CAPTURING SETS OF ORDINALS BY NORMAL ULTRAPOWERS

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Abstract. We investigate the extent to which ultrapowers by normal measures on κ can be correct about powersets $\mathcal{P}(\lambda)$ for $\lambda > \kappa$. We consider two versions of this question, the *capturing property* $\mathrm{CP}(\kappa, \lambda)$ and the *local capturing property* $\mathrm{LCP}(\kappa, \lambda)$. $\mathrm{CP}(\kappa, \lambda)$ holds if there is an ultrapower by a normal measure on κ which correctly computes $\mathcal{P}(\lambda)$. $\mathrm{LCP}(\kappa, \lambda)$ is a weakening of $\mathrm{CP}(\kappa, \lambda)$ which holds if every subset of λ is contained in some ultrapower by a normal measure on κ. After examining the basic properties of these two notions, we identify the exact consistency strength of $\mathrm{LCP}(\kappa, \kappa^+)$. Building on results of Cummings, who determined the exact consistency strength of $\mathrm{CP}(\kappa, \kappa^+)$, and using a forcing due to Apter and Shelah, we show that $\mathrm{CP}(\kappa, \lambda)$ can hold at the least measurable cardinal.

1. Introduction

It is well known that the ultrapower of the universe by a normal measure on some cardinal κ cannot be very close to V; for example, the measure itself never appears in the ultrapower. It follows that these ultrapowers cannot compute $V_{\kappa+2}$ correctly. In the presence of GCH, this is equivalent to saying that the ultrapower is incorrect about $\mathcal{P}(\kappa^+)$. But if GCH fails, it becomes conceivable that a normal ultrapower could compute additional powersets correctly. This conjecture turns out to be correct: Cummings [4], answering a question of Steel, showed that it is relatively consistent that there is a measurable cardinal κ with a normal measure whose ultrapower computes $\mathcal{P}(\kappa^+)$ correctly; in fact he showed that this situation is equiconsistent with a $(\kappa+2)$-strong cardinal κ. In this paper we will study this *capturing property* and its local variant further.

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Definition 1. Let κ and λ be cardinals. We say that the *local capturing property* $\mathrm{LCP}(\kappa,\lambda)$ holds if, for any $x \subseteq \lambda$, there is a normal measure U_x on κ such that $x \in \mathrm{Ult}(V, U_x)$. We shall say that U_x (or $\mathrm{Ult}(V, U_x)$) *captures* x.

The full capturing property will amount to having a uniform witness for the local version.

Definition 2. Let κ and λ be cardinals. We say that the *capturing property* $\mathrm{CP}(\kappa,\lambda)$ holds if there is a normal measure on κ that captures all subsets of λ; in other words, a normal measure U such that $\mathcal{P}(\lambda) \in \mathrm{Ult}(V, U)$.

Some quick and easy observations: increasing λ clearly gives us stronger properties, $\mathrm{CP}(\kappa,\lambda)$ implies $\mathrm{LCP}(\kappa,\lambda)$, and $\mathrm{CP}(\kappa,\kappa)$ holds for any measurable cardinal κ.

Using this language, we can summarize Cummings' result as follows:

Theorem 3 (Cummings)**.** *If κ is $(\kappa+2)$-strong, then there is a forcing extension in which $\mathrm{CP}(\kappa,\kappa^+)$ holds. Conversely, if $\mathrm{CP}(\kappa,\kappa^+)$ holds, then κ is $(\kappa+2)$-strong in an inner model.*

We should mention that $\mathrm{CP}(\kappa, 2^\kappa)$ is provably false: if it held, then some normal ultrapower would contain all families of subsets of κ, in particular the measure from which it arose, which is impossible. Therefore a failure of GCH is necessary for $\mathrm{CP}(\kappa,\kappa^+)$ to hold. By work of Gitik [7], this means that $\mathrm{CP}(\kappa,\kappa^+)$ has consistency strength at least that of a measurable cardinal κ with Mitchell rank $o(\kappa) = \kappa^{++}$, and the actual consistency strength of a $(\kappa+2)$-strong cardinal κ is only slightly beyond that.

The following are the main results of this paper. In section 2 we analyse the consistency strength of $\mathrm{LCP}(\kappa,\kappa^+)$ and show that it is only a small step below the strength of the full capturing property.

Main theorem 1. *Assuming GCH, if $\mathrm{LCP}(\kappa,\kappa^+)$ holds, then $o(\kappa) = \kappa^{++}$. Conversely, if $o(\kappa) \geq \kappa^{++}$, then $\mathrm{LCP}(\kappa,\kappa^+)$ holds in an inner model.*

In section 3 we continue the analysis in the case that GCH fails at κ and show that the first part of the previous theorem, namely that κ has high Mitchell rank, fails dramatically if $2^\kappa > \kappa^+$.

Main theorem 2. *If κ is $(\kappa+2)$-strong, then there is a forcing extension in which $\mathrm{CP}(\kappa,\kappa^+)$ holds and κ is the least measurable cardinal.*

This last theorem is a nontrivial improvement of Cummings' result. Since the forcing he used to achieve $\mathrm{CP}(\kappa,\kappa^+)$ was relatively mild, κ remained quite large in the resulting model; for example, it was still a measurable limit of measurable cardinals. Our theorem shows that, while $\mathrm{CP}(\kappa,\kappa^+)$ has nontrivial consistency strength, it does not directly imply anything about the size of κ in V (beyond κ being measurable).

We will list questions that we have left open wherever appropriate throughout the paper.

2. The local capturing property

Let us begin our analysis of the local capturing property with some simple observations.

Lemma 4. *If* $\mathrm{LCP}(\kappa, \lambda)$ *holds, then it can be witnessed by measures* U *for which* $\mathrm{Ult}(V, U)$ *and* V *agree on cardinals up to and including* λ.

Proof. Using a pairing function we can code a family of bijections $f_\alpha \colon \alpha \to |\alpha|$ for $\alpha \leq \lambda$ as a single subset $y \subseteq \lambda$. If we want to capture $x \subseteq \lambda$ in an ultrapower as in the lemma, we simply capture (a disjoint union of) x and y using $\mathrm{LCP}(\kappa, \lambda)$. $\square$

Proposition 5. $\mathrm{LCP}(\kappa, (2^\kappa)^+)$ *fails for any measurable* κ.

Proof. If $\mathrm{LCP}(\kappa, (2^\kappa)^+)$ held, there would have to be a normal measure ultrapower $j \colon V \to M$ with critical point κ such that M was correct about cardinals up to and including $(2^\kappa)^+$, by lemma 4. But no such ultrapower can exist, since the ordinals $j(\kappa)$ and $j(\kappa^+)$ are cardinals in M and both have size 2^κ in V. $\square$

The following lemma is quite well known, but it will be key in many of our observations.

Lemma 6. *Suppose that* $j \colon V \to M$ *is an elementary embedding with critical point* κ *and consider the diagram*

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "i"'] & M \\ & N \arrow[u, "k"'] \end{tikzcd}$$

where i *is the ultrapower by the normal measure on* κ *derived from* j *and* k *is the factor map. Then the critical point of* k *is strictly above* $(2^\kappa)^N$.

Proof. It is clear that the critical point of k is above κ. Consider some ordinal $\alpha \leq (2^\kappa)^N$. Fix a surjective map $f \colon \mathcal{P}(\kappa) \to \alpha$ in N (and note that both N and M compute $\mathcal{P}(\kappa)$ correctly). Since every ordinal up to and including κ is fixed by k, it follows that $k(f) = k \circ f$ is a surjection from $\mathcal{P}(\kappa)$ to $k(\alpha)$ and so $k \restriction \alpha$ is a surjection onto $k(\alpha)$. It follows that we must have $k(\alpha) = \alpha$. $\square$

Using an old argument of Solovay, we can see that the optimal local capturing property automatically holds at sufficiently large cardinals.

Proposition 7. *If a cardinal* κ *is* 2^κ*-supercompact, witnessed by an embedding* $j \colon V \to M$*, then* $\mathrm{LCP}(\kappa, 2^\kappa)$ *holds in both* V *and* M.

Proof. We first show that $\mathrm{LCP}(\kappa, 2^\kappa)$ holds in V. Suppose it fails. Then there is some $x \subseteq 2^\kappa$ which is not captured by any normal measure on κ. The model M agrees that this is the case, since it has all the normal measures on κ and all the functions $f \colon \kappa \to \mathcal{P}(\kappa)$ that could represent x. Let i and k be as in lemma 6. By that same lemma, the model N computes 2^κ correctly and it also believes that there is some $y \subseteq 2^\kappa$ which is not captured by any normal measure on κ. This y is fixed by k, so M also believes that y is not captured by any normal measure on κ, and V agrees. But this is a contradiction, since y is captured by the ultrapower N. Therefore $\mathrm{LCP}(\kappa, 2^\kappa)$ holds in V.

Observe that $\mathrm{LCP}(\kappa, 2^\kappa)$ only depends on $\mathcal{P}(2^\kappa)$, the normal measures on κ, and the representing functions $\kappa \to \mathcal{P}(\kappa)$. The ultrapower M has all of these objects, therefore M must agree that $\mathrm{LCP}(\kappa, 2^\kappa)$ holds. $\square$

In particular, if κ is 2^κ-supercompact, then there are many $\lambda < \kappa$ for which $\mathrm{LCP}(\lambda, 2^\lambda)$ holds.

