The tree property at the double successor of a singular cardinal with a larger gap[1]

SY-DAVID FRIEDMAN, RADEK HONZIK, ŠÁRKA STEJSKALOVÁ

Sy-D. Friedman: Kurt Gödel Research Center for Mathematical Logic, Währinger Strasse 25, 1090 Vienna Austria
sdf@logic.univie.ac.at

R. Honzik, Š. Stejskalová: Charles University, Department of Logic, Celetná 20, Praha 1, 116 42, Czech Republic
radek.honzik@ff.cuni.cz
sarka@logici.cz

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Abstract. Starting from a Laver-indestructible supercompact κ and a weakly compact λ above κ, we show there is a forcing extension where κ is a strong limit singular cardinal with cofinality ω, $2^\kappa = \kappa^{+3} = \lambda^+$, and the tree property holds at $\kappa^{++} = \lambda$. Next we generalize this result to an arbitrary cardinal μ such that $\kappa < \mathrm{cf}(\mu)$ and $\lambda^+ \leq \mu$. This result provides more information about possible relationships between the tree property and the continuum function.

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1 Introduction

In [1], Cummings and Foreman showed that starting from a Laver-indestructible supercompact cardinal κ and a weakly compact $\lambda > \kappa$, one can construct a generic extension where $2^\kappa = \lambda = \kappa^{++}$, κ is a singular strong limit cardinal with cofinality ω, and the tree property holds

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at κ^{++}. It is natural to try to generalize this result in at least two directions.

First, one can ask whether – in addition to the properties identified in the previous paragraph – κ can equal $\aleph_\omega$. Cummings and Foreman suggested in [1] that this is possible, but did not provide any details. A model with the tree property at $\aleph_{\omega+2}$, with $\aleph_\omega$ strong limit, was first constructed by Friedman and Halilović in [2], moreover from a significantly lower large cardinal assumption of hypermeasurability.[2] Shortly afterwards, Gitik, answering a question posed in [2], showed in [4] that the same result can be proved form a weaker and optimal assumption.

Second, one can ask whether it is possible to have 2^κ greater than κ^{++} with the tree property at κ^{++}. Using a variant of the Mitchell forcing, Friedman and Halilović [3] proved that starting from a sufficiently hypermeasurable κ, one can keep the measurability of κ together with $2^\kappa > \kappa^{++}$ and the tree property at κ^{++}.

In this paper, we generalize [1] in the second direction. In Theorem 2.1, we prove that starting from a Laver-indestructible supercompact κ and a weakly compact λ above, one can find a forcing extension where κ is strong limit singular with cofinality ω, $2^\kappa = \kappa^{+3} = \lambda^+$, and the tree property holds at κ^{++}. In Theorem 3.1 we give an outline of a generalisation in which the gap $(\kappa, 2^\kappa)$ can be arbitrarily large: $2^\kappa = \mu$ for any cardinal $\mu > \lambda$ with cofinality greater than κ. The method of the proof is based on the argument in [1], with reference to [6] which fills a gap in the final stage of that argument.

1.1 Preliminaries

We review some useful results regarding projections of partial orders and their Boolean completions. Let us recall that a projection π between a partial order $(\mathbb{P}, \leq_\mathbb{P})$ and $(\mathbb{Q}, \leq_\mathbb{Q})$ is an order-preserving function from $\mathbb{P}$ into $\mathbb{Q}$ such that $\pi(1_\mathbb{P}) = 1_\mathbb{Q}$, and for all $p \in \mathbb{P}$ and all $q \leq_\mathbb{Q} \pi(p)$ there is $\bar{p} \leq_\mathbb{P} p$ such that $\pi(\bar{p}) \leq_\mathbb{Q} q$. Note that the condition $\pi(1_\mathbb{P}) = 1_\mathbb{Q}$ ensures

[2] The technique of proof in [2] used the Sacks forcing to obtain the tree property, unlike the proof in [1] which is based on a Mitchell-style analysis.

that the range of π is dense in $\mathbb{Q}$, and is sometimes omitted from the definition of a projection. It is a standard fact (see for instance [5]) that if $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection, then $\mathbb{P}$ is forcing equivalent to an iteration $\mathbb{Q} * \mathbb{P}/\mathbb{Q}$ for a quotient forcing which we denote $\mathbb{P}/\mathbb{Q}$.

