

A Laver-like indestructibility for hypermeasurable cardinals

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Abstract: We show that if κ is $H(\mu)$ -hypermeasurable for some cardinal μ with $\kappa < \text{cf}(\mu) \leq \mu$ and GCH holds, then we can extend the universe by a cofinality-preserving forcing to obtain a model V^* in which the $H(\mu)$ -hypermeasurability of κ is indestructible by the Cohen forcing at κ of any length up to μ (in particular κ is $H(\mu)$ -hypermeasurable in V^*). The preservation of hypermeasurability (in contrast to preservation of mere measurability) is useful for subsequent arguments (such as the definition of Radin forcing). The construction of V^* is based on the ideas of Woodin (unpublished) and Cummings [3] for preservation of measurability, but suitably generalised and simplified to achieve a more general result.

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1 Introduction

Laver showed in [16] that one can extend any universe where κ is supercompact by a forcing of size κ such that in the resulting generic extension the supercompactness of κ is indestructible by any κ directed-closed forcing notion. It is natural to try to formulate an analogue of Laver's indestructibility for other large cardinals. There is ample literature on the subject, see for instance Gitik and Shelah [12] for strong cardinals (with respect to Prikry-style forcing notions), Hamkins [14] for strongly compact cardinals, Johnston [15] for strongly unfoldable cardinals and Apter [2] for a strong cardinal (the strong cardinal was supercompact in the ground model); these results usually try to achieve the greatest possible level of generality. However, it is often useful to have indestructibility for a particular forcing notion, such as the Cohen forcing at κ of a prescribed length α (we denote this forcing $\text{Add}(\kappa, \alpha)$).¹ With such a limited goal, one can study large cardinals for which a more general version of indestructibility may not be available.

Recall the definition of a hypermeasurable cardinal: If $\kappa < \mu$ are cardinals, we call κ $H(\mu)$ -hypermeasurable if there is $j : V \rightarrow M$ with critical point κ such that $j(\kappa) > \mu$ and $H(\mu) \subseteq M$, where $H(\mu)$ denotes the collection of sets whose transitive closure has size

¹This is for instance the case of obtaining the tree property at the double successor of a singular strong limit cardinal κ in [5] where the large cardinal assumption of supercompactness is invoked only to ensure that κ is still measurable after forcing with $\text{Add}(\kappa, \lambda)$, where λ is the least weakly compact above κ .

less than μ .² Hypermeasurable cardinals, and related notions, feature predominantly in many arguments involving the structure of the continuum function on all cardinals, and in arguments dealing with combinatorial properties at the double successor of a singular strong limit cardinal; see for instance [10], [11], [17], and [7]. In particular, the paper [7] provided the incentive for the results in this paper as the assumption of indestructibility simplified substantially the argument for the tree property (building on the indestructibility, [13] achieves a more general result).

Woodin first discovered an argument for preparing a universe with an $H(\kappa^{++})$ -hypermeasurable cardinal so that the further forcing with $\text{Add}(\kappa, \kappa^{++})$ yields the measurability (in fact $H(\kappa^{++})$ -hypermeasurability) of κ . There exist two different presentations of this argument: [4] and [3] (we will use the latter); both presentations force over the universe a generic object for the image of $\text{Add}(\kappa, \kappa^{++})$ by a measure ultrapower embedding which is then used to obtain the desired generic filter to be able to lift a certain embedding (the generic object needs to be modified; this step is often called the “surgery”). It seemed crucial in this argument that the image of $\text{Add}(\kappa, \kappa^{++})$ via a measure ultrapower embedding is in fact equivalent to $\text{Add}(\kappa^+, \kappa^{++})$ in the universe.

Building on the method described in [3], we reformulate and simplify the argument to work with an arbitrary $\kappa < \text{cf}(\mu) \leq \mu$ and to preserve the original degree of hypermeasurability. Since there seems to be no uniform representation of the image of $\text{Add}(\kappa, \mu)$ via a measure ultrapower embedding for larger μ (see Question **Q1** for more details), we use a lottery of forcing notions to circumvent this step (see Definition 2.2). This change necessitated another modification in the subsequent argument: it was necessary to avoid the use of the measure ultrapower embedding in the argument for Theorem 2.10 (see Remark 2.8 for more details). This modification led to a more elegant argument and allows one to handle more complicated forcing notions (see Question **Q2**).