The above argument seems to break down if κ is only θ-supercompact for some $\theta < 2^\kappa$, even if we are only aiming to capture subsets of θ; one simply cannot conclude that M has all the necessary measures to correctly judge whether a set is a counterexample to $\mathrm{LCP}(\kappa, \lambda)$ or not. Thus, the following question remains open.

Question 8. *Suppose that κ is θ-supercompact for some $\kappa < \theta < 2^\kappa$. Does it follow that* $\mathrm{LCP}(\kappa, \theta)$ *holds?*

The same conclusion as in proposition 7 follows even if κ is merely $(\kappa+2)$-strong.

Proposition 9. *If a cardinal κ is $(\kappa+2)$-strong, witnessed by an embedding $j\colon V \to M$, then* $\mathrm{LCP}(\kappa, 2^\kappa)$ *holds in both V and M.*

Proof. The argument works just like in proposition 7. Note that M has all the functions $\kappa \to \mathcal{P}(\kappa)$ and all the normal measures on κ. Furthermore, M has all the subsets of 2^κ (use a wellorder of $V_{\kappa+1}$ in $V_{\kappa+2}$ of ordertype 2^κ). It follows that V and M have all the same counterexamples to $\mathrm{LCP}(\kappa, 2^\kappa)$. $\square$

This last observation already implies that the consistency strength of $\mathrm{LCP}(\kappa, \kappa^+)$ is strictly lower than that of $\mathrm{CP}(\kappa, \kappa^+)$. Let us determine this consistency strength exactly.

Recall that the *Mitchell order* $\lhd$ on a measurable cardinal κ is a relation on the normal measures on κ, where $U \lhd U'$ if U appears in the ultrapower by U'. It is a standard fact that $\lhd$ is wellfounded, and the *Mitchell rank* of κ is the height $o(\kappa)$ of this order.

Proposition 10. *If* $\mathrm{LCP}(\kappa, 2^\kappa)$ *holds, then $o(\kappa) = (2^\kappa)^+$.*

Proof. This is essentially the proof that the large cardinals mentioned in the previous two propositions have maximal Mitchell rank. We shall recursively build a Mitchell-increasing sequence $\langle U_\alpha\,;\, \alpha < (2^\kappa)^+\rangle$ of normal measures on κ. So suppose that $\langle U_\alpha\,;\, \alpha < \delta\rangle$ has been constructed for some $\delta < (2^\kappa)^+$. Using a pairing function we can code each measure U_α as a subset of 2^κ, and then code the entire sequence $\langle U_\alpha\,;\, \alpha < \delta\rangle$ as a subset of 2^κ as well. By $\mathrm{LCP}(\kappa, 2^\kappa)$ there is a normal measure U on κ which captures this subset, and thus the whole sequence of measures. We can then simply let $U_\delta = U$. $\square$

To show that the lower bound from this proposition is sharp we will pass to a suitable inner model. Recall that a *coherent sequence of normal measures* $\mathcal{U}$ of length ℓ is given by a function $o^{\mathcal{U}}\colon \ell \to \mathrm{Ord}$ and a sequence

$$\mathcal{U} = \langle U_\alpha^\beta\,;\, \alpha < \ell, \beta < o^{\mathcal{U}}(\alpha)\rangle\,,$$

where each U_α^β is a normal measure on α and for each α, β, if j_α^β is the corresponding ultrapower map, we have

$$j_\alpha^\beta(\mathcal{U}) \restriction \alpha + 1 = \mathcal{U} \restriction (\alpha, \beta)\,.$$

Here $\mathcal{U} \restriction (\alpha, \beta) = \langle U_\gamma^\delta\,;\, (\gamma, \delta) <_{\mathrm{lex}} (\alpha, \beta)\rangle$ and $\mathcal{U} \restriction \alpha = \mathcal{U} \restriction (\alpha, 0)$.

Theorem 11. *Suppose that $V = L[\mathcal{U}]$ where $\mathcal{U}$ is a coherent sequence of normal measures with $o^{\mathcal{U}}(\kappa) = \kappa^{++}$. Then* $\mathrm{LCP}(\kappa, \kappa^+)$ *holds.*

Proof. We shall show that, given any $x \subseteq \kappa^+$, there is some $\beta < \kappa^{++}$ such that $x \in L[\mathcal{U} \restriction (\kappa, \beta)]$. The theorem then immediately follows since, given x, we can find a β as described, and the ultrapower by U^β_κ of $L[\mathcal{U}]$ contains $L[\mathcal{U} \restriction (\kappa, \beta)]$, and therefore x.

So fix some $x \subseteq \kappa^+$ and let ρ be a large regular cardinal so that $x \in L_\rho[\mathcal{U}]$. Since GCH holds, we can find an elementary submodel $M \prec L_\rho[\mathcal{U}]$ of size κ^+ such that $x, \mathcal{U} \in M$ and $\kappa^+, \mathcal{P}(\kappa) \subseteq M$. Let $\pi\colon M \to \overline{M}$ be the collapse map.

Note that $\delta = M \cap \kappa^{++} = \pi(\kappa^{++})$ is an ordinal below κ^{++}. It follows that $\pi(\mathcal{U})$ is (in $\overline{M}$) a coherent sequence of normal measures with $o^{\pi(\mathcal{U})}(\kappa) = \delta$, and moreover, that $\pi(\mathcal{U}) = \mathcal{U} \restriction (\kappa, \delta)$, since none of the measures U^β_α for $(\alpha, \beta) <_{\text{lex}} (\kappa, \delta)$ are moved by π. Therefore $\overline{M} = L_{\bar{\rho}}[\mathcal{U} \restriction (\kappa, \delta)]$ for some $\bar{\rho} < \rho$. Since $x \subseteq \kappa^+$ was fixed by π as well, we get $x \in \overline{M} \subseteq L[\mathcal{U} \restriction (\kappa, \delta)]$. □

Even if, starting from a measurable cardinal κ of Mitchell order κ^{++}, one could construct a coherent sequence $\mathcal{U}$ of normal measures with $o^{\mathcal{U}}(\kappa) = \kappa^{++}$, it seems to be an open question (according to [13]) whether it is necessarily the case that $\mathcal{U}$ remains coherent in $L[\mathcal{U}]$. We avoid this issue by using a result of Mitchell [12], who showed in ZFC that there is a sequence of filters $\mathcal{F}$ (possibly empty, possibly of length Ord, or anything in between) such that $\mathcal{F}$ is a coherent sequence of normal measures in $L[\mathcal{F}]$ and $o^{\mathcal{F}}(\alpha) = \min(o(\alpha)^V, (\alpha^{++})^{L[\mathcal{F}]})$. The inner model we need will be exactly this $L[\mathcal{F}]$.

Corollary 12. *Assume that $o(\kappa) \geq \kappa^{++}$. Then* $\mathrm{LCP}(\kappa, 2^\kappa)$ *holds in an inner model.*

Proof. Let $\mathcal{F}$ be the sequence of filters described above and work in $L[\mathcal{F}]$. By Mitchell's results we know that the sequence $\mathcal{F}$ is a coherent sequence of normal measures and $o^{\mathcal{F}} = \kappa^{++}$. It then follows from theorem 11 that $\mathrm{LCP}(\kappa, \kappa^+)$ holds. □

In fact, these canonical inner models satisfy a strong form of $\mathrm{LCP}(\kappa, \kappa^+)$, where there is a single function which represents any desired subset of κ^+ in an appropriate normal ultrapower.

Definition 13. Let κ be a measurable cardinal. An *$H_{\kappa^{++}}$-guessing Laver function* for κ is a function $\ell\colon \kappa \to V_\kappa$ with the property that for any $x \in H_{\kappa^{++}}$ there is an ultrapower embedding $j\colon V \to M$ by a normal measure on κ such that $j(\ell)(\kappa) = x$.

It is obvious that the existence of an $H_{\kappa^{++}}$-guessing Laver function for κ implies $\mathrm{LCP}(\kappa, \kappa^+)$. The first author [9, Theorem 28] showed that this stronger property holds in appropriate extender models, in particular the one from corollary 12.

Starting with a cardinal κ of high Mitchell rank, we obtained a model of the local capturing property by passing to an inner model. We are unsure whether one can obtain the local capturing property from the optimal hypothesis via forcing.

Question 14. *Suppose that GCH holds and $o(\kappa) = \kappa^{++}$. Is there a forcing extension in which* $\mathrm{LCP}(\kappa, \kappa^+)$ *holds?*

It is important to note that the hypothesis in proposition 10 is quite strong: we need to be able to capture all subsets of 2^κ in order to be able to conclude that the Mitchell rank of κ is large. One might wonder whether some strength can be derived even from weaker local capturing properties, for example $\mathrm{LCP}(\kappa, \kappa^+)$ assuming $\kappa^+ < 2^\kappa$. As we shall see in the following section, the answer is an emphatic no.

3. The capturing property at the least measurable cardinal

In this section we will give a proof of our second main theorem. Our argument owes a lot to Cummings' original proof of theorem 3 and to the forcing machinery introduced by Apter and Shelah. Nevertheless, we shall strive to give a mostly self-contained account, especially with regard to the forcing notions used.

