The tree property and the continuum function below $\aleph_\omega$

Radek Honzik, Šárka Stejskalová

Charles University in Prague, Department of Logic,
Celetná 20, Praha 1, 116 42, Czech Republic
radek.honzik@ff.cuni.cz
sarka.stejskalova@ff.cuni.cz

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Abstract: We say that a regular cardinal κ, $\kappa > \aleph_0$, has the tree property if there are no κ-Aronszajn trees; we say that κ has the weak tree property if there are no special κ-Aronszajn trees. Starting with infinitely many weakly compact cardinals, we show that the tree property at every even cardinal $\aleph_{2n}$, $0 < n < \omega$, is consistent with an arbitrary continuum function below $\aleph_\omega$ which satisfies $2^{\aleph_{2n}} > \aleph_{2n+1}$, $n < \omega$. Next, starting with infinitely many Mahlo cardinals, we show that the weak tree property at every cardinal $\aleph_n$, $1 < n < \omega$, is consistent with an arbitrary continuum function which satisfies $2^{\aleph_n} > \aleph_{n+1}$, $n < \omega$. Thus the tree property has no provable effect on the continuum function below $\aleph_\omega$ except for the trivial requirement that the tree property at κ^{++} implies $2^\kappa > \kappa^+$ for every infinite κ.

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1 Introduction

It is known that the usual large cardinals do not have any effect on the continuum function on small cardinals: in particular, the values of $2^{\aleph_n}$, $n < \omega$, do not depend in any way on the existence or non-existence of a large cardinal κ.[1] If we remove the assumption of inaccessibility from

[1]Except trivially, in the sense that $2^{\aleph_n}$ must be smaller than κ.

the definition of some large cardinals κ, κ may still retain some trace of "largeness", and yet be smaller than $\aleph_\omega$ and have influence on the continuum function below $\aleph_\omega$.

In this paper, we are interested in the *tree property*: we say that a regular $\kappa > \aleph_0$ has the tree property if there are no κ-Aronszajn trees. Thus to have the tree property is the same as to be weakly compact without the requirement for the inaccessibility of κ. If there are no special κ-Aronszajn trees, we refer to this property as the *weak tree property*. By a result of Jensen, the weak tree property at κ^+ is equivalent to the failure of the weak square at κ.

If $2^\kappa = \kappa^+$, $\kappa \geq \aleph_0$, then by Specker's result there are special κ^{++}-Aronszajn trees. Thus the tree property at $\aleph_2$ implies the failure of CH. It seems natural to ask whether the tree property at κ^{++} puts more restrictions the continuum function in addition to requiring $2^\kappa > \kappa^+$. We answer this question negatively for the continuum function below $\aleph_\omega$.[2]

The structure of the paper is as follows. First, in Theorem 2.5, we deal for simplicity with a single cardinal and show that the tree property at $\aleph_2$ is compatible with $2^{\aleph_0} = \aleph_3$ and $2^{\aleph_1} = \aleph_4$ (we use "gap three" for concreteness, there is nothing particular about it).[3] Theorem 2.5 is generalized in Theorem 3.1 where we show (starting with infinitely many weakly compact cardinals) that it is consistent that the tree property holds at every even cardinal larger than $\aleph_0$ below $\aleph_\omega$, $\aleph_\omega$ is strong limit, and the continuum function can be anything we want, providing $2^{\aleph_{2n}} \geq \aleph_{2n+2}$, $n < \omega$. In Theorem 3.4, we formulate an analogous result for the weak tree property: starting with infinitely many Mahlo cardinals, it is consistent that the weak tree property holds at every $\aleph_n$, $n > 1$, and the continuum function can be anything we want, providing $2^{\aleph_n} \geq \aleph_{n+2}$ for $n < \omega$.

Note that we use only modest large cardinal assumptions, i.e. weakly

[2]The study of the behaviour of the tree property below and close to $\aleph_\omega$ seems to be the standard test case for many of the results concerning the tree property (see for instance [2] or [6]).

[3]This result for $2^{\aleph_0}$ already follows from the "indestructibility" results presented in [7].

compact cardinals and Mahlo cardinals, and therefore we cannot get two successive cardinals with the tree property (while we can get two successive cardinals with the weak tree property). A generalization of this paper to successive cardinals requires larger cardinals; see Section 4 for more open questions in this direction.

1.1 Basic facts and notation

In general, if P is a forcing notion, we write $V[P]$ to denote a generic extension by P whenever the exact generic filter is not relevant. Let P, Q be forcing notions; if there is a projection from P onto Q, we denote by P/Q the quotient such that P is equivalent to $Q * P/Q$.

For a regular cardinal κ, we denote by $\mathrm{Add}(\kappa, \alpha)$ the Cohen forcing which adds α-many subsets of κ (we identify $p \in \mathrm{Add}(\kappa, \alpha)$ with a function of size $< \kappa$ from $\kappa \times \alpha$ to 2). If $A \subseteq \alpha$, we write $\mathrm{Add}(\kappa, A)$ for the forcing whose conditions only use coordinates in A. If $\beta < \alpha$, we write $\mathrm{Add}(\kappa, \alpha - \beta)$ for $\mathrm{Add}(\kappa, [\beta, \alpha))$.

Let κ be a regular cardinal, and $\lambda > \kappa$ an inaccessible cardinal.