The paper is structured as follows: In Section 2.2 we prepare the universe by forcing a generic object (denoted \tilde{g}^* in Corollary 2.7) which is then used in Section 2.3 and Theorem 2.10 to lift a hypermeasurable embedding. The preparation in Section 2.2 must be carefully set up so that the hypermeasurability of κ is preserved at this stage. In section 3, we list some open questions.

2 Construction of the model

2.1 Initial assumptions

Let us assume that GCH holds in V . Let us for the rest of the paper fix cardinals κ and μ with

$$\kappa < \text{cf}(\mu) \leq \mu$$

and assume that κ is $H(\mu)$ -hypermeasurable.

If $\mu = \kappa^+$, then κ is just measurable and the indestructibility argument is much simpler

²The same cardinal is also called $\kappa + \alpha$ -strong for an appropriate α , i.e. for an α such that $V_{\kappa+\alpha}$ is included in M if and only if $H(\mu)$ is included in M , where M is an inner model.

(see Remark 2.11). Let us therefore further assume

$$\kappa^{++} \leq \mu.$$

Let us first review some basic properties of embeddings associated with large cardinals (a general reference for this review is [4] which contains details regarding the lifting of embeddings).

Let us fix $j : V \rightarrow M$ which witnesses the $H(\mu)$ -hypermeasurability of κ and has the extender representation:

$$M = \{j(f)(\alpha) \mid f : \kappa \rightarrow V \text{ \& } \alpha < \mu\}.$$

In particular, $H(\mu)$ is included in M and M is closed under κ -sequences in V (this may be assumed since $\text{cf}(\mu) > \kappa$). Let us also assume there exists $\mathfrak{f} : \kappa \rightarrow \kappa$ such that $j(\mathfrak{f})(\kappa) = \mu$ and for each measurable $\alpha < \kappa$, $\mathfrak{f}(\alpha)$ has cofinality at least α^+ ; let us write μ_α for $\mathfrak{f}(\alpha)$.³

Let U be the normal measure derived from j , and let $i : V \rightarrow N$ be the ultrapower embedding generated by U . Let $k : N \rightarrow M$ be elementary so that $j = k \circ i$. Note that κ is the critical point of j, i and j, i have width κ , i.e. every element of M and N is of the form $j(f)(\alpha)$, for some $\alpha < \mu$, or $i(f)(\kappa)$ respectively, for some f with domain κ . In contrast, the critical point of k is $(\kappa^{++})^N$ and k has width $i(\mathfrak{f})(\kappa) < i(\kappa)$, i.e. every element of M can be written as $k(f)(\alpha)$ for some f in N with domain $i(\mathfrak{f})(\kappa)$;⁴ in particular $(\kappa^{++})^N \leq i(\mathfrak{f})(\kappa) < i(\kappa) < \kappa^{++}$.

We need the following simple lemma:

Lemma 2.1 *Assume GCH and that κ, μ, i, j are as above. Then*

$$(2.1) \quad \mu \leq i(\mu) < j(\kappa) < \mu^+ \text{ and } j(\kappa) < j(\mu) < \mu^{++};$$

more precisely:

- (i) *If μ is the successor of a singular cardinal ν with cofinality κ , then $\mu \leq i(\mu) < j(\kappa)$.*
- (ii) *Otherwise $\mu = i(\mu) < j(\kappa)$.*

PROOF. All inequalities in (2.1) – except for $i(\mu) < j(\kappa)$ – follow by simple counting arguments and elementarity using the fact that the cofinality μ is at least κ^+ . The claim $i(\mu) < j(\kappa)$ will follow from the proofs of (i) and (ii).

(i). Every $\alpha < i(\nu)$ in the ultrapower can be written as $[f]_U$ for some $f : \kappa \rightarrow \nu$ (where $[f]_U$ denotes the collapse of the equivalence class of f). Since M contains U as an element and is closed under κ -sequences in V , it can compute that $|i(\nu)| \leq \nu^\kappa = (\nu^+)^M = \mu$ so that:

$$i(\nu) < (\nu^{++})^M.$$

³Such an \mathfrak{f} can be forced if necessary using the fast-function forcing of Woodin; see [14].

⁴Every $j(f)(\alpha)$ can be written as $k(i(f))(\alpha)$; if we set $f^* = i(f) \upharpoonright i(\mathfrak{f})(\kappa)$, the domain of $k(f^*)$ is equal to $k(i(\mathfrak{f})(\kappa)) = \mu$, and $k(f^*)(\alpha) = j(f)(\alpha)$.

