κ is still κ^{+n}-hypermeasurable, and for some reverse Easton iteration P_κ defined in V^1, κ stops being measurable in $V^* = V^1[P_\kappa]$. In V^*, the measurability of κ is resurrected by Cohen forcing $\mathrm{Add}(\kappa,\alpha)$ for any $\kappa^+ \le \alpha \le \kappa^{+n}$.*

Proof. Let j witness the $H(\kappa^{+n})$-hypermeasurability of κ, and let i be a normal embedding generated by the normal measure U derived from j. Recall Lemma 3.2 from [6] which implies that if $i : V \to N$ is an embedding generated by a normal measure on κ, then

$$\mathrm{Add}(i(\kappa), i(\kappa)^{+n})^N \cong \mathrm{Add}(\kappa^+, \kappa^{+n}). \tag{2.4}$$

This allows us to define P^1 more simply as follows: Define P^1 is an Easton-supported iteration

$$\langle (P^1_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa, \alpha \text{ is inaccessible}\rangle * \dot{Q}_\kappa,$$

where for an inaccessible $\beta \le \kappa$, $\dot{Q}_\beta$ is $\mathrm{Add}(\beta^+, \beta^{+n})$ of $V[P^1_\beta]$. Lemma 2.1 now follows easily since $\dot{Q}_\beta$ is just a Cohen forcing; Lemma 2.2 is also easier since is now a standard lifting argument (it is not necessary to work below a specific condition); the generic g of (B) in Lemma 2.2 is available by (2.4).

Let $G_\kappa * g$ be $P^1_\kappa * \dot{Q}_\kappa$-generic over V, and denote $V[G_\kappa * g]$ by V^1. Let j^1 and i^1 be the liftings of j and i, respectively, as in Lemma 2.2.

In V^1 define P_κ as an Easton supported iteration:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \,|\, \alpha < \kappa \text{ is inaccessible} \rangle, \tag{2.5}$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathrm{Add}(\alpha, \alpha^{+n})$ of $V^1[P_\alpha]$.

First note that κ stops being measurable in $V^* = V^1[P_\kappa]$ by the application of the gap-forcing theorem of [8]: a hypothetical embedding k with critical point κ found in V^* could be written as an embedding from $V^1[P_\kappa]$ to some $N[j(P_\kappa)]$, with $N \subseteq V^1$; in particular a generic filter for $j(P_\kappa)$ would need to add a non-trivial generic filter at stage κ which cannot be found in $V^1[P_\kappa]$; contradiction.

The rest of the Theorem follows from the following Claim:

Claim 2.7 *Let α be an ordinal, $\kappa^+ \le \alpha \le \kappa^{+n}$. Then κ is still measurable in $V^1[P_\kappa][\mathrm{Add}(\kappa, \alpha)]$, where $\mathrm{Add}(\kappa, \alpha)$ is defined in $V^1[P_\kappa]$.*

Proof. It suffices to show the Claim for α's which are cardinals. So assume $\kappa^{+m} = |\alpha|$ for some $1 \le m \le n$. Choose in V^1 an embedding $j_m : V^1 \to M_m$ which witnesses the $H(\kappa^{+m})$-hypermeasurability of κ with $\kappa^{+m} < j_m(\kappa) < \kappa^{+m+1}$ (this is possible since $2^\kappa = \kappa^+$ in V^1). By the definition of P_κ, $j_m(P_\kappa)(\kappa)$ is equal to $\mathrm{Add}(\kappa, \kappa^{+n})^{M_m[P_\kappa]}$. Since $(\kappa^{+n})^{M_m}$ has size κ^{+m} in V^1, $\mathrm{Add}(\kappa, \kappa^{+m})^{V^1[P_\kappa]}$ is equivalent to $\mathrm{Add}(\kappa, \kappa^{+n})^{M_m[P_\kappa]}$, and therefore the generic for $\mathrm{Add}(\kappa, \kappa^{+m})^{V^1[P_\kappa]}$ provides a generic for $\mathrm{Add}(\kappa, \kappa^{+n})^{M_m[P_\kappa]}$. The argument is then finished as in Theorem 2.4, using the fact that the generic g for $i^1(\mathrm{Add}(\kappa, \kappa^{+n}))$ is also generic for $i^1(\mathrm{Add}(\kappa, \kappa^{+m}))$. □

This concludes the proof of Theorem 2.6. □

Note that the method in the proof of Theorem 2.6 does not work for the case of α smaller than κ^+: every elementary embedding $k : V^1 \to M$ with critical point κ sends κ above κ^+ and therefore $\kappa^+ \leq |\kappa^{+n}|$ in V^1; thus $k(P_\kappa)(\kappa)$, which is $\mathrm{Add}(\kappa, \kappa^{+n})^{M[P_\kappa]}$, is in $V^1[P_\kappa]$ equivalent to the Cohen forcing at κ of length at least κ^+. It follows that to lift the embedding, we need to force over $V^1[P_\kappa]$ with a Cohen forcing at κ of length at least κ^+. If $\alpha < \kappa^+$, this condition is not satisfied. We remedy this by a more complicated construction in Theorem 2.8.

Theorem 2.8 *With the assumptions and the notation as in Theorem 2.6, one can define P_κ so that κ is measurable in V^*, and its measurability is indestructible by $\mathrm{Add}(\kappa, \alpha)$ for any $0 < \alpha \leq \kappa^{+n}$.*

Proof. Modify the definition of P_κ in (2.5) so that at an inaccessible $\alpha < \kappa$, $\dot{Q}_\alpha$ is chosen generically[12] amongst the following forcings: $\{1\}$ (the trivial forcing), and $\mathrm{Add}(\alpha, \alpha^{+k})$, for $0 \leq k \leq m$.

Then one can argue that κ is still measurable in V^*: while lifting the embedding j^1, it suffices to work below a condition in $j^1(P_\kappa)$ which chooses the trivial forcing $\{1\}$ at stage κ.

To argue that for any $0 < \alpha \leq \kappa^{+n}$, κ is still measurable in $V^*[\mathrm{Add}(\kappa, \alpha)]$, work below a condition in $j^1(P_\kappa)$ which chooses the right forcing at stage κ. □

2.4 Open questions

Is it possible to get an analogue of Theorem 2.8 for any $\lambda > \kappa$? It might be easier to show this if λ has a "nice definition" (such as the first inaccessible above κ, etc.)

[12]The "lottery preparation"; see Footnote 9.

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