Definition 1.1 *We denote by $\mathbb{M}(\kappa, \lambda)$ the forcing which is defined as follows: (p, q) is in $\mathbb{M}(\kappa, \lambda)$ if p is a condition in $\mathrm{Add}(\kappa, \lambda)$ and q is a function of size at most κ with $\mathrm{dom}(q) \subseteq \lambda$ and for all $\alpha \in \mathrm{dom}(q)$, $q(\alpha)$ is an $\mathrm{Add}(\kappa, \alpha)$-name for a condition in $\mathrm{Add}(\kappa^+, 1)^{V[\mathrm{Add}(\kappa,\alpha)]}$. The ordering is defined as follows: $(p_1, q_1) \leq (p_2, q_2)$ if $p_1 \leq p_2$ and for all $\alpha \in \mathrm{dom}(q_2)$, $q_1|\alpha \Vdash_{\mathrm{Add}(\kappa,\alpha)} q_1(\alpha) \leq q_2(\alpha)$.*

By an analysis of Abraham [1], there is a κ^+-closed term forcing which we denote ${}^1\mathbb{M}(\kappa, \lambda)$ such that there is a projection from $\mathrm{Add}(\kappa, \lambda) \times {}^1\mathbb{M}(\kappa, \lambda)$ onto $\mathbb{M}(\kappa, \lambda)$. This projection property carries over to quotients: for $\alpha < \lambda$, the quotient $\mathbb{M}(\kappa, \lambda)/\mathbb{M}(\kappa, \alpha)$ is a projection in $V[\mathbb{M}(\kappa, \alpha)]$ of $\mathrm{Add}(\kappa, \lambda - \alpha) \times {}^1\mathbb{M}(\kappa, \lambda - \alpha)$ for a certain term forcing ${}^1\mathbb{M}(\kappa, \lambda - \alpha)$ which is κ^+-closed in $V[\mathbb{M}(\kappa, \alpha)]$.

If κ is regular, we write $\mathrm{TP}(\kappa)$ to denote that the *tree property* holds at κ, i.e. every κ-tree has a cofinal branch (that is, there are no κ-Aronszajn trees). We write $\mathrm{wTP}(\kappa)$ to denote that the *weak tree property* holds at κ, i.e. there are no special κ-Aronszajn trees. Note that by a result of Jensen $\mathrm{wTP}(\kappa^+)$ is equivalent to $\neg\Box^*_\kappa$.

Fact 1.2 (Mitchell) *If GCH holds and $\kappa < \lambda$ are regular, then:*

(i) If λ is weakly compact, then in $V[\mathbb{M}(\kappa,\lambda)]$, $\mathrm{TP}(\lambda)$ holds.
(ii) If λ is Mahlo, then in $V[\mathbb{M}(\kappa,\lambda)]$, $\mathrm{wTP}(\lambda)$ holds.

In either case $2^\kappa = \lambda = \kappa^{++}$ in $V[\mathbb{M}(\kappa,\lambda)]$.

We shall use following facts for arguments that certain forcings do not add branches to λ^{++}-trees.[4]

Fact 1.3 *If P is a λ^+-closed forcing, $2^\lambda > \lambda^+$, and T is a λ^{++}-tree, then in $V[P]$ there are no new branches in T.*

Fact 1.4 *If P is λ^{++}-Knaster, and T is a tree of height λ^{++}, then in $V[P]$, there are no new branches in T.*

These facts can be generalized as follows (see [7]):

Fact 1.5 *If $P \times P = P^2$ is λ^{++}-cc, and T is a tree of height λ^{++}, then in $V[P]$, there are no new branches in T.*

Fact 1.6 *Suppose P is λ^+-cc and preserves λ, Q is λ^+-closed, and $2^\lambda > \lambda^+$. If T is a λ^{++}-tree in $V[P]$, then in $V[P][Q]$, T has no new branches.*

The following lemma is easy, but will be useful:

Lemma 1.7 *Assume $\kappa \geq \aleph_0$ is regular and $\lambda > \kappa$ Mahlo. Assume P is κ^+-cc and Q is κ^+-closed. Then in $V[P \times \mathbb{M}(\kappa,\lambda)]$, Q is still κ^+-distributive.*

[4] We use λ^{++} because this is the relevant context for the paper; with appropriate modifications, the facts hold for any regular λ as well.

Proof. Let $\mathrm{Add}(\kappa,\lambda)\times {}^1\mathbb{M}(\kappa,\lambda)$ be the product forcing which projects onto $\mathbb{M}(\kappa,\lambda)$, where ${}^1\mathbb{M}(\kappa,\lambda)$ is a κ^+-closed term forcing. The product ${}^1\mathbb{M}(\kappa,\lambda)\times Q$ is κ^+-distributive over $V[P\times\mathrm{Add}(\kappa,\lambda)]$ by Easton lemma, and thus

$$\text{(1.1)}\quad \text{all } \kappa\text{-sequences of ordinals in } V[P\times\mathrm{Add}(\kappa,\lambda)\times {}^1\mathbb{M}(\kappa,\lambda)\times Q] \text{ are already in } V[P\times\mathrm{Add}(\kappa,\lambda)].$$

There is a natural projection

$$\pi : P\times\mathrm{Add}(\kappa,\lambda)\times {}^1\mathbb{M}(\kappa,\lambda)\times Q \to P\times\mathbb{M}(\kappa,\lambda)\times Q. \tag{1.2}$$

If there were a condition r in $P\times\mathbb{M}(\kappa,\lambda)\times Q$ forcing a counterexample to the κ^+-distributivity of Q over $V[P\times\mathbb{M}(\kappa,\lambda)]$, one could pick a generic filter F for $P\times\mathrm{Add}(\kappa,\lambda)\times {}^1\mathbb{M}(\kappa,\lambda)\times Q$ such that for some $r'\in F$, $\pi(r')\le r$. In $V[F]$, the κ-sequence of ordinals forced by r to violate the κ^+-distributivity would contradict (1.1). □

2 Large $2^{\aleph_0}$ and $2^{\aleph_1}$ with $\mathrm{TP}(\aleph_2)$

In this section we provide a proof of a special case of Theorem 3.1. It illustrates the main idea behind the construction with more clarity than the proof of Theorem 3.1 which needs to deal with infinitely many cardinals.

Note that in this section we use a measurable cardinal for ease of exposition; a modification with a weakly compact cardinal is straightforward.