Recall that if $\mathbb{P}$ is a separative partial order, we can identify $\mathbb{P}$ with a dense suborder in the canonical Boolean completion of $\mathbb{P}$ without the least element, denoted by $\mathrm{RO}^+(\mathbb{P})$.

Lemma 1.1 *Let $\mathbb{P}$ and $\mathbb{Q}$ be two separative partial orders. Assume that for every $\mathbb{P}$-generic filter G over V, there is in $V[G]$ a $\mathbb{Q}$-generic filter F over V. Let $\dot{F}$ be a $\mathbb{P}$-name such that $\mathbb{P} \Vdash (\dot{F}$ is a $\mathbb{Q}$-generic filter). Then the following hold:*

(i) Define $\pi : \mathbb{P} \to \mathrm{RO}^+(\mathbb{Q})$ by

$$\pi(p) = \bigwedge\{b \in \mathrm{RO}^+(\mathbb{Q}) \mid p \Vdash b \in \dot{F}\}. \tag{1.1}$$

Set $b_{\mathbb{P}} = \pi(1) = \bigwedge\{b \in \mathrm{RO}^+(\mathbb{Q}) \mid 1 \Vdash b \in \dot{F}\}$. Let $\mathrm{RO}^+(\mathbb{Q})|b_{\mathbb{P}}$ denote the partial order $\{b \in \mathrm{RO}^+(\mathbb{Q}) \mid b \leq b_{\mathbb{P}}\}$. Then

$$\pi : \mathbb{P} \to \mathrm{RO}^+(\mathbb{Q})|b_{\mathbb{P}} \text{ is a projection} \tag{1.2}$$

and the range of π is dense in $\mathrm{RO}^+(\mathbb{Q})|b_{\mathbb{P}}$.

(ii) Suppose π is as in (i), $\pi(1) = 1$ and every q_1, q_2 in $\mathbb{Q}$ have the supremum in $\mathbb{Q}$. Then π can be defined just using $-\mathbb{Q} = \{-q \mid q \in \mathbb{Q}\}$:

$$\pi(p) = \bigwedge\{-q \mid q \in \mathbb{Q} \,\&\, p \Vdash -q \in \dot{F}\} = \bigwedge\{-q \mid q \in \mathbb{Q} \,\&\, p \Vdash q \notin \dot{F}\}. \tag{1.3}$$

Proof. (i). We first show (1.2). The preservation of the ordering is easy. We check the density condition, i.e. for every $p \in \mathbb{P}$ and every $c \leq \pi(p)$, there is $p' \leq p$ such that $\pi(p') \leq c$. Let p and c be given. If $c = \pi(p)$, we are trivially done. So suppose $c < \pi(p)$. If for every $p' \leq p$, $p' \nVdash c \in \dot{F}$, then $p \Vdash \pi(p) - c \in \dot{F}$, which contradicts the fact that $\pi(p)$

is the infimum of $\{b \in \mathrm{RO}^+(\mathbb{Q}) \,|\, p \Vdash b \in \dot{F}\}$. It follows that there is some $p' \leq p$, $p' \Vdash c \in \dot{F}$. Then $\pi(p') \leq c$ as required.

We now show that the range of π is dense. Suppose for contradiction there is $b \leq b_{\mathbb{P}}$ such that the range of π is disjoint from $\{b' \in \mathrm{RO}^+(\mathbb{Q})|b_{\mathbb{P}} \,|\, b' \leq b\}$. Then $b_{\mathbb{P}} - b$ is forced by 1 to be in $\dot{F}$, a contradiction because $b_{\mathbb{P}} - b < b_{\mathbb{P}}$.

(ii). Let p be fixed and let a_p denote $\bigwedge\{-q \,|\, q \in \mathbb{Q} \;\&\; p \Vdash -q \in \dot{F}\}$. We wish to show that $\pi(p)$ as in (1.1) is equal to a_p. Clearly $\pi(p) \leq a_p$. Suppose for contradiction $\pi(p) < a_p$ and set $b = a_p - \pi(p)$. By density, there is $q_b \in \mathbb{Q}$ such that $q_b \leq b$; in particular

$$a_p - q_b < a_p. \tag{1.4}$$

Since $a_p - q_b \geq \pi(p)$ and $p \Vdash \pi(p) \in \dot{F}$, we have

$$p \Vdash a_p - q_b \in \dot{F}. \tag{1.5}$$

Now,

$$a_p - q_b = a_p \wedge -q_b = \bigwedge\{-q \wedge -q_b \,|\, q \in \mathbb{Q} \;\&\; p \Vdash -q \in \dot{F}\}. \tag{1.6}$$

By (1.5), $p \Vdash -q \wedge -q_b = -(q \vee q_b) \in \dot{F}$, and hence $-(q \vee q_b)$ is an element of $\{-q \,|\, q \in \mathbb{Q} \;\&\; p \Vdash -q \in \dot{F}\}$ whenever p forces $-q \in \dot{F}$. It follows $a_p - q_b = a_p$, contradicting (1.4), and so $\pi(p) = a_p$ as desired.