Since ν is less than $j(\kappa)$, $j(\kappa)$ is inaccessible in M , and $i(\mu)$ is the successor of $i(\nu)$ we obtain:

$$i(\mu) = (i(\nu^+))^M < (\nu^{+++})^M < j(\kappa).$$

(ii). Every $\alpha < i(\mu)$ can be written as $[f]_U$ for some $f : \kappa \rightarrow \mu$, and this gives $|i(\mu)| \leq \mu^\kappa = \mu$. To argue that $i(\mu) = \mu$, it suffices to show that $\langle i(\alpha) \mid \alpha < \mu \rangle$ is cofinal in $i(\mu)$ and its limit is μ . For the first claim, note that if $[f]_U < i(\mu)$, then without the loss of generality there is some $\alpha < \mu$ such that the range of f is included in α (since the cofinality of μ is at least κ^+), and therefore $[f]_U \leq i(\alpha)$. For the second claim notice that each $i(\alpha)$ has size strictly less than μ and therefore $i(\alpha) < \mu$. (The second claim does not go through for case (i).) \square

Note that since U is in M , an argument similar to Lemma 2.1(i), (ii) implies that the forcing $i(\text{Add}(\kappa, \mu)) = \text{Add}(i(\kappa), i(\mu))^N$ computes the same way in V and M , and is an element of $H(j(\kappa))^M$.

2.2 Preparation

Let $C(f)$ be the closed unbounded set of all closure points of f : if α is in $C(f)$ and $\beta < \alpha$, then $f(\beta) < \alpha$. Let \mathcal{M} be the unbounded set in κ which contains the measurable cardinals in $C(f)$, and let \mathcal{M}^+ be the set $\mathcal{M} \cup \{\kappa\}$. Recall that we write μ_α instead of $f(\alpha)$.

Let us review the notion of the *sum of forcing notions* or the *lottery sum of forcing notions*, in the terminology of [14]. Let I be a non-empty set and $\{P_i \mid i \in I\}$ a set of forcing notions. Then the lottery

$$(2.2) \quad \bigoplus \{P_i \mid i \in I\}$$

is a forcing notion with the domain $\{(i, p) \mid i \in I \ \& \ p \in P_i\} \cup \{1\}$, where 1 is not in $\bigcup \{P_i \mid i \in I\}$, and the ordering defined as follows: (i) 1 is the greatest element and (ii) $(i, p) \leq (j, q)$ if and only if $i = j$ and $p \leq q$ in P_i . The intuition is that the generic filter first chooses a forcing notion, and then forces with it.

We will use the lottery twice: in Definition 2.2 and in Definition 2.9.

Definition 2.2 *Define in V an Easton product*

$$(2.3) \quad P_\kappa^1 = \prod_{\alpha \in \mathcal{M}} Q_\alpha^1$$

of length κ such that at every $\alpha \in \mathcal{M}$,

$$Q_\alpha^1 = \bigoplus \{R \mid R \text{ satisfies } (*)\},$$

where $()$ is the following condition*

- (*) *There is a normal measure W on α in V and R is equal to $i_W(\text{Add}(\alpha, \mu_\alpha))$, where i_W is the embedding generated by W .*

Let us further define

$$(2.4) \quad P^1 = P_\kappa^1 \times i(\text{Add}(\kappa, \mu)).$$

Let us write $i(\text{Add}(\kappa, \mu))$ as Q_κ^1 , and let us use μ_κ to denote μ in the next paragraph and in Lemma 2.3 to simplify notation.

For $\alpha \in \mathcal{M}^+$, let us write P_α^1 for the product P^1 below α , and $P_{>\alpha}^1$ for the product indexed above α so that $P^1 = P_\alpha^1 \times Q_\alpha^1 \times P_{>\alpha}^1$.⁵ For $\alpha \in \mathcal{M}^+$, let G_α^1 , $G_{>\alpha}^1$ and g_α^1 denote generic filters for P_α^1 , $P_{>\alpha}^1$, and Q_α^1 over V , respectively. For $\alpha \in \mathcal{M}$, let us write α^* for the next element in \mathcal{M} above α .

We state some facts concerning P^1 .