We assume that the reader is familiar with the usual argument which shows that $\mathbb{M}(\kappa,\lambda)$ forces the tree property at λ, whenever $\kappa<\lambda$ and λ is weakly compact.

For concreteness of the construction in this section we will force "gap three" on $\aleph_0$ and $\aleph_1$, i.e. get $2^{\aleph_0}=\aleph_3$ and $2^{\aleph_1}=\aleph_4$ with the tree property at $\aleph_2$. Other values of the continuum functions are easily obtainable; see Theorem 3.1.

Let κ be a measurable cardinal. Denote

$$\mathbb{P}=\mathbb{M}(\aleph_0,\kappa)\times\mathrm{Add}(\aleph_0,\kappa^+)\times\mathrm{Add}(\aleph_1,\kappa^{++}). \tag{2.3}$$

Remark 2.1 Note that $\mathbb{M}(\aleph_0, \kappa)$ forces $2^{\aleph_0} = \aleph_2$, and therefore to increase the value of $2^{\aleph_1}$, we need to use some kind of product because the forcing $\mathrm{Add}(\aleph_1, 1)$ defined in $V[\mathbb{M}(\aleph_0, \kappa)]$ collapses $2^{\aleph_0}$ to $\aleph_1$.

Lemma 2.2 *(GCH). In $V[\mathbb{P}]$, $\kappa = \aleph_2$, $2^{\aleph_0} = \aleph_3$, $2^{\aleph_1} = \aleph_4$.*

PROOF. Obvious. $\square$

Lemma 2.3 *If $\mathbb{P}$ forces that $\dot{S}$ is a κ-Aronszajn tree, then there are $A \subseteq \kappa^+$ and $B \subseteq \kappa^{++}$, both size κ, and some name $\dot{T}$, such that $\mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_0, A) \times \mathrm{Add}(\aleph_1, B)$ forces that $\dot{T}$ is a κ-Aronszajn tree.*

PROOF. $\mathbb{P}$ is κ-cc, and therefore we can assume that $\dot{S}$ is a nice name for a subset of κ which contains at most κ-many conditions in $\mathbb{P}$; the supports of these conditions in $\mathrm{Add}(\aleph_0, \kappa^+)$ and $\mathrm{Add}(\aleph_1, \kappa^{++})$ determine the sets A, B. $\square$

Corollary 2.4 *If $\mathbb{P}$ adds a κ-Aronszajn tree, so does*

$$\mathbb{P}|\kappa =_{df} \mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_1, \kappa).$$

PROOF. Any bijection between A, B and κ determines an isomorphism between $\mathrm{Add}(\aleph_0, A)$ and $\mathrm{Add}(\aleph_0, \kappa)$, and similarly for B. $\square$

Theorem 2.5 *(GCH). Assume κ is measurable and $\mathbb{P}$ is as in (2.3). Then in $V[\mathbb{P}]$, $2^{\aleph_0} = \aleph_3$, $2^{\aleph_1} = \aleph_4$, and TP($\aleph_2$).*

PROOF. By Corollary 2.4, it suffices to show that $\mathbb{P}|\kappa$ cannot add a κ-Aronszajn tree. Suppose for contradiction there is a condition (r_1, r_2) in $\mathbb{P}|\kappa = \mathbb{M}(\aleph_0, \kappa) \times (\mathrm{Add}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_1, \kappa))$ which forces there is a κ-Aronszajn tree.

Let $j : V \to M$ be a measure ultrapower embedding with critical point κ.

Let $G*(H_1\times H_2)$ denote a generic filter over V for $\mathbb{M}(\aleph_0,\kappa)*(\mathrm{Add}(\aleph_0,j(\kappa)-\kappa)\times{}^1\mathbb{M}(\aleph_0,j(\kappa)-\kappa))$, where the product $\mathrm{Add}(\aleph_0,j(\kappa)-\kappa)\times{}^1\mathbb{M}(\aleph_0,j(\kappa)-\kappa)$ projects to $j(\mathbb{M}(\aleph_0,\kappa))/\mathbb{M}(\aleph_0,\kappa)$. Denote $G*H$ the $j(\mathbb{M}(\aleph_0,\kappa))$-generic obtained from $G*(H_2\times H_1)$, so that $j"G\subseteq G*H$. Assume further that $r_1\in G$.

Now we can lift in $V[G*(H_1\times H_2)]$ to

$$j:V[G]\to M[G*H].$$

Let $x^*\times y^*$, with $x^*=x_0\times x_1$ and $y^*=y_0\times y_1$, be $V[G*(H_1\times H_2)]$-generic for $\mathrm{Add}(\aleph_1,j(\kappa))^V\times\mathrm{Add}(\aleph_0,j(\kappa))$, with $x_0\times y_0$ being $\mathrm{Add}(\aleph_1,\kappa)\times\mathrm{Add}(\aleph_0,\kappa)$-generic over $V[G*(H_1\times H_2)]$ so that

$$j"(x_0\times y_0)\subseteq x^*\times y^*.$$

Assume further that $r_2\in x_0\times y_0$.

Remark 2.6 It is worth noting that $\mathrm{Add}(\aleph_1,j(\kappa))^V\times\mathrm{Add}(\aleph_0,j(\kappa))$ lives in $V[G]$ (actually already in V), so $x^*\times y^*\times H_1\times H_2$ is generic filter over $V[G]$ for the product forcing $\mathrm{Add}(\aleph_1,j(\kappa))^V\times\mathrm{Add}(\aleph_0,j(\kappa))\times\mathrm{Add}(\aleph_0,j(\kappa)-\kappa)\times{}^1\mathbb{M}(\aleph_0,j(\kappa)-\kappa)$, and therefore x^*, y^*, H_1, and H_2 are mutually generic over $V[G]$.