Note that if $\mathbb{Q}$ is not closed under the suprema of q_1, q_2 in $\mathbb{Q}$, then the proof still provides a simplification of the definition of $\pi(p)$:

$$\pi(p) = \bigwedge\{-b \,|\, (\exists n \in \omega)(\exists q_1, \ldots, q_n \in \mathbb{Q})(b = q_1 \vee \ldots \vee q_n \;\&\; p \Vdash -b \in \dot{F})\}. \tag{1.7}$$

□

It would be tempting to try and prove that $\pi(p)$ is equivalent to

$$\bigwedge\{q \in \mathbb{Q} \,|\, p \Vdash q \in \dot{F}\}, \tag{1.8}$$

and not the rather unintuitive (1.3) or (1.7).[3] However, (1.8) does not work in general.

The following folklore results will be used tacitly later on when we deal with projections on Boolean completions:

Lemma 1.2 *Assume $\mathbb{P}$ and $\mathbb{Q}$ are separative partial orders and $\pi : \mathbb{P} \to \mathrm{RO}^+(\mathbb{Q})$ is a projection.*

(i) If $\mathbb{P}'$ is dense in $\mathbb{P}$, then $\pi|\mathbb{P}' : \mathbb{P}' \to \mathrm{RO}^+(\mathbb{Q})$ is a projection.

(ii) (a) Assume $\mathbb{P}'$ is forcing equivalent with $\mathbb{P}$. Then there is a projection $\pi' : \mathbb{P}' \to \mathrm{RO}^+(\mathbb{Q})$.

(b) Moreover, if $\mathbb{P}$ is dense in $\mathbb{P}'$, then there is $\pi' \supseteq \pi$ such that $\pi' : \mathbb{P}' \to \mathrm{RO}^+(\mathbb{Q})$ is a projection.

(iii) Let $\mathbb{R}$ be a $\mathbb{P}$-name for a forcing notion. Then π naturally extends to a projection $\pi' : \mathbb{P} * \mathbb{R} \to \mathrm{RO}^+(\mathbb{Q})$.

Proof. (i). Obvious.

(ii)(a). We show that if G' is $\mathbb{P}'$-generic, then we can find a $\mathbb{Q}$-generic F in $V[G]$. Since $\mathbb{P}$ and $\mathbb{P}'$ are equivalent, there is some P-generic G such that $V[G'] = V[G]$. By virtue of the projection π, there is a $\mathbb{Q}$-generic F in $V[G]$, and therefore in $V[G']$. The rest follows by Lemma 1.1.

(ii)(b). For $p' \in \mathbb{P}'$ define

$$\pi'(p') = \bigvee\{\pi(p) \,|\, p \le p'\}.$$

By density of $\mathbb{P}$ in $\mathbb{P}'$, $\{\pi(p) \,|\, p \le p'\}$ is non-empty for every p' and therefore $\pi'(p')$ is in $\mathrm{RO}^+(\mathbb{Q})$. If $p' \le q'$ in $\mathbb{P}$, then clearly $\pi(p') \le \pi(q')$. Suppose $p' \in \mathbb{P}'$ is arbitrary and $b \le \pi'(p')$. By the definition of $\pi'(p)$, there is $b' \le b$ such that for some $p \le p'$, $p \in \mathbb{P}$, $b' \le \pi(p)$. It follows there is some $q \le p \le p'$, $q \in \mathbb{P}$, such that $\pi(q) = \pi'(q) \le b' \le b$ as desired.

[3] Perhaps some intuition is salvaged by considering that if $p \Vdash q \in \dot{F}$, $q \in \mathbb{Q}$, and $q' \in \mathbb{Q}$ is incompatible with q, then $p \Vdash -q' \in \dot{F}$ by the upwards closure of $\dot{F}$. Thus using density, q is captured as the infimum of all complements of q' which are incompatible with q: $q = \bigwedge\{-q' \,|\, q' \in \mathbb{Q} \;\&\; q' \perp q\}$.