Let us first explain why we cannot simply use the proof from theorem 3 and afterwards make κ into the least measurable cardinal just by applying the standard methods of destroying measurable cardinals, such as iterated Prikry forcing or adding nonreflecting stationary sets. In his argument, Cummings starts with a (κ, κ^{++})-extender embedding, lifts it through a certain iteration of Cohen forcings (which will, among other things, ensure that $2^\kappa > \kappa^+$, a necessary condition as we explained), and concludes that the lifted embedding $j\colon V[G] \to M[j(G)]$ is in fact equal to the ultrapower by some normal measure on κ and $M[j(G)]$ captures all the subsets of κ^+ in the extension. One would now hope to be able to lift this new embedding further, through any of the usual forcings which would make κ into the least measurable cardinal. However, this strategy can only hope to work if κ is not measurable in $M[j(G)]$; otherwise the measurability of κ would have to be destroyed over $M[j(G)]$, and there is enough agreement between $V[G]$ and $M[j(G)]$ that κ would necessarily be nonmeasurable in the extension of $V[G]$ as well. Since κ is very much measurable in $M[j(G)]$ after the forcing done by Cummings, a different approach is necessary.

Instead of first forcing the capturing property and then making κ into the least measurable, the solution is to destroy all the measurable cardinals below κ and blow up 2^κ at the same time. The tools to make this approach work are due to Apter and Shelah [1, 2].

3.1. The forcing notions.

Let us review the particular forcing notions that will go into building our final forcing iteration. The material in this subsection is contained, in some form or another, in sections 1 of [1, 2].

Since we will be discussing the strategic closure of some of these posets, let us fix some terminology. If $\mathbb{P}$ is a poset and α is an ordinal, the *closure game* for $\mathbb{P}$ of length α consists of two players alternately playing conditions $p \in \mathbb{P}$ in a descending sequence of length α, with player II playing at limit steps. Player II loses the game if at any stage she is unable to make a move; otherwise she wins. If $\mathbb{P}$ is a poset and κ is a cardinal, we shall say that:

- $\mathbb{P}$ is $\leq \kappa$-strategically closed if player II has a winning strategy in the closure game for $\mathbb{P}$ of length $\kappa + 1$.
- $\mathbb{P}$ is $\prec \kappa$-strategically closed if player II has a winning strategy in the closure game for $\mathbb{P}$ of length κ.

- $\mathbb{P}$ is $< \kappa$-strategically closed if it is $\leq \lambda$-strategically closed for all $\lambda < \kappa$.

If $\delta \geq \omega_2$ is a regular cardinal, we let $\mathbb{S}_\delta$ be the forcing to add a nonreflecting stationary subset of δ, consisting of points of countable cofinality.[1] A condition in $\mathbb{S}_\delta$ is simply a bounded subset of $x \subseteq \delta$, consisting of points of countable cofinality and satisfying the property that $x \cap \alpha$ is nonstationary in α for every limit $\alpha < \delta$ of uncountable cofinality. The conditions in $\mathbb{S}_\delta$ are ordered by end-extension. It is a standard fact that $\mathbb{S}_\delta$ is $\prec \delta$-strategically closed and, if $2^{<\delta} = \delta$, is δ^+-cc (see [5, Section 6] for more details). Note that the generic stationary set added will also be costationary, since it avoids all ordinals of uncountable cofinality.

If $S \subset \delta$ is a costationary set, let $\mathbb{C}(S)$ be the forcing to shoot a club through $\delta \setminus S$; conditions are closed bounded subsets of $\delta \setminus S$. Again, if $2^{<\delta} = \delta$, then $\mathbb{C}(S)$ will be δ^+-cc ([5, Section 6] has more details).

Before we continue with the exposition, let us fix some terminology.

Definition 15. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets. We say that $\mathbb{P}$ and $\mathbb{Q}$ are *forcing equivalent* if they have isomorphic dense subsets.

This is not the most general definition of forcing equivalence that has appeared in the literature, but it has the advantage of being obviously upward absolute between transitive models of set theory.

Lemma 16. *If δ is a cardinal satisfying $\delta^{<\delta} = \delta$ then $\mathbb{S}_\delta * \mathbb{C}(\dot{S})$, where $\dot{S}$ is the name for the generic nonreflecting stationary set added by $\mathbb{S}_\delta$, is forcing equivalent to* $\mathrm{Add}(\delta, 1)$.

Proof. This is standard; the iteration has a dense $< \delta$-closed subset of size δ, which is equivalent to $\mathrm{Add}(\delta, 1)$ by [5, Theorem 14.1]. ☐

Suppose that γ and δ are cardinals, $I \subseteq \delta$, and $\vec{X} = \langle x_\alpha \,;\, \alpha \in I\rangle$ is a ladder system (meaning that each $x_\alpha \subseteq \alpha$ is a $\mathrm{cf}(\alpha)$-sequence cofinal in α). The *Apter–Shelah forcing*[2] $\mathbb{A}(\gamma, \delta, \vec{X})$ consists of conditions (p, Z) where

1. p is a condition in the Cohen forcing $\mathrm{Add}(\gamma, \delta)$, seen as filling in δ many columns of height γ with 0s and 1s. We will denote by $\mathrm{supp}(p) \subseteq \delta$ the set of indices of the nonempty columns of p.
2. p is a uniform condition, meaning that all of its nonempty columns have the same height.[3]
3. Z is a subset of the ladder system $\vec{X}$ and each ladder $z \in Z$ is a subset of $\mathrm{supp}(p)$.

The conditions in $\mathbb{A}(\gamma, \delta, \vec{X})$ are ordered by letting $(p', Z') \leq (p, Z)$ if $p' \leq p$ and $Z' \supseteq Z$, and for any $z \in Z$ the extended part $(p' \setminus p) \restriction z$ above z has unboundedly many 0s and 1s in each row.

[1] In our argument we could use any other fixed cofinality below the large cardinal in question. We sacrifice a bit of generality in order to avoid carrying an extra parameter with us throughout the proof. The specific choice of countable cofinality also simplifies some arguments.

[2] We chose the letter $\mathbb{A}$ without prejudice against Shelah, but rather to emphasize that the forcing is derived from the Cohen forcing $\mathrm{Add}(\gamma, \delta)$ by adding some side conditions.

[3] This requirement is not crucial for the argument, but it does make the poset slightly nicer than otherwise, for example $< \kappa$-closed.

Some comments are in order regarding the forcing $\mathbb{A}(\gamma,\delta,\vec{X})$. It is similar enough to the Cohen poset $\mathrm{Add}(\gamma,\delta)$ that one would hope that is is just as simple to show that this forcing also adds δ new subsets of γ and so on. But with the addition of the side conditions this is no longer clear. It is not even immediate that generically we will fill out the entire γ-by-δ matrix. On the other hand, if we want to use this forcing as the main part of our construction to destroy many measurable cardinals, then it cannot be too close to plain Cohen forcing after all. This tension between the Apter–Shelah poset and the Cohen poset is controlled by the ladder system $\vec{X}$, so we will have to choose these ladder systems carefully in our proof.

The following facts are due to Apter and Shelah, and we omit most of their proofs.

Lemma 17. *Suppose $\gamma<\delta$ are regular cardinals, with γ inaccessible. Fix a set $I\subseteq\delta$ and let $\vec{X}$ be a ladder system on I. Then the forcing $\mathbb{A}(\gamma,\delta,\vec{X})$ is γ^+-Knaster (meaning that any set of γ^+ many conditions has a subset of γ^+ many pairwise compatible conditions).*

Proof. This fact is implicit in [1], but is never spelled out, so we give the straightforward proof. Suppose that (p_α,Z_α) for $\alpha<\gamma^+$ are conditions in $\mathbb{A}(\gamma,\delta,\vec{X})$. We may assume that all of the working parts p_α have the same height. Since the poset $\mathrm{Add}(\gamma,\delta)$ is γ^+-Knaster, we can find a subset $J\subseteq\gamma^+$ so that the conditions p_α for $\alpha\in J$ are pairwise compatible. But it then easily follows that the full conditions (p_α,Z_α) for $\alpha\in J$ are also pairwise compatible. $\square$

Lemma 18. *Suppose $\gamma<\delta$ are regular cardinals, with γ inaccessible. Suppose that $I\subseteq\delta$ consists of points of countable cofinality and that all of its initial segments are nonstationary. Let $\vec{X}$ be a ladder system on I. Then a generic for $\mathbb{A}(\gamma,\delta,\vec{X})$ is a total function on $\delta\times\gamma$ and each of its columns is a new subset of γ.*

If δ is a regular cardinal and $S\subseteq\delta$ is stationary, recall that a $\clubsuit_\delta(S)$-sequence is a ladder system $\langle x_\alpha\,;\,\alpha\in S\rangle$ such that for any unbounded $A\subseteq\delta$ there is some $\alpha\in S$ such that $x_\alpha\subseteq A$.

Lemma 19. *Suppose that $\gamma<\delta$ are regular cardinals, with γ inaccessible. Let $S\subseteq\delta$ be a nonreflecting stationary set consisting of points of countable cofinality, and let $\vec{X}$ be a $\clubsuit_\delta(S)$-sequence. Then $\mathbb{A}(\gamma,\delta,\vec{X})$ forces that γ is not measurable.*

Just to give the briefest of sketches of the proof of this lemma, starting from a condition and a name for an ultrafilter on γ, we use a Δ-system argument to find an unbounded subset I of δ and compatible stronger conditions forcing that the sets added by the Ith slices of the generic (or their complements) are in the ultrafilter. We then use $\clubsuit_\delta(S)$ to find a single stronger condition that forces that the intersection of the Ith slices of the generic (or even just countably many of them) is bounded in γ, and therefore the ultrafilter could not have been complete.