Now we can lift in $V[G*(H_1\times H_2)][x^*\times y^*]$ to

$$j:V[G][x_0\times y_0]\to M[G][H][x^*\times y^*].$$

Since (r_1,r_2) is in $G*(x_0\times y_0)$, there should be a κ-Aronszajn tree T in $V[G][x_0\times y_0]$, and hence also in $M[G][x_0\times y_0]$. Also, because $j(T)|\kappa=T$, we know that T has a cofinal branch in $M[G][H][x^*\times y^*]$.

By Remark 2.6, the relevant filters are mutually generic over $V[G]$, and also over $M[G]$, hence the following is a model of ZFC:

$$M[G][x_0][y_0][H_1][H_2][x_1][y_1]. \tag{2.4}$$

We finish the proof by showing that the generic filter $H_1\times H_2\times x_1\times y_1$ cannot add a branch to T, and therefore a branch existing in the model (2.4) must already exist in $M[G][x_0\times y_0]$, which contradicts our initial assumption that T is a κ-Aronszajn tree in $M[G][x_0\times y_0]$.

Claim 2.7 *The square of* $\mathrm{Add}(\aleph_1, j(\kappa) - \kappa)^V$ *is* κ*-cc in* $M[G][x_0][y_0]$.

PROOF. Obvious because $\mathrm{Add}(\aleph_1, j(\kappa)-\kappa)^V$ is κ-Knaster and $\mathbb{M}(\aleph_0, \kappa) \times \mathrm{Add}(\aleph_1, \kappa)^V \times \mathrm{Add}(\aleph_0, \kappa)$ is κ-Knaster. $\square$

By Claim 2.7, there are no new branches in $M[G][x_0][y_0][x_1]$ in comparison to $M[G][x_0][y_0]$. By Lemma 1.7, $M[G]$ and $M[G][x_0][x_1]$ have the same countable sequences of ordinals, and therefore ${}^1\mathbb{M}(\aleph_0, j(\kappa) - \kappa)$ – which adds H_2 – is still σ-closed in $M[G][x_0][x_1]$, and by Fact 1.6 there are no new branches in $M[G][x_0][y_0][x_1][H_2]$ in comparison to $M[G][x_0][x_1][y_0] = M[G][x_0][y_0][x_1]$. Finally, since $H_1 \times y_1$ is added by a $\aleph_1$-Knaster forcing, and since κ has cofinality $\aleph_1$ in $M[G][x_0][y_0][x_1][H_2]$ (and therefore T has cofinality $\aleph_1$), there are no new branches in the final model $M[G][x_0][y_0][x_1][H_2][H_1][y_1]$. $\square$

3 Main theorems

In this section, we prove a more general version of Theorem 2.5, both for the tree property (Theorem 3.1), and the weak tree property (Theorem 3.4).

3.1 The tree property

Let $\kappa_1 < \kappa_2 < \cdots$ be an ω-sequence of weakly compact cardinals with limit λ. Let κ_0 denote $\aleph_0$. In Theorem 3.1, we control the continuum function below $\aleph_\omega = \lambda$, λ strong limit, while having the tree property at all even aleph's.

Let A denote the set $\{\kappa_i \,|\, i < \omega\} \cup \{\kappa_i^+ \,|\, i < \omega\}$, and let $f : A \to A$ be a function which satisfies for all α, β in A:

(i) $\alpha < \beta \to f(\alpha) \leq f(\beta)$.
(ii) If $\alpha = \kappa_i$, then $f(\alpha) \geq \kappa_{i+1}$.

We say that f is an Easton function on A which respects the κ_i's (condition (ii)).

Theorem 3.1 *Assume GCH and let $\langle \kappa_i \,|\, i < \omega \rangle$, λ, and A be as above. Let f be an Easton function on A which respects the κ_i's. Then there is a forcing notion $\mathbb{S}$ such that if G is a $\mathbb{S}$-generic filter, then in $V[G]$:*

(i) Cardinals in A are preserved, and all other cardinals below λ are collapsed; in particular, for all $n < \omega$, $\kappa_n = \aleph_{2n}$, and $\kappa_n^+ = \aleph_{2n+1}$,
(ii) For all $0 < n < \omega$, the tree property holds at $\aleph_{2n}$,
(iii) The continuum function on $A = \{\aleph_n \,|\, n < \omega\}$ is controlled by f.

Proof. Let $\mathbb{P}$ be a reverse Easton iteration of the Cohen forcing $\mathrm{Add}(\alpha, 1)$ for every inaccessible $\alpha < \lambda$. Let $\mathbb{M}(\kappa_n, \kappa_{n+1})$ denote the Mitchell forcing which makes $2^{\kappa_n} = \kappa_{n+1}$ and forces the tree property at κ_{n+1}. Set $\mathbb{Q}$ to be the full support product

$$\mathbb{Q} = \prod_{n<\omega} \mathbb{M}(\kappa_n, \kappa_{n+1}).$$

Finally, let $\mathbb{R}$ be the standard Easton product to force the prescribed behaviour of the continuum function below $\aleph_\omega$ (taking into account that the cardinals below $\aleph_\omega$ will be equal to cardinals in A):

$$\mathbb{R} = \prod_{n<\omega} \big(\mathrm{Add}(\kappa_n, f(\kappa_n)) \times \mathrm{Add}(\kappa_n^+, f(\kappa_n^+)\big).$$

For simplicity of notation, let us write $\mathbb{R}^0(n) = \mathrm{Add}(\kappa_n, f(\kappa_n))$, $\mathbb{R}^1(n) = \mathrm{Add}(\kappa_n^+, f(\kappa_n^+))$, and $\mathbb{R}(n) = \mathbb{R}^0(n) \times \mathbb{R}^1(n)$. Thus $\mathbb{R} = \prod_{n<\omega} \mathbb{R}(n)$.

We define the forcing $\mathbb{S}$ as follows:

$$\mathbb{S} = \mathbb{P} * (\mathbb{Q} \times \mathbb{R}). \tag{3.5}$$

We leave it as an exercise for the reader to verify that $\mathbb{S}$ preserves all cardinals in A and forces the prescribed continuum function (the argument is routine). We will check that the tree property holds at every $\aleph_{2n}$, $0 < n < \omega$.