Lemma 2.3 *Assume GCH.*

- (i) For every $\alpha \in \mathcal{M}$, Q_α^1 has size less than α^* .
- (ii) For every $\alpha \in \mathcal{M}^+$, Q_α^1 preserves cofinalities over V .
- (iii) P^1 preserves cofinalities over V .
- (iv) For every $\alpha \in \mathcal{M}^+$, G_α^1 , g_α^1 , and $G_{>\alpha}^1$ are pairwise mutually generic.

PROOF. (i). There are at most α^{++} -many normal measures W at stage α and each R in the lottery has size less than α^* (by an analogue of Lemma 2.1, $|i_W(\mu_\alpha)| = \mu_\alpha$, and therefore $|R| = \mu_\alpha$).

(ii). Let R be a variable for the forcings in the lottery Q_α^1 for $\alpha \in \mathcal{M}$ and for the forcing Q_κ^1 for $\alpha = \kappa$. It suffices to show that R is α^+ -closed and α^{++} -cc in V , and therefore is cofinality-preserving. Closure is obvious by the fact that a measure ultrapower via a normal measure at α is closed under α -sequences in V . Regarding the chain condition, notice that every element of R can be identified with the equivalence class of some function $f : \alpha \rightarrow \text{Add}(\alpha, \mu_\alpha)$. For $f, f' : \alpha \rightarrow \text{Add}(\alpha, \mu_\alpha)$, set $f \leq f'$ if for all $i < \alpha$, $f(i) \leq f'(i)$; it suffices to check that the ordering \leq on these f 's is α^{++} -cc. Let A be a maximal antichain in this ordering; take an elementary substructure \bar{M} in some large enough $H(\theta)$ of V which contains all relevant data, has size α^+ and is closed under α -sequences. Then it is not hard to check that $A \cap \bar{M}$ is maximal in the ordering (and so $A \subseteq \bar{M}$), and therefore has size at most α^+ .

(iii). For every $\alpha \in \mathcal{M}$, P_α^1 is α -cc, Q_α^1 is α^+ -closed and has size less than α^* , and $P_{>\alpha}^1$ is $(\alpha^*)^+$ -closed. For $\alpha = \kappa$, the same facts are true (and the size of Q_κ^1 is just μ). By standard arguments, it suffices to show that each R (R is a variable as in (ii)) is cofinality preserving over $V[G_\alpha^1]$. Since P_α^1 is α -cc, R remains α^+ -distributive in $V[G_\alpha^1]$; R is α^{++} -cc in V , and since P_α^1 has size just α , it forces that R is still α^{++} -cc (if there were an antichain in $V[G_\alpha^1]$ of size α^{++} , a single condition in P_α^1 would determine a cofinal part of it, which would yield an antichain in V of size α^{++}).

(iv). This follows by the combination of the right chain condition and closure using the Easton's lemma (which implies that if P is α -cc and Q is α -closed, then the respective generic filters are mutually generic). \square

⁵For $\alpha = \kappa$, we consider $P_{>\kappa}^1$ as the trivial forcing.

We will show now that Q_κ^1 provides an important generic object (which we denote \tilde{g}) which will be later used for the lifting of the Cohen forcing at κ (see Theorem 2.10).

Notational remark. From now on, let us write Q instead of Q_κ^1 , and g instead of g_κ^1 for easier reading.

Let us fix a $P_\kappa^1 \times Q$ -generic filter $G_\kappa^1 \times g$.

Lemma 2.4 *Assume GCH. The following hold:*

- (i) j lifts to $j^1 : V[g] \rightarrow M[j^1(g)]$, where j^1 restricted to V is the original j .
- (ii) i lifts to $i^1 : V[g] \rightarrow N[i^1(g)]$, where i^1 restricted to V is the original i . $N[i^1(g)]$ is the measure ultrapower obtained from j^1 .
- (iii) k lifts to $k^1 : N[i^1(g)] \rightarrow M[j^1(g)]$, where k^1 restricted to N is the original k .
- (iv) g is Q -generic over $N[i^1(g)]$.
- (v) There is \tilde{g} in $V[g]$ such that \tilde{g} is $k(Q) = j(\text{Add}(\kappa, \mu))$ -generic over $M[j^1(g)]$.

PROOF. (i) and (ii). These follow the by κ^+ -closure of Q in V and the fact that j, i have width κ : the pointwise image of g generates a generic filter for $j(Q)$ and $i(Q)$, respectively.