The proof of the following lemma is much like the proof that $\mathrm{Add}(\omega_1,1)$ forces $\diamondsuit$.

Lemma 20. *Let δ be a regular cardinal satisfying $\delta^\omega = \delta$. Then $\mathbb{S}_\delta$ forces that $\clubsuit_\delta(S)$ holds, where S is the generic stationary set added.*

Since we now know that $\mathbb{S}_\delta$ adds a $\clubsuit_\delta(S)$-sequence, it makes sense to consider the iteration $\mathbb{S}_\delta * \mathbb{A}(\gamma,\delta,\vec{X})$, where $\vec{X}$ is a $\clubsuit_\delta(S)$-sequence added by the first stage of forcing. Lemma 19 implies that this iteration will definitely make γ nonmeasurable. The following lemma is a complement to that result and shows that the measurability of γ may be resurrected.

Lemma 21. *Let $\gamma < \delta$ be regular cardinals with γ inaccessible and δ satisfying $\delta^{<\delta} = \delta$. Then the iteration $\mathbb{S}_\delta * (\mathbb{A}(\gamma,\delta,\vec{X}) \times \mathbb{C}(\dot{S}))$, where $\vec{X}$ is an arbitrary ladder system on S, is equivalent to* $\mathrm{Add}(\delta,1) * \mathrm{Add}(\gamma,\delta)$.

It follows from this lemma (or just by manual calculation) that the iteration $\mathbb{S}_\delta * \mathbb{A}(\gamma,\delta,\vec{X})$ is a reasonable forcing: it is δ^+-cc and $<\gamma$-strategically closed, and it forces $2^\gamma = \delta$.

3.2. Some additional facts about forcing and elementary embeddings.

In this subsection we collect some facts about forcing and ultrapowers that we will need in our proof of the main theorem.

Recall that if $\mathbb{P}$ is a poset and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a poset, the *term forcing* $\mathrm{Term}(\mathbb{P},\dot{\mathbb{Q}})$ consists of $\mathbb{P}$-names for elements of $\dot{\mathbb{Q}}$, ordered by letting $\sigma \leq \tau$ if $\mathbb{P} \Vdash \sigma \leq \tau$. It is easy to see that if $G \subseteq \mathbb{P}$ and $H \subseteq \mathrm{Term}(\mathbb{P},\dot{\mathbb{Q}})$ are generic over V, then $\{\sigma^G\,;\,\sigma \in H\} \subseteq \dot{\mathbb{Q}}^G$ is generic over $V[G]$.

Lemma 22. *Suppose that κ is a cardinal satisfying $\kappa^{<\kappa} = \kappa$ and let $\mathbb{P}$ be a κ-cc forcing of size κ. Let $\dot{\mathbb{Q}}_\lambda$ be the $\mathbb{P}$-name for* $\mathrm{Add}(\kappa,\lambda)$ *in the extension. Then* $\mathrm{Term}(\mathbb{P},\dot{\mathbb{Q}}_\lambda)$ *is forcing equivalent, in V, to* $\mathrm{Add}(\kappa,\lambda)$.

For a proof, see [3, Section 1.2.5].

Lemma 23. *Let κ be a measurable cardinal satisfying $2^\kappa = \kappa^+$ and let $j\colon V \to M$ be the ultrapower by a normal measure on κ. Given any finite $n \geq 1$, the forcings $j(\mathrm{Add}(\kappa,\kappa^{+n}))$ and $\mathrm{Add}(\kappa^+,\kappa^{+n})$ are equivalent in V.*

Cummings gave a proof of this lemma for $n = 2$ in [3] (attributing the proof to Woodin), and Gitik and Merimovich proved the generalization to all n in [8, Lemma 3.2].

Lemma 24. *Let κ be a regular cardinal, let $\mathbb{P}$ be a $<\kappa$-distributive forcing notion, and suppose that $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is a κ-cc forcing which is a subset of H_κ. Let $G * H$ be generic for $\mathbb{P} * \dot{\mathbb{Q}}$. Then any bounded subset of κ in $V[G][H]$ has a $\dot{\mathbb{Q}}^G$-name in H_κ^V.*

We should point out that V will not, in general, be aware that the object it has is a $\dot{\mathbb{Q}}^G$-name, since the poset $\dot{\mathbb{Q}}^G$ does not exist yet in V. But the point is that all the conditions of $\dot{\mathbb{Q}}^G$ are already in V, and the name as a set exists already in V.

Proof. Since, in $V[G]$, the poset $\mathbb{Q} \subseteq H_\kappa$ is κ-cc, any bounded subset of κ in $V[G][H]$ has a nice name which is also an element of H_κ. But note that $H_\kappa^{V[G]} = H_\kappa^V$ because of the distributivity of $\mathbb{P}$. $\square$

The following key observation was already implicit in Cummings' proof of theorem 3. It shows that, as long as one can arrange the value of 2^κ appropriately, the apparently difficult part of the capturing property tends to follow for free from the construction.

Lemma 25. *Suppose that $j\colon V \to M$ is a (κ,λ)-extender embedding and $2^\kappa \geq \lambda$. Then j is the ultrapower by a normal measure on κ.*

Proof. Let $i\colon V \to N$ be the ultrapower by the normal measure derived from j and let $k\colon N \to M$ be the factor embedding. Consider some $x \in M$. Since j is a (κ,λ)-extender embedding, we can write $x = j(f)(\alpha)$ for some $\alpha < \lambda$ and some function with domain κ. By lemma 6 the critical point of k is above λ and therefore

$$x = j(f)(\alpha) = k(i(f))(\alpha) = k(i(f)(\alpha))\,,$$

which shows that k is surjective. It follows that k is an isomorphism of transitive structures and thus trivial, so we can conclude that $j = i$. □

3.3. **The proof.** We are now ready to prove the second main theorem. We restate it here for convenience.

Theorem 26. *If κ is $(\kappa+2)$-strong, then there is a forcing extension in which $\mathrm{CP}(\kappa,\kappa^+)$ holds and κ is the least measurable.*

This theorem shows that the hypothesis in proposition 10 is in some sense optimal: if $2^\kappa > \kappa^+$ then $\mathrm{LCP}(\kappa,\kappa^+)$ is not enough to conclude that the Mitchell rank of κ is large. In fact, even $\mathrm{CP}(\kappa,\kappa^+)$ can hold at the least measurable cardinal.

Proof. We make some simplifying assumptions to start with. We may assume that GCH holds and that the $(\kappa+2)$-strongness of κ is witnessed by a (κ,κ^{++})-extender embedding $j\colon V \to M$. We have the usual diagram

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "i"'] & M \\ & N \arrow[u, "k"'] \end{tikzcd}$$

where i is the induced normal ultrapower map. Using the GCH and lemma 6, we can see that the critical point of k is $(\kappa^{++})^N$. Using the argument from [4], we may also assume that, in V, there is an $i(\mathrm{Add}(\kappa,\kappa^{++}))$-generic filter over N.

We now specify the forcing we will use. Let $\mathbb{P}_\kappa$ be the Easton support iteration of length κ which forces at inaccessible $\gamma < \kappa$ with $\mathbb{S}_{\gamma^{++}} * \mathbb{A}(\gamma,\gamma^{++},\vec{X})$, where $\vec{X}$ is some $\clubsuit_{\gamma^{++}}(S)$-sequence added by $\mathbb{S}_{\gamma^{++}}$.[4] Let G_κ be $\mathbb{P}_\kappa$-generic over V. We shall try to lift the embeddings i and j through this forcing.

We can factor $j(\mathbb{P}_\kappa)$ as

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{S}_{\kappa^{++}} * \mathbb{A}(\kappa,\kappa^{++},\vec{Y}) * \mathbb{P}_{\mathrm{tail}}\,,$$

[4] It does not matter much how we pick these $\clubsuit$-sequences. One possible way is to fix in advance a wellordering of some large H_θ and always pick the least appropriate name.

where $\vec{Y}$ is the $\clubsuit_{\kappa^{++}}$-sequence used by the forcing at stage κ and $\mathbb{P}_{\text{tail}}$ is the remainder of the forcing between κ and $j(\kappa)$. Similarly, we can rewrite $i(\mathbb{P}_\kappa)$ as

$$i(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * (\mathbb{S}_{\kappa^{++}} * \mathbb{A}(\kappa, \kappa^{++}, \vec{Y}'))^{N^{\mathbb{P}_\kappa}} * \mathbb{P}'_{\text{tail}}\,,$$

where $\vec{Y}'$ and $\mathbb{P}'_{\text{tail}}$ are defined analogously. Since G_κ is generic over all of V, it is definitely generic over N and M. The forcing $\mathbb{P}_\kappa$ is below the critical point of the embedding k, so we can easily lift it to $k\colon N[G_\kappa] \to M[G_\kappa]$. Moreover, since $\mathbb{P}_\kappa$ is κ-cc, $N[G_\kappa]$ will be closed under κ-sequences in $V[G_\kappa]$.