Let us denote for $0 < n < \omega$:

$$\mathbb{T}(n) = \mathbb{R}^0(n+1) \times \prod_{m \le n+1} \mathbb{M}(\kappa_m, \kappa_{m+1}) \times \prod_{m \le n} \mathbb{R}(m), \tag{3.6}$$

and

$$\mathbb{T}(n)_{\mathrm{tail}} = \mathbb{R}^1(n+1) \times \prod_{m>n+1} \mathbb{M}(\kappa_m, \kappa_{m+1}) \times \prod_{m>n} \mathbb{R}(m), \tag{3.7}$$

so that $\mathbb{Q} \times \mathbb{R} = \mathbb{T}(n) \times \mathbb{T}(n)_{\mathrm{tail}}$.

Suppose $\mathbb{S}$ adds a κ_{n+1}-tree T. Then T is added by

$$\mathbb{P} * \mathbb{T}(n) \tag{3.8}$$

because $\mathbb{T}(n)_{\mathrm{tail}}$ is κ_{n+1}^+-closed in $V[\mathbb{P}]$, and using Lemma 1.7, viewing $\mathbb{T}(n)$ as a product of a κ_{n+1}^+-cc forcing and $\mathbb{M}(\kappa_{n+1}, \kappa_{n+2})$, it follows that $\mathbb{T}(n)_{\mathrm{tail}}$ is still κ_{n+1}^+-distributive over the forcing (3.8), and hence does not add any κ_{n+1}-trees.

The forcing $\mathbb{T}(n)$ is κ_{n+2}-Knaster in $V[\mathbb{P}]$, and therefore T has a name $\dot{T}$ which can be taken to be a $< \kappa_{n+2}$-sequence of elements in $V[\mathbb{P}]$. This name is already present in $\mathbb{P}(< \kappa_{n+2})$ (the iteration $\mathbb{P}$ below κ_{n+2}). It follows that

$$\mathbb{P}(< \kappa_{n+2}) * \mathbb{T}(n) \tag{3.9}$$

already adds T.

Let us define

$$\mathbb{T}(n)^- = \prod_{m \le n} \mathbb{M}(\kappa_m, \kappa_{m+1}) \times \prod_{m \le n} \mathbb{R}(m).$$

Thus we can write the forcing in (3.9) as

$$\mathbb{P}(< \kappa_{n+2}) * (\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}) \times \mathbb{R}^0(n+1) \times \mathbb{T}(n)^-). \tag{3.10}$$

This forcing is equivalent to

$$\mathbb{P}(< \kappa_{n+2}) * \big(\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}) \times \mathbb{R}^0(n+1)\big) * \mathbb{T}(n)^-. \tag{3.11}$$

because $\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}) \times \mathbb{R}^0(n+1)$ does not change $H(\kappa_{n+1})$ where the rest of the forcing lives.

We claim that T is in fact added by

$$\mathbb{P}(<\kappa_{n+2}) * \mathrm{Add}(\kappa_{n+1}, 1) * \mathbb{T}(n)^-. \tag{3.12}$$

This is true because T has a name in the forcing

$$\mathbb{P}(<\kappa_{n+2}) * \big(\mathrm{Add}(\kappa_{n+1}, \kappa_{n+2}) \times \mathbb{R}^0(n+1)\big) * \mathbb{T}(n)^- \tag{3.13}$$

of size at most κ_{n+1} and therefore a name in the forcing (3.12).

$\mathbb{P}(<\kappa_{n+2}) * \mathrm{Add}(\kappa_{n+1}, 1)$ preserves the weak compactness of κ_{n+1} (since we prepared by the Cohen forcing below), so it remains to show that $\mathbb{T}(n)^-$ forces the tree property at κ_{n+1} for a weakly compact κ_{n+1}.

Let us write $\mathbb{T}(n)^-$ as:

$$\mathbb{T}(n)^- = \mathbb{M}(\kappa_n, \kappa_{n+1}) \times \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{T}_3, \tag{3.14}$$

where $\mathbb{T}_1 = \prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1})$, $\mathbb{T}_2 = \mathbb{R}^0_n \times \prod_{m<n} \mathbb{R}(m)$, and $\mathbb{T}_3 = \mathbb{R}^1(n)$. These forcings have the following basic properties which are relevant for the proof:

- $\mathbb{M}(\kappa_n, \kappa_{n+1})$ is κ_{n+1}-Knaster, and there is a projection to it from a product $\mathrm{Add}(\kappa_n, \kappa_{n+1}) \times {}^1\mathbb{M}(\kappa_n, \kappa_{n+1})$, where ${}^1\mathbb{M}(\kappa_n, \kappa_{n+1})$ is a κ_n^+-closed term forcing.
- $\mathbb{T}_1$ is κ_n-Knaster, and bounded in $H(\kappa_{n+1})$.
- $\mathbb{T}_2$ is κ_n^+-Knaster.
- $\mathbb{T}_3$ is κ_n^+-closed.

Denote $\kappa_{n+1} = \kappa$. Exactly as in the proof of Theorem 2.5, using the chain condition of $\mathbb{T}_2$ and $\mathbb{T}_3$, if $\mathbb{T}(n)^-$ adds a κ-Aronszajn tree, so does the forcing

$$\mathbb{M}(\kappa_n, \kappa_{n+1}) \times \mathbb{T}_1 \times \mathbb{T}_2|\kappa \times \mathbb{T}_3|\kappa,$$

where $\mathbb{T}_2|\kappa$ and $\mathbb{T}_3|\kappa$ denote the restrictions of all of the Cohen products in $\mathbb{T}_2$ and $\mathbb{T}_3$ to length κ.