(iii). $i(Q)$ is $i(\kappa^+)$ -closed in N , and since $i(\text{f})(\kappa) < i(\kappa) < i(\kappa^+)$, we use the closure of $i(Q)$ and the fact that k has width $i(\text{f})(\kappa)$ to argue that the pointwise image $k''(i^1(g))$ generates a generic filter which is equal to the generic filter generated by $j''g$ by commutativity of j, i, k .

(iv). Q is $i(\kappa^+)$ -cc in N and $i(Q)$ is $i(\kappa^+)$ -closed in N . Therefore g and $i^1(g)$ are mutually generic over N by Easton's lemma.

(v). Q is $i(\kappa)$ -closed in $N[i^1(g)]$ since the generic $i^1(g)$ does not add new sequences of length $i(\kappa)$; it follows as in (iii) that $k^{1''}g$ generates a $j(\text{Add}(\kappa, \mu))$ -generic filter \tilde{g} over $M[j^1(g)]$. \square

After using k to secure the generic object \tilde{g} , we will forget i and k , and only lift j^1 further to $V[g][G_\kappa^1]$ (which is the same as $V[G_\kappa^1][g]$ by mutual genericity). For convenience, let us write h^* for $j^1(g)$.

Lemma 2.5 *Assume j^1 is as in Lemma 2.4.*

- (i) j^1 lifts in $V[g][G_\kappa^1]$ to

$$(2.5) \quad j^2 : V[g][G_\kappa^1] \rightarrow M[h^*][G_\kappa^1][g][h],$$

for some h .

- (ii) There exists in $V[g][G_\kappa^1]$ an object \tilde{g}^* which is $j^2(\text{Add}(\kappa, \mu)^{V[g][G_\kappa^1]})$ -generic over the model $M[h^*][G_\kappa^1][g][h]$.

PROOF. In $V[G_\kappa][g]$, lift j^1 to

$$(2.6) \quad j^2 : V[g][G_\kappa^1] \rightarrow M[h^*][G_\kappa^1][g][h],$$

choosing in the lottery at stage κ of $j(P_\kappa^1)$ the forcing Q which is available here since $H(\mu)$ is included in M (see the paragraph below the proof of Lemma 2.1). The generic object h is

constructed using the extender representation of M and the fact that $j(P_\kappa^1)$ in the interval $(\kappa, j(\kappa))$ is κ^+ -closed in V , and more than μ^+ -closed in the sense of M . In more detail: every dense open set can be written as $j(f)(\alpha)$ for some $f : \kappa \rightarrow H(\kappa^+)$ and $\alpha < \mu$. For every such f , the collection of all dense open sets $j(f)(\alpha)$, $\alpha < \mu$, has size at most μ , and its intersection is therefore dense. Since there are only κ^+ -many such f , h can be constructed recursively in κ^+ -many steps.

By mutual genericity, h constructed in V as generic over M is actually generic over $M[h^*][G_\kappa^1][g]$. Let us denote $M[h^*]$ by M^1 , and $M[h^*][G_\kappa^1][g][h]$ by M^2 .

Note that $\text{Add}(\kappa, \mu) = \text{Add}(\kappa, \mu)^{V[g]}$, but $\text{Add}(\kappa, \mu)^{V[g][G_\kappa^1]}$ properly contains $\text{Add}(\kappa, \mu)$. We know from Lemma 2.4 that the object \tilde{g} is $j(\text{Add}(\kappa, \mu))$ -generic over M^1 . We wish to find an object which is generic for $j^2(\text{Add}(\kappa, \mu)^{V[g][G_\kappa^1]})$ over M^2 . We use the following general Fact to achieve this aim:

Fact 2.6 *Let S be a κ -cc forcing notion of cardinality κ , $\kappa^{<\kappa} = \kappa$. Then for any γ , the term forcing $Q_S = \text{Add}(\kappa, \mu)^{V[S]}/S$ is isomorphic to $\text{Add}(\kappa, \gamma)$.*

PROOF. Recall that Q_S is the term forcing defined as follows: the elements of Q_S are names τ such that τ is an S -name and it is forced by 1_S to be in $\text{Add}(\kappa, \mu)$ of $V[S]$. The ordering is $\tau \leq \sigma \leftrightarrow 1_S \Vdash \tau \leq \sigma$. For the proof, see Fact 2 in [3]. \square

The import of Fact 2.6 is that any generic filter for $\text{Add}(\kappa, \gamma)$ adds a generic filter for $\text{Add}(\kappa, \mu)$ as defined in $V[S]$, provided that S is κ -cc and has size κ .