We now claim that, in $V[G_\kappa]$, there is an $\mathbb{S}^{N[G_\kappa]}_{\kappa^{++}}$-generic over $N[G_\kappa]$, and moreover that this generic amounts to a nonstationary set in $V[G_\kappa]$. This follows from lemma 16, which tells us that the iteration $\mathbb{S}_{\kappa^{++}} * \mathbb{C}(\dot{S})$ is equivalent to $\mathrm{Add}(\kappa^{++}, 1)$. Since $V[G_\kappa]$ has an $\mathrm{Add}(\kappa^{++}, 1)^{N[G_\kappa]}$-generic over $N[G_\kappa]$ (as this forcing is $\leq\kappa$-closed in $V[G_\kappa]$ and only has κ^+ many dense subsets from $N[G_\kappa]$), we can also extract the generic for $\mathbb{S}^{N[G_\kappa]}_{\kappa^{++}}$. Furthermore, this generic stationary set will be nonstationary in $V[G_\kappa]$, as witnessed by the generic club added by $\mathbb{C}(S)$.

So let $S' \in V[G_\kappa]$ be $\mathbb{S}^{N[G_\kappa]}_{\kappa^{++}}$-generic over $N[G_\kappa]$. Note that S' is a condition in the real $\mathbb{S}_{\kappa^{++}}$, since it is not actually stationary in $(\kappa^{++})^N$. Let S be some $\mathbb{S}_{\kappa^{++}}$-generic over $V[G_\kappa]$ extending S'. The embedding k lifts easily again to $k\colon N[G_\kappa][S'] \to M[G_\kappa][S]$.

Now consider the $\clubsuit_{(\kappa^{++})^N}$-sequence $\vec{Y}'$ used by $i(\mathbb{P}_\kappa)$ at stage κ. Since the critical point of k is $(\kappa^{++})^N$, the sequence $\vec{Y}'$ is simply an initial segment of the sequence $\vec{Y} = k(\vec{Y}')$ used by $j(\mathbb{P}_\kappa)$ at stage κ.[5] It follows that, if we look at the forcing $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})$ in $V[G_\kappa][S]$, we can write it as a product

$$\mathbb{A}(\kappa, \kappa^{++}, \vec{Y}) \cong \mathbb{A}(\kappa, (\kappa^{++})^N, \vec{Y}') \times \mathbb{A}(\kappa, \kappa^{++} \setminus (\kappa^{++})^N, \vec{Y})\,, \tag{1}$$

where we allow ourselves some abuse of notation in the second factor by not modifying the ladder system $\vec{Y}$. Observe also that, since $\mathbb{S}_{\kappa^{++}}$ does not add bounded subsets to κ^{++}, we know

$$\mathbb{A}(\kappa, (\kappa^{++})^N, \vec{Y}')^{V[G_\kappa][S]} = \mathbb{A}(\kappa, (\kappa^{++})^N, \vec{Y}')^{V[G_\kappa]} = \mathbb{A}(\kappa, \kappa^{++}, \vec{Y}')^{N[G_\kappa][S']}\,.$$

Let g' be $\mathbb{A}(\kappa, (\kappa^{++})^N, \vec{Y}')$-generic over $V[G_\kappa][S]$; in particular, it is also generic over $N[G_\kappa][S']$. Since g' is generic for a forcing that is κ^+-cc in $V[G_\kappa]$, it follows that $N[G_\kappa][S'][g']$ is still closed under κ-sequences in $V[G_\kappa][g']$. This, together with the fact that $\mathbb{P}'_{\text{tail}}$ is quite strategically closed and has only few dense subsets from $N[G_\kappa][S'][g']$, allows us to build, in $V[G_\kappa][g']$, a $\mathbb{P}'_{\text{tail}}$-generic G'_{tail} over $N[G_\kappa][S'][g']$ and lift the embedding i to

$$i\colon V[G_\kappa] \to N[G_\kappa][S'][g'][G'_{\text{tail}}]\,.$$

We can now force over $V[G_\kappa][S]$, using the factorization (1), to complete g' to g which is fully $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})$-generic over $V[G_\kappa][S]$. In the extension $V[G_\kappa][S][g]$ we can finally also lift the map k to

$$k\colon N[G_\kappa][S'][g'][G'_{\text{tail}}] \to M[G_\kappa][S][g][G_{\text{tail}}]\,,$$

[5] We could have arranged matters so that $\vec{Y}$ was also a $\clubsuit_{\kappa^{++}}(S)$-sequence in $V[G_\kappa][S]$, but this will not be important for the argument.

where G_{tail} is the filter generated by the pointwise image of G'_{tail}; this is generic since the forcing $\mathbb{P}_{\mathrm{tail}}$ is quite distributive in $N[G_\kappa][S'][g']$. Composing the two lifts of i and k gives us a lift of j. The situation is summarized in the following diagram; we should keep in mind that the pictured embeddings exist in $V[G_\kappa][S][g]$.

$$\begin{tikzcd} V[G_\kappa] \arrow[rr, "j"] \arrow[dr, "i"'] & & M[G_\kappa][S][g][G_{\mathrm{tail}}] \\ & N[G_\kappa][S'][g'][G'_{\mathrm{tail}}] \arrow[ur, "k"'] & \end{tikzcd}$$

As the final act of forcing, let C be $\mathbb{C}(S)^{V[G_\kappa][S]}$-generic over $V[G_\kappa][S][g]$. We claim that $V[G_\kappa][S][g \times C]$ is our desired final extension. Recall that lemma 16 tells us that we can also write this extension as $V[G_\kappa][H^0 \times H^2]$ for some generic $H^0 \subseteq \mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]}$ and $H^2 \subseteq \mathrm{Add}(\kappa^{++}, 1)^{V[G_\kappa]}$. We will work for a minute with this alternative representation of the extension and try to lift the embedding j.

By lemma 22 we know that $\mathrm{Term}(\mathbb{P}_\kappa, \mathrm{Add}(\kappa, \kappa^{++}))$ is forcing equivalent to $\mathrm{Add}(\kappa, \kappa^{++})$ in V. It follows from this by elementarity that the poset $\mathrm{Term}(i(\mathbb{P}_\kappa), i(\mathrm{Add}(\kappa, \kappa^{++})))$ is equivalent to $i(\mathrm{Add}(\kappa, \kappa^{++}))$ in N. Now we return to an assumption we made at the start of the proof. Since V has an $i(\mathrm{Add}(\kappa, \kappa^{++}))$-generic over N, we can use this equivalence and facts about term forcing to extract an $i(\mathrm{Add}(\kappa, \kappa^{++})^{V[G_\kappa]})$-generic K' over $N[G_\kappa][S'][g'][G'_{\mathrm{tail}}]$ in $V[G_\kappa][g']$. Since the forcing $i(\mathrm{Add}(\kappa, \kappa^{++}))$ is sufficiently distributive, the pointwise image $k[K']$ generates a generic filter $\tilde{K}^0$ over $M[G_\kappa][S][g][G_{\mathrm{tail}}]$. It is not necessarily the case that $j[H^0] \subseteq \tilde{K}^0$, but we can surgically alter $\tilde{K}^0$ (exactly as described in [4]) to obtain another generic K^0 over $M[G_\kappa][S][g][G_{\mathrm{tail}}]$ for which this will be the case, and we are able to lift j to

$$j \colon V[G_\kappa][H^0] \to M[G_\kappa][S][g][G_{\mathrm{tail}}][K^0]\,.$$

We can now forget about the maps i and k and focus solely on j. To complete the lift, observe that $\mathrm{Add}(\kappa^{++}, 1)^{V[G_\kappa]}$ remains $\leq \kappa^+$-distributive in $V[G_\kappa][H^0]$ by Easton's lemma, and so the filter $j[H^2]$ generates a generic K^2 over $M[G_\kappa][S][g][G_{\mathrm{tail}}][K^0]$, which gives us our final lift

$$j \colon V[G_\kappa][H^0 \times H^2] \to M[G_\kappa][S][g][G_{\mathrm{tail}}][K^0 \times K^2]\,.$$

Since j was originally a (κ, κ^{++})-extender embedding, the same remains true for the lifted embedding. Since we clearly have $2^\kappa = \kappa^{++}$ in the final model, lemma 25 tells us that the lift j is the ultrapower by a normal measure.

Claim. *The embedding j witnesses $\mathrm{CP}(\kappa, \kappa^+)$ in $V[G_\kappa][H^0][H^2]$.*

Proof. Let us write $M^* = M[G_\kappa][S][g][G_{\mathrm{tail}}][K^0][K^2]$. Since the part of forcing over M after stage κ is sufficiently distributive, M^* and $M[G_\kappa][S][g]$ agree on $\mathcal{P}(\kappa^+)$. Next, notice that $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})^{V[G_\kappa][S]}$ is κ^+-cc in $V[G_\kappa][S][C]$. This is because it is κ^+-cc in $V[G_\kappa][S]$ and the forcing $\mathbb{C}(S)$ is $\leq \kappa^+$-distributive in that model, so it cannot add any antichains of size κ^+ to $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})$.