Let $j : N \to M$ be a weakly compact embedding with critical point κ, where N contains all the relevant parameters. Pursuing the analogy with Theorem 2.5, and the notation in that proof, consider the model $M[G][x_0][y_0][x_1][H_2][H_1][y_1]$, where in our case we have:

– $G = G_0 \times G_1$ is $\mathbb{M}(\kappa_n, \kappa_{n+1}) \times \mathbb{T}_1$-generic.
– x_0 is $\mathbb{T}_3|\kappa$-generic.
– x_1 is such that $x_0 \times x_1$ is $j(\mathbb{T}_3|\kappa)$-generic. Let us denote the relevant forcing as $\hat{\mathbb{T}}_3$: $j(\mathbb{T}_3|\kappa) = \mathbb{T}_3|\kappa \times \hat{\mathbb{T}}_3$.
– y_0 is $\mathbb{T}_2|\kappa$-generic.
– y_1 is such that $y_0 \times y_1$ is $j(\mathbb{T}_2|\kappa)$-generic. Let us denote the relevant forcing as $\hat{\mathbb{T}}_2$: $j(\mathbb{T}_2|\kappa) = \mathbb{T}_2|\kappa \times \hat{\mathbb{T}}_2$.
– H_1 is $\mathrm{Add}(\kappa_n, j(\kappa) - \kappa)^V$-generic.
– H_2 is ${}^1\mathbb{M}(\kappa_n, j(\kappa) - \kappa)$-generic, where ${}^1\mathbb{M}(\kappa_n, j(\kappa) - \kappa)$ is the term forcing which is κ_n^+-closed in $M[G_0]$.

Note that we can view

$$M[G][x_0][y_0][x_1][H_2][H_1][y_1]$$

as the model

$$M[G_0][G_1][x_0][y_0][x_1][H_2][H_1][y_1],$$

where

$$G_1 \times x_0 \times y_0 \times x_1 \times H_2 \times H_1 \times y_1$$

is a generic filter for a product defined in $M[G_0]$.

Also note that $j(\mathbb{T}_3|\kappa)$ and $j(\mathbb{T}_2|\kappa)$ are the products of the respective Cohen forcings as defined in V, only extended to length $j(\kappa)$ (because all these forcings have equivalent definitions in V and M).

We also have the analogue of Claim 2.7 (with the same proof):

Claim 3.2 *The square of $j(\mathbb{T}_3|\kappa)$ is κ-cc in in $M[G_0][x_0][y_0]$.*

The proof is now finished as in the last paragraph of the proof of Theorem 2.5, except for the last part which concerns the κ_n^+-Knasterness of the forcing $\mathrm{Add}(\kappa_n, j(\kappa)-\kappa)^V \times \hat{\mathbb{T}}_2$; in Theorem 2.5, this was trivial because the forcing was the Cohen forcing at $\aleph_0$. In the present situation, we argue that the square of the forcing is still κ_n^+-cc in $M[G][x_0][y_0][x_1][H_2]$ (which suffices by Fact 1.5). We note that $\mathrm{Add}(\kappa_n, j(\kappa) - \kappa)^V \times \hat{\mathbb{T}}_2$ is isomorphic to its square, so it suffices to show that $\mathrm{Add}(\kappa_n, j(\kappa)-\kappa)^V \times \hat{\mathbb{T}}_2$ is κ_n^+-cc in $M[G][x_0][y_0][x_1][H_2]$.

Let us denote

– $P_1 = {}^1\mathbb{M}(\kappa_n, \kappa_{n+1}) \times \mathbb{T}_3|\kappa \times \hat{\mathbb{T}}_3$,
– $P_2 = \mathrm{Add}(\kappa_n, \kappa)^V$,
– $P_3 = \mathbb{T}_1 \times \mathbb{T}_2|\kappa \times \mathrm{Add}(\kappa_n, j(\kappa) - \kappa)^V \times \hat{\mathbb{T}}_2$.

By Easton lemma, P_1 forces that $P_2 \times P_3$ is κ_n^+-cc, and therefore $P_1 \times P_2$ forces that P_3 is κ_n^+-cc. As $P_1 \times P_2$ projects to the forcing corresponding to the generic filter $G_0 \times x_0 \times x_1$, it follows that P_3 is κ_n^+-cc in $M[G_0][x_0][x_1]$.

Now notice that ${}^1\mathbb{M}(\kappa_n, j(\kappa) - \kappa)$ is still κ_n^+-closed in $M[G_0][x_0][x_1]$ by Claim 3.2. Therefore by Easton lemma, P_3 is κ_n^+-cc in $M[G_0][x_0][x_1][H_2]$. We can view $M[G_0][G_1][x_0][y_0][x_1][H_2]$ as a model which is a generic extension of $M[G_0]$ by the following product defined in $M[G_0]$:

$$ {}^1\mathbb{M}(\kappa_n, j(\kappa) - \kappa) \times \mathbb{T}_1 \times \mathbb{T}_2|\kappa \times \mathbb{T}_3|\kappa \times \hat{\mathbb{T}}_3, $$

and therefore all generic filters for the components of the product are mutually generic over $M[G_0]$. It follows that $\mathrm{Add}(\kappa_n, j(\kappa) - \kappa)^V \times \hat{\mathbb{T}}_2$ is κ_n^+-cc in

$$ M[G_0][x_0][x_1][H_2][G_1][y_0] = M[G][x_0][y_0][x_1][H_2], $$

as desired. $\square$

3.2 The weak tree property

For the sake of completeness, we also address the question of the weak tree property and the continuum function below $\aleph_\omega$.

Let $\kappa_2 < \kappa_3 < \cdots$ be an ω-sequence of Mahlo cardinals with limit λ. Let κ_0 denote $\aleph_0$, and κ_1 denote $\aleph_1$. In Theorem 3.4, we control the continuum function below $\aleph_\omega = \lambda$, λ strong limit, while having the weak tree property at all $\aleph_n$, $n \geq 2$.