We wish to apply this fact to $j(\text{Add}(\kappa, \mu)) = \text{Add}(j(\kappa), j(\mu))^{M^1}$ and $j(P_\kappa^1)$ in the model M^1 . $j(\kappa)^{<j(\kappa)}$ holds in M^1 , $j(P_\kappa^1)$ has size $j(\kappa)$, and is $j(\kappa)$ -cc. It follows by Fact 2.6 that \tilde{g} generates an object \tilde{g}^* which is $\text{Add}(j(\kappa), j(\mu))^{M^2} = j^2(\text{Add}(\kappa, \mu)^{V[g][G_\kappa^1]})$ -generic over M^2 .

Since G_κ^1 and g are in M^2 , j^2 witnesses that κ is still $H(\mu)$ -hypermeasurable in $V[g][G_\kappa^1]$. \square

We can summarise the construction above as follows (removing the superscripts for clarity):

Corollary 2.7 *Assume V' is a ground model where GCH holds and κ is $H(\mu)$ -hypermeasurable for some $\kappa < \text{cf}(\mu) \leq \mu$, and $\kappa^{++} \leq \mu$. Then we can assume that there is a cofinality-preserving generic extension V of V' and an embedding $j : V \rightarrow M$ with critical point κ such that the following hold:*

- (i) *GCH holds on an unbounded set of measurable cardinals below κ , $2^\kappa = \kappa^+$, and $2^{\kappa^+} = \mu$.*
- (ii) *j is a (κ, μ) -extender embedding, i.e.*

$$M = \{j(f)(\alpha) \mid f : \kappa \rightarrow V \ \& \ \alpha < \mu\},$$

M is closed under κ -sequences, $\mu < j(\kappa) < \mu^+$, and $H(\mu)$ is included in M .

- (iii) *There exists in V a function $\mathfrak{f} : \kappa \rightarrow \kappa$ such that $\mu = j(\mathfrak{f})(\kappa)$ (we denote $\mathfrak{f}(\alpha)$ by μ_α).*
- (iv) *There exists in V an object \tilde{g}^* which is $j(\text{Add}(\kappa, \mu))$ -generic over M .*

Remark 2.8 Notice that unlike in [3], Corollary 2.7 does not mention the lifting of k . In fact k may not lift in this context because the forcing $k(P^1)$ cannot choose the right forcing in the lottery at stage κ : the normal measure U generating k is certainly not available in the measure ultrapower via U . The dropping of k from the statement of Corollary 2.7 simplifies the argument in Theorem 2.10 since we deal directly with \tilde{g}^* (unlike [3] which, roughly speaking, uses g of Lemma 2.4 instead).

2.3 The final model V^*

We start with the model V from Corollary 2.7 and define a reverse Easton iteration of length κ which will provide the desired model V^* in which the $H(\mu)$ -hypermeasurability of κ is indestructible by $\text{Add}(\kappa, \gamma)$ for any $0 \leq \gamma \leq \mu$, where we construe $\text{Add}(\kappa, 0)$ as the trivial forcing.

Let \mathcal{I} denote the set of inaccessible cardinals below κ which are in $C(f)$ (the closure points of f).

Definition 2.9 *Define*

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \mid \alpha \in \mathcal{I} \rangle$$

as a reverse Easton iteration such that at each $\alpha \in \mathcal{I}$,

$$\dot{Q}_\alpha = \bigoplus \{ \text{Add}(\alpha, \beta)^{V^{[P_\alpha]}} \mid 0 \leq \beta \leq \mu_\alpha \}.$$

Let G_κ be P_κ -generic over V , and denote $V^* = V[G_\kappa]$.

Theorem 2.10 *V^* is the desired model where the hypermeasurability of κ is indestructible under the Cohen forcing at κ of any length up to μ : For any $0 \leq \gamma \leq \mu$, κ remains $H(\mu)$ -hypermeasurable in the generic extension of V^* by $\text{Add}(\kappa, \gamma)^{V^*}$.*

PROOF. Let us fix γ , and let g_γ be $\text{Add}(\kappa, \gamma)^{V^*}$ -generic over V^* . We will show that j lifts to $V[G_\kappa][g_\gamma]$.