Consider the state of affairs in $V[G_\kappa]$: the forcing $\mathbb{S}_{\kappa^{++}} * \mathbb{C}(S)$ is $< \kappa^{++}$-distributive and it forces that $\mathbb{A}(\kappa, \kappa^{++}, \vec{Y})^{V[G_\kappa][S]}$ is κ^+-cc and a subset of $H_{\kappa^{++}}$. Lemma 24 now implies that every subset x of κ^+ in $V[G_\kappa][S][C][g]$ has a name $\sigma \in H_{\kappa^{++}}^{V[G_\kappa]}$ such that $\sigma^g = x$. Since $\mathbb{P}_\kappa$ is κ-cc, we know that $H_{\kappa^{++}}^{M[G_\kappa]} = H_{\kappa^{++}}^{V[G_\kappa]}$, so these names also appear in $M[G_\kappa]$. It follows from this that $M[G_\kappa][S][g]$ has all the subsets of κ^+ from $V[G_\kappa][S][C][g]$. ■

We have shown that $\mathrm{CP}(\kappa, \kappa^+)$ holds in $V[G_\kappa][H^0 \times H^2]$. To finish the proof we also need to see that κ is the least measurable cardinal in that model. This follows easily from the way we designed the forcing $\mathbb{P}_\kappa$. If $\gamma < \kappa$ were measurable in $V[G_\kappa][H^0 \times H^2]$, it must definitely be inaccessible in V. It follows that we did some nontrivial forcing at stage γ in the iteration $\mathbb{P}_\kappa$ and lemma 19 implies that after the stage γ forcing γ is not measurable. The remaining forcing to get from that model to the model $V[G_\kappa][H^0 \times H^2]$ is at least $\leq 2^{2^\gamma}$-closed, which means that it could not have possibly added any measures on γ. We can therefore conclude that γ remains nonmeasurable in $V[G_\kappa][H^0 \times H^2]$. □

The iteration we used is essentially the one described in [1, Section 4]. It follows from the results proved there that, had we assumed in theorem 26 that κ was κ^+-supercompact, this would remain true in the resulting extension.

Corollary 27. *If κ is κ^+-supercompact, then there is a forcing extension in which* $\mathrm{CP}(\kappa, \kappa^+)$ *holds, and κ is κ^+-supercompact and the least measurable.*

By starting with a stronger large cardinal hypothesis and modifying the forcing iteration appropriately, we can push up the value of 2^κ beyond just κ^{++} and capture even more powersets.

Theorem 28. *Assume GCH holds and suppose that κ is H_λ-strong for some regular cardinal $\lambda \geq \kappa^{++}$. Then there is a forcing extension in which κ is the least measurable cardinal, $2^\kappa = \lambda$, and* $\mathrm{CP}(\kappa, < \lambda)$ *holds (meaning that a single normal measure on κ captures every $\mathcal{P}(\mu)$ for $\mu < \lambda$).*

Proof. The argument is much like the proof of theorem 26, with a slight modification to the forcing used. Furthermore, instead of preparing the model as in [4], we use a result of the second author from [11] and pass to a forcing extension in order to be able to assume that the following hold in V:

(1) $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \lambda$.
(2) κ is H_λ-strong and this is witnessed by a (κ, λ)-extender embedding $j \colon V \to M$; moreover, M is closed under κ-sequences.
(3) There is a function $\ell \colon \kappa \to \kappa$ such that $j(\ell)(\kappa) = \lambda$.
(4) There is in V an M-generic filter for the poset $j(\mathrm{Add}(\kappa, \lambda))$.

Since λ is regular, we may even assume that $\ell(\gamma)$ is regular whenever γ is an inaccessible cardinal. The initial iteration $\mathbb{P}_\kappa$ will now be an Easton-support iteration which forces at inaccessible cardinals $\gamma < \kappa$ with the forcing $\mathbb{S}_{\ell(\gamma)} * \mathbb{A}(\gamma, \ell(\gamma), \vec{X})$, *provided that γ is inaccessible in $V^{\mathbb{P}_\gamma}$.*

Note that, since $\mathbb{P}_\kappa$ is κ-cc, there will be nontrivial forcing at stage κ of the iteration $j(\mathbb{P}_\kappa)$ and we can write

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{S}_\lambda * \mathbb{A}(\kappa, \lambda, \vec{Y}) * \mathbb{P}_{\mathrm{tail}}\,.$$

The full forcing that will give us the theorem is then

$$\mathbb{P} = \mathbb{P}_\kappa * \mathbb{S}_\lambda * (\mathbb{A}(\kappa, \lambda, \vec{Y} \times \mathbb{C}(\dot{S}))) .$$

The argument now proceeds very much like the proof of theorem 26, except that we do not need to deal with the maps i and k at all, since our preparation gave us a better generic than the one from the proof of theorem 26. We replace κ^{++} everywhere in that argument with λ and notice that the part of $j(\mathbb{P})$ above κ now only has nontrivial forcing beyond stage λ; this means that, as before, only the part of the forcing $j(\mathbb{P})$ up to and including stage κ is relevant as far as subsets of λ are concerned. $\square$

Conversely, we can extend Cummings' argument to show that the large cardinal hypothesis we used above is optimal.

Theorem 29. *Suppose that* $\mathrm{CP}(\kappa, < \lambda)$ *holds for some regular cardinal* $\lambda \geq \kappa^{++}$. *Write* $\lambda = \kappa^{+\alpha}$ *for some ordinal* α. *Then* κ *is* $(\kappa + \alpha)$*-strong (or, equivalently,* H_λ*-strong) in an inner model.*

Proof. This is essentially standard. Suppose that $j : V \to M$ is an elementary embedding with critical point κ such that $H_\lambda \in M$. We assume that there is no inner model with a strong cardinal and let K be the core model with the (nonoverlapping) extender sequence $\vec{E}$. It follows that $j \restriction K$ is the result of a normal iteration of $\vec{E}$ and, since the critical point of j is κ, the first extender applied in this iteration must have index (κ, η) for some η. Since $\vec{E}$ is coherent, the sequence $j(\vec{E})$ has no extenders with indices (κ, β) for $\beta \geq \eta$. But since M captured all of H_λ, we must have $K \restriction \lambda = K^M \restriction \lambda$, and so $\vec{E}$ and $j(\vec{E})$ must agree up to λ. It follows that $\eta \geq \lambda$ and so $o(\kappa) \geq \lambda + 1$ (and κ is H_λ-strong) in K. $\square$

The preparation from [11] works even for singular λ of cofinality strictly above κ (if the cofinality of λ is equal to κ^+, we get $2^{\kappa^+} = \lambda^+$ in (1) above). It is unclear, however, whether theorem 28 can allow for this weaker hypothesis (in particular, lemma 21 seems to rely crucially on the second parameter in the Apter–Shelah forcing being regular).

Question 30. *Can theorem 28 be improved to allow for arbitrary* λ *of cofinality strictly above* κ*?*

Another question raised by theorem 28 is whether $\mathrm{CP}(\kappa, \lambda)$ can fail for the first time at some $\kappa^+ < \lambda < 2^\kappa$. The following theorem shows that the answer is yes.

Theorem 31. *Suppose GCH holds and* κ *is* $H_{\kappa^{+3}}$*-strong. Then there is a forcing extension in which* κ *is the least measurable,* $2^\kappa = \kappa^{+3}$ *and* $\mathrm{CP}(\kappa, \kappa^+)$ *hold, while* $\mathrm{LCP}(\kappa, \kappa^{++})$ *fails.*

One would expect that it should be possible to force $2^\kappa = \kappa^{+3}$ and $\mathrm{CP}(\kappa, \kappa^+)$ starting from a large cardinal hypothesis weaker than an $H_{\kappa^{+3}}$-strong cardinal κ; an $H_{\kappa^{+2}}$-strong and κ^{+3}-tall cardinal κ likely suffices (and this would be optimal). However, the proof that we are about to give seems to require a stronger hypothesis in order to deduce a connection between the forcings at stage κ over V and over the target model (in particular, the forcings to add a nonreflecting stationary subset to κ^{+3} should look sufficiently similar). It is nevertheless plausible, if unclear, that the required constellation of properties can be forced using a weaker hypothesis.

Proof. The proof of this theorem is similar to the proof of theorem 26, but with some technical complications. The idea should, nevertheless, be clear. Ideally we would like to force as in theorem 28 and at the end of the iteration add, in a product manner, a new subset to κ^{++}. It is not too hard to argue that this new subset cannot be captured by any normal ultrapower on κ in the extension. However, the problem arises since already the preparatory forcing we used in that previous theorem necessarily forces $2^{\kappa^+} = \kappa^{+3}$, which means that our adding a new subset to κ^{++} at the end will more than likely collapse κ^{+3}. Our solution to this problem will be to delve to some extent into the preparatory forcing (which we have so far valiantly managed to avoid) and essentially fold the preparation into the main forcing itself. This will ensure that, at stage κ, enough of the GCH is maintained for us to be able to preserve κ^{+3} and arrange matters as required in the statement of the theorem.

As always, let $j\colon V \to M$ be a (κ, κ^{+3})-extender embedding witnessing $H_{\kappa^{+3}}$-strongness, where M is closed under κ-sequences. We draw the usual diagram

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "i"'] & M \\ & N \arrow[u, "k"'] \end{tikzcd}$$

where i is the induced normal ultrapower. Note that the critical point of k is $(\kappa^{++})^N$ and that $(\kappa^{+3})^N$ is an ordinal of size κ^+ in V. Let us write $\nu = (\kappa^{+3})^N$.