Let A denote the set $\{\kappa_i \,|\, i < \omega\}$, and let $f : A \to A$ be a function which satisfies for all α, β in A:

(i) $\alpha < \beta \to f(\alpha) \leq f(\beta)$.
(ii) If $\alpha = \kappa_i$, then $f(\alpha) \geq \kappa_{i+2}$.

We say that f is an Easton function on A which respects the κ_i's (condition (ii)).

The following natural modification of the Mitchell forcing first appeared in [8].

Definition 3.3 *Let $0 \leq n < \omega$ be given. We define $\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ as a collection of pairs (p, q) which satisfy the same conditions as in $\mathbb{M}(\kappa_n, \kappa_{n+2})$ with the difference that instead of $\mathrm{Add}(\kappa_n^+, 1)$ for collapsing, we use $\mathrm{Add}(\kappa_{n+1}, 1)$, and the size of the domain of q is now $< \kappa_{n+1}$. In particular, $\mathbb{M}(\kappa_n, \kappa_{n+2})$ is equal to $\mathbb{M}(\kappa_n, \kappa_n^+, \kappa_{n+2})$. Note that $\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ is a projection of the product of the Cohen forcing $\mathrm{Add}(\kappa_n, \kappa_{n+2})$ and of a certain term forcing ${}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})$ which κ_{n+1}-closed.*

The following theorem is a generalization of Theorem 4.11 in [8].

Theorem 3.4 *Assume GCH and let $\langle \kappa_i \,|\, i < \omega \rangle$, λ, and A be as above. Let f be an Easton function on A which respects the κ_i's. Then there is a forcing notion $\mathbb{S}$ such that if G is a $\mathbb{S}$-generic filter, then in $V[G]$:*

(i) Cardinals in A are preserved, and all other cardinals below λ are collapsed; in particular, for all $n < \omega$, $\kappa_n = \aleph_n$,
(ii) The continuum function on $A = \{\aleph_n \,|\, n < \omega\}$ is controlled by f.
(iii) The weak tree property holds on every $\aleph_n$, $2 \leq n < \omega$.

PROOF. Set $\mathbb{Q}$ to be the full support product

$$\mathbb{Q} = \prod_{n<\omega} \mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2}).$$

Let $\mathbb{R}$ be the standard Easton product to force the prescribed behaviour of the continuum function below $\aleph_\omega$ (taking into account that the cardinals below $\aleph_\omega$ will be equal to cardinals in A):

$$\mathbb{R} = \prod_{n<\omega} \mathrm{Add}(\kappa_n, f(\kappa_n)).$$

For simplicity of notation, let us write $\mathbb{R}(n) = \mathrm{Add}(\kappa_n, f(\kappa_n))$.

We define the forcing $\mathbb{S}$ as follows:

$$\mathbb{S} = \mathbb{Q} \times \mathbb{R}. \tag{3.15}$$

Again, we leave it as an exercise for the reader to verify that the cardinals in A are preserved, $\kappa_n = \aleph_n$, and the continuum function below $\aleph_\omega$ is controlled by f.

Let $n < \omega$ be fixed. We show that there are no special κ_{n+2}-Aronszajn trees in $V[\mathbb{S}]$.

Let us denote:

$$\mathbb{T}(n) = \prod_{m \le n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m \le n+1} \mathbb{R}(m), \tag{3.16}$$

and

$$\mathbb{T}(n)_{\mathrm{tail}} = \prod_{m > n+2} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m > n+2} \mathbb{R}(m), \tag{3.17}$$

so that

$$\begin{aligned} \mathbb{S} = {} & \mathbb{T}(n) \times \mathbb{R}(n+2) \\ & \times \mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \times \mathbb{M}(\kappa_{n+2}, \kappa_{n+3}, \kappa_{n+4}) \times \mathbb{T}(n)_{\mathrm{tail}}. \end{aligned} \tag{3.18}$$

Suppose for contradiction $\mathbb{S}$ adds a special κ_{n+2}-Aronszajn tree. Then also the forcing

$$\begin{aligned} & \mathbb{T}(n) \times \mathbb{R}(n+2) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3}) \times {}^1\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \\ & \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4}) \times {}^1\mathbb{M}(\kappa_{n+2}, \kappa_{n+3}, \kappa_{n+4}) \times \mathbb{T}(n)_{\mathrm{tail}} \end{aligned} \tag{3.19}$$

adds a special κ_{n+2}-Aronszajn tree because it projects onto $\mathbb{S}$. Denote the tree T.

Then T is added by

$$\begin{aligned} & \mathbb{T}(n) \times \mathbb{R}(n+2) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3}) \\ & \times {}^1\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4}) \end{aligned} \tag{3.20}$$

because ${}^1\mathbb{M}(\kappa_{n+2}, \kappa_{n+3}, \kappa_{n+4}) \times \mathbb{T}(n)_{\rm tail}$ is κ_{n+2}^+-closed in V, and using an obvious analogue of Lemma 1.7, it follows that ${}^1\mathbb{M}(\kappa_{n+2}, \kappa_{n+3}, \kappa_{n+4}) \times \mathbb{T}(n)_{\rm tail}$ is still κ_{n+2}^+-distributive over the forcing (3.20), and hence does not add any κ_{n+2}-trees.