We follow closely the argument in Cummings [3] but with the important simplification that we do not use a measure ultrapower embedding obtained from j and a generic filter over the measure ultrapower, but use the generic object \tilde{g}^* directly.

Using standard methods, lift j to

$$j^* : V[G_\kappa] \rightarrow M[G_\kappa][g_\gamma][H]$$

as follows:

- At stage κ of $j(P_\kappa)$ work below a condition which chooses in the lottery the forcing $\text{Add}(\kappa, \gamma)$.

- The generic object H is constructed using the extender representation of M : the dense open sets in the forcing $j(P_\kappa)$ in the interval $(\kappa, j(\kappa))$ can be grouped into κ^+ -many groups each of size μ in M ; these groups are of the form $\{j(f)(\alpha) \mid \alpha < \mu\}$, where f is a function from κ to $H(\kappa^+)$. The intersection of each group is a dense set because the forcing $j(P_\kappa)$ in the interval $(\kappa, j(\kappa))$ is μ^+ -closed in M . Since there are only κ^+ -many of these groups, a generic H can be constructed in $V[G_\kappa][g_\gamma]$ which meets them all.

It remains to find a generic filter for the j^* -image of g_γ . We use at this crucial step the generic object \tilde{g}^* which we prepared in V . Let us denote by \tilde{g}_γ^* the truncation of \tilde{g}^* to $j(\gamma)$. Notice that \tilde{g}_γ^* is generic for the wrong forcing: it is $j(\text{Add}(\kappa, \gamma)^V)$ -generic over M , but we need a generic object for $j^*(\text{Add}(\kappa, \gamma)^{V^*})$ over $M[G_\kappa][g_\gamma][H]$. Fact 2.6 comes to the rescue again: By elementarity and the fact that $j(P_\kappa)$ has size $j(\kappa)$ and is $j(\kappa)$ -cc in M , and $j(\kappa)^{<j(\kappa)} = j(\kappa)$, Fact 2.6 implies that in $V[G_\kappa][g_\gamma]$, \tilde{g}_γ^* yields a generic object g_γ^* over $M[G_\kappa][g_\gamma][H]$ for $j^*(\text{Add}(\kappa, \gamma)^{V^*})$.

The object g_γ^* is still not good enough to lift j^* to g_γ because it may not contain the pointwise image $j^{**}g_\gamma$. Using the method of surgery (see [3]), we modify g_γ^* to g_γ^{**} which is still $j^*(\text{Add}(\kappa, \gamma)^{V^*})$ -generic, but in addition contains the pointwise image $j^{**}g_\gamma$. It follows we can lift to

$$j^{**} : V[G_\kappa][g_\gamma] \rightarrow M[G_\kappa][g_\gamma][H][g_\gamma^{**}].$$

The last lifting shows that κ remains measurable, in fact $H(\mu)$ -hypermeasurable, in the model $V[G_\kappa][g_\gamma]$ as desired. \square

Remark 2.11 We have proved Theorem 2.10 under the assumption $\mu \geq \kappa^{++}$. Strictly speaking, our argument does not apply to the case $\mu = \kappa^+$ since the measure U is no longer present in M (since M can be identified with a normal measure ultrapower in this case), and the lifting argument for P^1 immediately below (2.6) will not work. However, a much easier argument is available. Suppose $j : V \rightarrow M$ is a normal measure embedding generated by a normal measure at κ ; it suffices to lift the embedding (the inclusion of $H(\kappa^+)$ in the target model is then automatic). There is no need for the preparation forcing P^1 since it is possible to construct the generic object for $j(\text{Add}(\kappa, \kappa^+))$ – which is used in Theorem 2.10 – directly in V : There are only $j(\kappa^+)$ -many maximal antichains of $j(\text{Add}(\kappa, \kappa^+))$ in M , and the size of $j(\kappa^+)$ is just κ^+ in V ; it follows the generic object can be directly constructed in V by a recursion of length κ^+ .