Let $\mathbb{P}_\kappa$ be an Easton-support iteration which forces at inaccessible cardinals $\gamma < \kappa$ with the forcing $\mathbb{S}_{\gamma^{+3}} * (\mathbb{A}(\gamma, \gamma^{+3}, \vec{X}) \times \mathrm{Add}(\gamma^+, \gamma^{+3}))$. The images of this forcing via j and i can be written as

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{S}_{\kappa^{+3}} * (\mathbb{A}(\kappa, \kappa^{+3}, \vec{Y}) \times \mathrm{Add}(\kappa^+, \kappa^{+3})) * \mathbb{P}_{\mathrm{tail}}$$

and

$$i(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \big(\mathbb{S}_{\kappa^{+3}} * (\mathbb{A}(\kappa, \kappa^{+3}, \vec{Y}') \times \mathrm{Add}(\kappa^+, \kappa^{+3}))\big)^{N^{\mathbb{P}_\kappa}} * \mathbb{P}'_{\mathrm{tail}}\,.$$

Let G_κ be $\mathbb{P}_\kappa$-generic over V. We proceed similarly to the proof of theorem 26. In $V[G_\kappa]$ there is an unbounded $S' \subseteq \nu$ which is $(\mathbb{S}_{\kappa^{+3}})^{N[G_\kappa]}$-generic over $N[G_\kappa]$. Moreover, S' is nonstationary in ν. Since the map k is progressive (meaning that $k(\alpha) \geq \alpha$ for ordinals α) and injective, Fodor's lemma implies that $k[S']$ is also nonstationary in its supremum and its proper initial segments are nonstationary in their suprema as well. Since $k[\nu]$ is bounded in κ^{+3}, as ν has size only κ^+, the set $k[S']$ is a condition in $\mathbb{S}_{\kappa^{+3}}$. Let S be a generic for this forcing over $V[G_\kappa]$ such that $k[S']$ is an initial segment of S.

Let $g' \times h'$ be generic for $\mathbb{A}(\kappa, \nu, \vec{Y}') \times \mathrm{Add}(\kappa^+, \nu)$ over $V[G_\kappa][S]$. Observe that $2^\kappa = \kappa^+$ still holds in $V[G_\kappa][S][g' \times h']$ and the model $N[G_\kappa][S'][g' \times h']$ remains closed under κ-sequences. It follows that we may build a $\mathbb{P}'_{\mathrm{tail}}$-generic G'_{tail} over $N[G_\kappa][S'][g' \times h']$ in $V[G_\kappa][S][g' \times h']$. This already allows us to partially lift the embedding i to

$$i\colon V[G_\kappa] \to N[G_\kappa][S'][g' \times h'][G'_{\text{tail}}]$$

in the model $V[G_\kappa][S][g' \times h']$.

Before we continue lifting the entire diagram, we make a small digression which will be useful later in the argument.

Claim. *The embedding i can be lifted in the model $V[G_\kappa][S][g' \times h']$ to an embedding i^+ defined on this whole model.*

Proof. For the duration of this proof let us write $V^+ = V[G_\kappa]$ and $N^+ = N[G_\kappa][S'][g' \times h'][G'_{\text{tail}}]$. Until now we have built an embedding $i\colon V^+ \to N^+$ in $V^+[S][g' \times h']$. We should note that $2^\kappa = \kappa^+$ holds in $V^+[S][g' \times h']$ and that N^+ is closed under κ-sequences in this model. Since the forcing to add S is sufficiently distributive, we can simply transfer the generic S along i and get a further lift $i\colon V^+[S] \to N^+[i(S)]$. Furthermore, $N^+[i(S)]$ remains closed under κ-sequences in $V^+[S][g' \times h']$.

Consider now the forcing $\mathbb{A}(\kappa, \nu, \vec{Y}')$ that added g'. Since ν has size κ^+, we can replace this poset with an isomorphic one of the form $\mathbb{A}(\kappa, \kappa^+, \vec{Y}'')$, but we allow ourselves a small abuse of notation and just assume that g' was added by $\mathbb{A}(\kappa, \kappa^+, \vec{Y}')$.

The forcing $i(\mathbb{A}(\kappa, \kappa^+, \vec{Y}'))$ is $i(\kappa^+)$-cc, has size $i(\kappa^+)$, and is $< i(\kappa)$-closed in $N^+[i(S)]$. Using the fact that $2^\kappa = \kappa^+$ in $V^+[S][g' \times h']$, we can enumerate all of the maximal antichains of $i(\mathbb{A}(\kappa, \kappa^+, \vec{Y}'))$ from $N^+[i(S)]$ as A_α for $\alpha < \kappa^+$. Moreover, the support of each of these maximal antichains is bounded in $i(\kappa^+)$. In fact, since i is continuous at κ^+, there are ordinals $\beta_\alpha < \kappa^+$ so that A_α is contained in $i(\mathbb{A}(\kappa, \beta_\alpha, \vec{Y}'))$. We may assume that the β_α form an increasing sequence converging to κ^+. We shall use these maximal antichains to build an appropriate generic object over $N^+[i(S)]$.

Let us start with p_0 being the trivial condition. As $N^+[i(S)]$ is closed under κ-sequences, we have $i[g' \restriction \beta_0] \in N^+[i(S)]$, and this is actually a condition in $i(\mathbb{A}(\kappa, \beta_0, \vec{Y}'))$. Let $q_0 \leq p_0$ be the union of this condition with p_0 and then let $p_1 \leq q_0$ be some condition in $i(\mathbb{A}(\kappa, \beta_0, \vec{Y}'))$ deciding the maximal antichain A_0. The next step works much the same: $i[g' \restriction \beta_1]$ is a condition in $i(\mathbb{A}(\kappa, \beta_1, \vec{Y}'))$. By an argument as in the proof of lemma 18, this condition is compatible with p_1, so let q_1 be a common lower bound in $i(\mathbb{A}(\kappa, \beta_1, \vec{Y}'))$. To finish the step, let p_2 be an extension of q_1 in this poset that decides the maximal antichain A_1.

We can continue in this way for κ^+ many steps, using the closure of $i(\mathbb{A}(\kappa, \kappa^+, \vec{Y}))$ in $V^+[S][g' \times h']$ to pass through limit steps. Let g^+ be the filter generated by the descending sequence of the p_α. We ensured that g^+ is generic over $N^+[i(S)]$, and, since we fed information about $i[g']$ into the conditions during the construction, we get $i[g'] \subseteq g^+$. It follows that we may lift the embedding i to $i\colon V^+[S][g'] \to N^+[i(S)][g^+]$.

To finish the proof, we simply observe that the forcing to add h' over $V^+[S][g']$ is $\leq \kappa$-distributive, which means that we can lift i again by simply transferring the generic h'. This final lift

$$i^+\colon V^+[S][g'][h'] \to N^+[i(S)][g^+][i^+(h')]$$

is the lift we required. ■

Let us now complete the generics g' and h' to fully fledged generics. First, observe that, since $k[S']$ is an initial segment of S, the only ladders appearing in $\vec{Y}$ below $\sup k[\nu]$ are the pointwise images of the ladders in $\vec{Y}'$, and we may assume that none of the other ladders in $\vec{Y}$ have points below $\sup k[\nu]$. It follows that we may factor the forcing $\mathbb{A}(\kappa, \kappa^{+3}, \vec{Y})$ as

$$\mathbb{A}(\kappa, \kappa^{+3}, \vec{Y}) = \mathbb{A}(\kappa, k[\nu], k[\vec{Y}']) \times \mathbb{A}(\kappa, \kappa^{+3} \setminus k[\nu], \vec{Y})\,.$$

Since $k \restriction \nu$ is an element of both V and M, it follows that $k[g']$ is $V[G_\kappa][S]$-generic for the first factor above. If we let g'' be $V[G_\kappa][S][g' \times h']$-generic for the second factor, we get a generic $g = k[g'] \times g''$ over $V[G_\kappa][S]$. In a similar fashion we can complete $k[h']$ to an $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa]}$-generic $h = k[h'] \times h''$ over $V[G_\kappa][S][g]$.

With all these generics in hand, and using k to transfer the generic G'_{tail} to a generic G_{tail} for $\mathbb{P}_{\text{tail}}$, we can lift the entire diagram in $V[G_\kappa][S][g \times h]$.

$$\begin{tikzcd} V[G_\kappa] \arrow[rr, "j"] \arrow[dr, "i"'] & & M[G_\kappa][S][g \times h][G_{\text{tail}}] \\ & N[G_\kappa][S'][g' \times h'][G'_{\text{tail}}] \arrow[ur, "k"'] & \end{tikzcd}$$

Now force over the model $V[G_\kappa][S][g \times h]$ to add a $\mathbb{C}(S)^{V[G_\kappa][S]}$-generic C. By lemma 21 we can rewrite the resulting extension $V[G_\kappa][S][g \times h \times C]$ as $V[G_\kappa][H^0 \times h \times H^3]$ where H^0 is $\mathrm{Add}(\kappa, \kappa^{+3})$-generic and H^3 is $\mathrm{Add}(\kappa^{+3}, 1)$-generic.