We finish the proof by arguing that the forcing in (3.20) cannot add T. First note that $\mathbb{R}(n+2) \times {}^1\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4})$ is κ_{n+2}-closed in V and therefore preserves the Mahloness of κ_{n+2}, and also the chain and closure properties of $\mathbb{T}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$. Let us work in

$$V^* = V[\mathbb{R}(n+2) \times {}^1\mathbb{M}(\kappa_{n+1}, \kappa_{n+2}, \kappa_{n+3}) \times \mathrm{Add}(\kappa_{n+2}, \kappa_{n+4})].$$

Let $\dot{T}$ and $\dot{g}$ be $\mathbb{T}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$-names for the tree and a specializing function. We can identify $\dot{T}$ with a name for a subset of κ_{n+2} and $\dot{g}$ with a function from κ_{n+2} to κ_{n+1}. Since the forcing $\mathbb{T}(n) \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+3})$ is κ_{n+2}-cc, we may assume that already $\mathbb{T}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})$ adds $\dot{T}$ and $\dot{g}$, where $\mathbb{T}(n)|\kappa_{n+2}$ is the forcing

$$\prod_{m\le n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \prod_{m\le n+1} \mathbb{R}(m)|\kappa_{n+2},$$

where $\prod_{m\le n+1} \mathbb{R}(m)|\kappa_{n+2}$ is the restriction of the Cohen forcings to length κ_{n+2}.

Since $\mathbb{T}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})$ is κ_{n+2}-cc over V^*, there is a V-inaccessible δ, $\kappa_{n+1} < \delta < \kappa_{n+2}$, such that $T|\delta, g|\delta$ are added by the forcing

$$\mathbb{T} = \prod_{m<n} \mathbb{M}(\kappa_m, \kappa_{m+1}, \kappa_{m+2}) \times \mathbb{M}(\kappa_n, \kappa_{n+1}, \delta) \times \prod_{m\le n} \mathbb{R}(m)|\delta \times \mathrm{Add}(\kappa_{n+1}, \delta) \tag{3.21}$$

which is δ-cc. Also note that $\mathbb{R}(n+1)|\delta$ is the same forcing as $\mathrm{Add}(\kappa_{n+1}, \delta)$, and therefore we removed $\mathbb{R}(n+1)|\delta$ from (3.21).

We finish the proof by arguing that over V^*, the forcing from $V^*[\mathbb{T}]$ to $V^*[\mathbb{T}(n)|\kappa_{n+2} \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2})]$ cannot add a branch to $T|\delta$. Let us denote by ${}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2} - \delta)$ the term forcing which is κ_{n+1}-closed in $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$ such that in $V^*[\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)]$,

$$\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2})/\mathbb{M}(\kappa_n, \kappa_{n+1}, \delta)$$

is a projection of $\mathrm{Add}(\kappa_n, \kappa_{n+2} - \delta) \times {}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2} - \delta)$.

Thus it suffices to show that over $V^*[\mathbb{T}]$, the forcing

$$\begin{aligned} \text{(3.22)} \quad & \mathrm{Add}(\kappa_n, \kappa_{n+2} - \delta) \times {}^1\mathbb{M}(\kappa_n, \kappa_{n+1}, \kappa_{n+2} - \delta) \\ & \qquad \times \mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta) \times \prod_{m \le n} \mathbb{R}(m) | (\kappa_{n+2} - \delta) \end{aligned}$$

does not add a branch to $T|\delta$, where $\prod_{m \le n} \mathbb{R}(m)|(\kappa_{n+2} - \delta)$ is the restriction of the Cohen forcings to the interval $[\delta, \kappa_{n+2})$.

This is shown exactly as at the end of Theorem 3.1,[5] with the forcing $\mathrm{Add}(\kappa_{n+1}, \kappa_{n+2} - \delta)$ being κ_{n+1}-closed in V^*, and $\prod_{m \le n} \mathbb{R}(m)|(\kappa_{n+2} - \delta)$ being κ_{n+1}-Knaster in V^*. □

4 Open questions

Q1. Is it possible to get the results of Theorem 3.1 and 3.4 with a strong limit $\aleph_\omega$ with $2^{\aleph_\omega} > \aleph_{\omega+1}$? Or even stronger, with the tree property at $\aleph_{\omega+2}$?

Note that with the failure of GCH at $\aleph_\omega$, the situation is much more complex because we can no longer use a simple product construction. In [5], Gitik and Merimovich show that an arbitrary continuum function below $\aleph_\omega$ is compatible with the failure of GCH at $\aleph_\omega$, but in their model they do not discuss the tree property. Note that in the paper [4], a model is constructed with the tree property at the $\aleph_{2n}$'s, $0 < n < \omega$, with the failure of GCH at $\aleph_\omega$, but in that model, $2^{\aleph_{2n}} = \aleph_{2n+2}$ for every $n < \omega$,

[5]It is immaterial to the argument whether we work in a generic extension of M as in Theorem 3.1, and discuss the ordinals $\kappa < j(\kappa)$, or work in a generic extension of V, and discuss the ordinals $\delta < \kappa_{n+2}$. Note that the proof of Theorem 3.1 could also have been formulated with some $\delta < \kappa$ without mentioning an elementary embedding.

and GCH holds at the remaining cardinals below $\aleph_\omega$; the construction in [4] does not seem to admit an easy generalization along the lines of Theorem 3.1.

Cummings and Foreman [2] proved that starting with infinitely many supercompact cardinals, there is a model where the tree property holds at every $\aleph_n$, $2 \leq n < \omega$, with $2^{\aleph_n} = \aleph_{n+2}$ for all $n < \omega$.

Q2. Starting with infinitely many supercompact cardinals, is it consistent that the tree property holds at every $\aleph_n$, $2 \leq n < \omega$, and the continuum function is arbitrary such that $2^{\aleph_n} \geq \aleph_{n+2}$, $n < \omega$?

There is a notion of a *super tree property* which captures the combinatorial essence of a supercompact cardinal (see for instance [9], or [3], for definitions). Weiss noticed that Mitchell's forcing over a supercompact cardinal yields the super tree property. Later, Fontanella [3] and Unger independently proved that starting with infinitely many supercompact cardinals, the super tree property can hold at every $\aleph_n$, $2 \leq n < \omega$.

Q3. Starting with infinitely many supercompact cardinals, is it consistent that the super tree property holds at every $\aleph_n$, $2 \leq n < \omega$, and the continuum function is arbitrary such that $2^{\aleph_n} \geq \aleph_{n+2}$, $n < \omega$?

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