3 Open questions

In the presentation in [3], the use of the lottery in Definition 2.2 is avoided as one can show that $i(\text{Add}(\kappa, \kappa^{++}))$ (in the notation in Definition 2.2) is in V isomorphic to $\text{Add}(\kappa^+, \kappa^{++})$, and therefore in [3] the preparation is carried out with $\text{Add}(\alpha^+, \alpha^{++})$ at inaccessible $\alpha < \kappa$. Gitik and Merimovich [11] generalised this result to $i(\text{Add}(\kappa, \kappa^{+n}))$ for any $1 \leq n < \omega$, showing that $i(\text{Add}(\kappa, \kappa^{+n}))$ is isomorphic to $\text{Add}(\kappa^+, \kappa^{+n})$. It is likely that this may be extended to $i(\text{Add}(\kappa, \mu))$ for any μ which is below the first singular cardinal of cofinality κ above κ . But the general case is not clear:

Q1 Suppose $i : V \rightarrow N$ is an elementary embedding with critical point κ generated by a normal measure at κ , $2^\kappa = \kappa^+$ and $\mu > \kappa$ is a cardinal. Is $i(\text{Add}(\kappa, \mu))$ isomorphic in V to $\text{Add}(\kappa^+, i(\mu))$?

The positive answer would eliminate the use of lottery in Definition 2.2.

It is natural to ask whether the indestructibility can be extended to include more forcing notions. We mention just a couple of examples before we state the general question **Q2**:

Example 1. With a slight modification of the preparation in Definition 2.9, one can ensure that the hypermeasurability of κ is indestructible under Mitchell forcing $\mathbb{M}(\kappa, \lambda)$ at κ , where λ is the least weakly compact above κ in the original ground model before the preparation P^1 (notice that P^1 with $i(\text{Add}(\kappa, \lambda))$ destroys the strong limitness of λ as it adds λ many new subsets of κ^+); see [1] or [7] for definitions and details regarding the Mitchell forcing. Note that the indestructibility is straightforward if the Mitchell forcing is used only at κ (because it decomposes into $\text{Add}(\kappa, \lambda)$ followed by a κ^+ -distributive forcing notion); however, in many applications (such as [7] or [13]) it necessary to prepare in the forcing P_κ in Definition 2.9 with the Mitchell forcing below κ . As it turns out, the detailed argument in this case hinges on the omission of k in the argument in Theorem 2.10 (see Remark 2.8) – the point being that k moves non-trivially the collapsing part of the Mitchell forcing at κ .

Example 2. If we are willing to use more complicated forcings than the Cohen forcing, one can achieve indestructibility of the hypermeasurability of κ by much simpler means: using the methods introduced in [9], and developed in [6] and [8], one can show that just preparing with a version of the forcing P_κ in Definition 2.9 (entirely omitting P^1 from Definition 2.2), one can for instance obtain V^* where the $H(\mu)$ -hypermeasurability is indestructible by the generalised Sacks forcing at κ of any length less than μ (both product and iteration). Notice that by omitting P^1 , GCH still holds above κ after forcing with P_κ .⁶

Example 3. Recall that κ is called μ -tall for a cardinal $\mu > \kappa$ if there exists an elementary embedding $j : V \rightarrow M$ such that $j(\kappa) > \mu$ and ${}^\kappa M \subseteq M$. The indestructibility of μ -tallness of κ by the Cohen forcing at κ is easier to obtain than the indestructibility of hypermeasurability: the definition of P^1 in Definition 2.2 can be simplified to include just the forcing Q_κ^1 (the whole forcing P_κ^1 can be omitted – its only purpose is to preserve the hypermeasurability). However, the following version of the question **Q2** is open for tall and hypermeasurable cardinals alike. Suppose κ is a strong cardinal (i.e. κ is $H(\mu)$ -hypermeasurable for every $\mu > \kappa$); is it possible to prepare V^* so that the strongness of κ is indestructible by the forcing $\text{Add}(\kappa, \mu)$ for *any* μ ? Note that the current method cannot be used as this would require to add proper-class many subsets of κ^+ in the preparation P^1 . Also note that this contrasts with the Sacks forcing mentioned in Example 2: it is relatively easy to check that it is possible to prepare V^* so that the Sacks forcing at κ of any length preserves the hypermeasurability of κ .

The general form of the question is as follows:

⁶It is known that the iteration of the Sacks forcing at κ also forces the tree property (if the iteration goes up a weakly compact cardinal above κ). However, such an iteration always forces $2^\kappa = \kappa^{++}$. In [7] we worked with any finite gap at $\kappa = \aleph_\omega$, and therefore we needed to use the Mitchell forcing instead (at the price of a more complicated preparation).

Q2 Is there a comprehensive and well-defined class of forcing notions which add new subsets of κ for which the hypermeasurability of κ can be made indestructible?

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