We first work to find an $i(\mathrm{Add}(\kappa, \kappa^{+3}))$-generic over $N[G_\kappa][S'][g' \times h'][G'_{\text{tail}}]$. Note that this forcing is the same as $i^+(\mathrm{Add}(\kappa, \kappa^{+3}))$, where i^+ is the embedding from the claim above. Lemma 23 implies that this forcing is equivalent to $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa][S][g']}$. What we have is the $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa][S][g \times h']}$-generic h'', but these posets do not quite match up. However, all is not lost. Lemma 22 tells us that the term forcing $\mathrm{Term}(\mathbb{A}(\kappa, \nu, \vec{Y}'), \mathrm{Add}(\kappa^+, \kappa^{+3}))$ in the model $V[G_\kappa][S][h']$ is equivalent to $\mathrm{Add}(\kappa^+, \kappa^{+3})$ in that model. It follows that, in the model $V[G_\kappa][S][g' \times h]$, we can extract a generic for this term forcing from h'' and, combined with g', this gives us a $\mathrm{Add}(\kappa^+, \kappa^{+3})^{V[G_\kappa][S][g']}$-generic over $V[G_\kappa][S][g' \times h']$. This is exactly what we wanted.

The generic over $N[G_\kappa][S'][g' \times h'][G'_{\text{tail}}]$ can be transferred along k to give a $j(\mathrm{Add}(\kappa, \kappa^{+3}))$-generic over $M[G_\kappa][S][g \times h][G_{\text{tail}}]$. Let K^0 be the result of a surgical modification to this generic to ensure that $j[H^0] \subseteq K^0$. This allows us to lift j to

$$j\colon V[G_\kappa][H^0] \to M[G_\kappa][S][g \times h][G_{\text{tail}}][K^0]\,.$$

At this point we can again forget about i and k and just work with j. The forcing to add $h \times H^3$ was sufficiently distributive that the generic can simply be transferred along j to yield $K^1 \times K^3$ and a final lift

$$j\colon V[G_\kappa][H^0 \times h \times H^3] \to M[G_\kappa][S][g \times h][G_{\text{tail}}][K^0 \times K^1 \times K^3]\,.$$

The iteration $\mathbb{P}_\kappa$ destroyed the measurability of all $\gamma < \kappa$, so κ is now the least measurable cardinal. Furthermore, since we clearly have $2^\kappa = \kappa^{+3}$ in this final extension, the lifted j is a normal ultrapower by lemma 25.

The embedding j witnesses $\mathrm{CP}(\kappa, \kappa^{++})$, which can be seen using lemma 24. More precisely, the lemma shows that any subset $x \subseteq \kappa^{++}$ in the model $V[G_\kappa][S][C \times g \times h]$ has a name $\sigma \in H_{\kappa^{+3}}^{V[G_\kappa]}$ such that $x = \sigma^{g \times h}$. But since $V[G_\kappa]$ and $M[G_\kappa]$ agree on $H_{\kappa^{+3}}$, the name σ must appear in $M[G_\kappa]$ as well, and therefore x appears in the codomain of j.

Finally, let H^2 be $\mathrm{Add}(\kappa^{++}, 1)^{V[G_\kappa]}$-generic over this final model. We can transfer H^2 along j to obtain another generic K^2 and lift j again to

$$j\colon V[G_\kappa][H^0 \times h \times H^2 \times H^3] \to M[G_\kappa][S][g \times h][G_{\mathrm{tail}}][K^0 \times K^1 \times K^2 \times K^3]$$

Adding H^2 did not add any new subsets to κ^+, so κ is still the least measurable cardinal and the lifted j still witnesses $\mathrm{CP}(\kappa, \kappa^+)$.

Claim. $\mathrm{LCP}(\kappa, \kappa^{++})$ *fails in the model* $V[G_\kappa][H^0 \times h \times H^2 \times H^3]$.

Proof. Let us write $H = H^0 \times h \times H^2 \times H^3$ and let $\mathbb{P}$ be the entire forcing to add $G_\kappa * H$ over V. Assume that $\mathrm{LCP}(\kappa, \kappa^{++})$ holds. Then there is a normal ultrapower $j^*\colon V[G_\kappa][H] \to M^*[G^*][H^*]$ on κ which captures H^2 and $\mathcal{P}(\kappa^+)^V$. In particular, this implies that $M^*[G^*][H^*]$ computes κ^{++} correctly.

Let us write $G^* = G_\kappa * (S^* * (g^* \times h^*)) * G^*_{\mathrm{tail}}$; note that G_κ really is an initial segment of G^*, since we necessarily have $j(G_\kappa) = G^*$ and thus $p = j(p) \in G^*$ for any $p \in G_\kappa$. Let γ_0 be the least inaccessible cardinal in V. First, observe that $\mathbb{P}$ has a low gap: we can factor $\mathbb{P}$ as $\mathbb{P} = \mathbb{Q}_0 * \mathbb{Q}^0$ where $\mathbb{Q}_0$ is nontrivial of size less than γ_0^{+5} and $\mathbb{Q}^0$ is $\leq \gamma_0^{+5}$-strategically closed. It follows from Hamkins' gap forcing theorem [10] that $M^* = V \cap M^*[G^*][H^*] \subseteq V$. This implies that $\mathcal{P}(\kappa^+)^V \in M^*$ and it follows that $\mathcal{P}(\kappa^+)^{V[G_\kappa]} \in M^*[G_\kappa]$. Moreover, it means that $H^2 \notin M^*[G_\kappa]$, since H^2 is generic over $V[G_\kappa] \supseteq M^*[G_\kappa]$. Additionally, the further extension $M^*[G_\kappa][S^*]$ does not add any new subsets of κ^{++}, so H^2 does not appear there either.

Using lemma 17 we can see that the square of $\mathbb{A}(\kappa, \kappa^{+3}, \vec{X}) \times \mathrm{Add}(\kappa^+, \kappa^{+3})$ is κ^{++}-cc. It follows from this and a result of Unger [14, Lemma 2.4] that passing to $M^*[G_\kappa][S^*][g^* \times h^*]$ does not add any new fresh subsets of κ^{++} to $M^*[G_\kappa][S^*]$ (recall that a set of ordinals is *fresh* over a model if it is not in that model but all of its initial segments are). Of course, H^2 is a fresh subset of κ^{++} over $V[G_\kappa]$, and since $V[G_\kappa]$ and $M^*[G_\kappa][S^*]$ have the same bounded subsets of κ^{++}, it is also fresh over $M^*[G_\kappa][S^*]$. Therefore H^2 does not appear in $M^*[G_\kappa][S^*][g^* \times h^*]$. To conclude the proof, notice that the remainder of the forcing to go from $M^*[G_\kappa][S^*][g^* \times h^*]$ to $M^*[G^*][H^*]$ does not add any subsets of κ^{++}, so it definitely cannot add H^2. But this contradicts our assumption that $M^*[G^*][H^*]$ captured H^2. ■

To summarize, the model $V[G_\kappa][H^0 \times h \times H^2 \times H^3]$ satisfies $2^\kappa = \kappa^{+3}$ and the lifted embedding j witnesses $\mathrm{CP}(\kappa, \kappa^+)$, but, as the last claim shows, $\mathrm{LCP}(\kappa, \kappa^{++})$ fails. This finishes the proof of the theorem. □

At the end of the paper, let us give another example of the power of lemma 25 in showing that $\mathrm{CP}(\kappa, \kappa^+)$ holds in known forcing extensions. As we have seen, $\mathrm{CP}(\kappa, \kappa^+)$ does not have any implications for the outright size of κ, since it may consistently hold at the least measurable cardinal κ. But one might try to measure its effects slightly differently. While the capturing property says that there is a normal measure on κ which is quite "fat", in the sense that it captures all subsets of κ^+, perhaps κ must inevitably also carry some, or many, "thin" measures which do not capture much at all. In other words, perhaps $\mathrm{CP}(\kappa,\kappa^+)$ has some implications about the number of normal measures on κ. A combination of lemma 25 and a theorem of Friedman and Magidor will show us that this is not the case.

Theorem 32. *If V is the minimal extender model with a $(\kappa+2)$-strong cardinal κ and $\lambda \leq \kappa^{++}$ is a cardinal, then there is a forcing extension in which κ carries exactly λ many normal measures and each of them witnesses $\mathrm{CP}(\kappa,\kappa^+)$. In particular, it is consistent that κ has a unique normal measure and $\mathrm{CP}(\kappa,\kappa^+)$ holds.*

Proof. The hard part of the proof was done by Friedman and Magidor [6], who showed that, starting from the listed hypotheses, there is a forcing extension $V[G]$ satisfying $2^\kappa = \kappa^{++}$ in which κ carries exactly λ many normal measures. They also show that each of these normal measures is derived from a lift of the ground model extender embedding $j\colon V \to M$ witnessing the $(\kappa+2)$-strongness of κ. However, lemma 25 implies that these lifts are themselves already ultrapowers by a normal measure on κ. Finally, an analysis of their proof shows that the forcing used to obtain the model $V[G]$ can be written as $\mathbb{P} * \dot{\mathbb{Q}}$ where $\mathbb{P} \subseteq H_{\kappa^{++}}$ is a κ^{++}-cc poset which is regularly embedded in $j(\mathbb{P})$, and $\dot{\mathbb{Q}}$ is forced to be $\leq \kappa^+$-distributive. It follows that every subset of κ^+ in $V[G]$ has a nice name in $H^V_{\kappa^{++}} \in M$ and therefore appears in $M[j(G)]$. □

It is unclear whether one can obtain similar results at the least measurable cardinal κ. It seems likely that, to do so, it would be necessary to adapt the Apter–Shelah forcing to incorporate the Sacks forcing machinery that Friedman and Magidor used in their arguments.

Question 33. *Is it consistent that the least measurable cardinal κ carries a unique normal measure and $\mathrm{CP}(\kappa,\kappa^+)$ holds?*

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