(iii). Define

$$\pi'(p, r) = \pi(p), \tag{1.9}$$

or every (p, r) in $\mathbb{P} * \mathbb{R}$. If $(p_1, r_1) \le (p_2, r_2)$, then in particular $p_1 \le p_2$, and so $\pi'(p_1, r_1) \le \pi'(p_2, r_2)$ because π is order-preserving. If (p, r) is arbitrary and $b \le \pi'(p, r) = \pi(p)$, then since π is a projection, there is $p' \le p$ such that $\pi(p') \le b$. Since $(p', r) \le (p, r)$, $\pi'(p', r) \le b$ is as required. □

2 Gap three

Let μ be a regular cardinal. We write $\mathrm{TP}(\mu)$ to say that μ satisfies the tree property. If $\mathbb{P}$ is a forcing notion, we write $V[\mathbb{P}]$ to denote an arbitrary generic extension by the forcing $\mathbb{P}$. We say that a supercompact cardinal κ is *Laver-indestructible* if it remains supercompact in any forcing extension by a forcing which is κ-directed closed (where $\mathbb{P}$ is κ-directed closed if for every $D \subseteq \mathbb{P}$ of size less than κ, if for all p_1, p_2 in D there is $e \in D$ such that $e \le p_1$ and $e \le p_2$, then there is $p \in \mathbb{P}$, with $p \le d$ for all $d \in D$).

Theorem 2.1 *Assume GCH and let κ be a Laver-indestructible supercompact cardinal and $\kappa < \lambda$, λ weakly compact. Then there is a forcing notion $\mathbb{R}$ such that the following hold:*

(i) $\mathbb{R}$ *preserves cardinals* $\le \kappa^+$ *and* $\ge \lambda$.
(ii) $V[\mathbb{R}] \models (\kappa^{++} = \lambda \;\&\; 2^\kappa = \lambda^+ \;\&\; \mathrm{cf}(\kappa) = \omega \;\&\; \kappa \textit{ is strong limit})$.
(iii) $V[\mathbb{R}] \models \mathrm{TP}(\lambda)$.

The proof will be given in a sequence of lemmas, and is divided into two stages. Stage 1 defines $\mathbb{R}$, verifies some basic properties for (i) and (ii) of Theorem 2.1 and shows that if $\mathbb{R}$ adds an Aronszajn tree on λ, then already a regular subforcing, which we denote $\mathbb{R}^*$, adds an Aronszajn tree on λ. The forcing $\mathbb{R}^*$ is designed to be very similar to the forcing used in [1]. In stage 2, we show that indeed $\mathbb{R}^*$ allows a very similar analysis to [1] (with correction according to [6]), and therefore cannot add an Aronszajn tree on λ, which finishes the proof.

2.1 Stage 1

Definition 2.2 *Let $\mathbb{P}$ denote the Cohen forcing* $\mathrm{Add}(\kappa, \lambda^+)$ *and for $\alpha < \lambda^+$, let $\mathbb{P}|\alpha$ denote* $\mathrm{Add}(\kappa, \alpha)$.

The following lemma will be useful.

Lemma 2.3 *Let $\dot{U}$ be a $\mathbb{P}$-name such that*

$$1_{\mathbb{P}} \Vdash \dot{U} \textit{ is a normal measure on } \kappa.$$

Then there is a set A of unboundedly many $\alpha < \lambda^+$ containing its limit points of cofinality $> \kappa$ such that for every $\alpha \in A$ and every $\mathbb{P}$-generic filter G,

$$\dot{U}^G \cap V[G|\alpha] \in V[G|\alpha].$$

PROOF. Let $\alpha_0 < \lambda^+$ be given, we show how to find $\alpha \geq \alpha_0$ in A. Let $\langle \dot{x}_i \,|\, i < \nu < \lambda^+ \rangle$ be some enumeration of all nice $\mathbb{P}|\alpha_0$-names for subsets of κ. Note that there are at most λ-many such names so we can indeed choose $\nu < \lambda^+$. For every $i < \nu$, let X_i be a maximal antichain in $\mathbb{P}$ of conditions deciding the statement $\dot{x}_i \in \dot{U}$; by the κ^+-cc of $\mathbb{P}$, the size of X_i is at most κ. Let $\beta_0 \geq \alpha_0$ be such that the supports of all conditions in $\bigcup_{i<\nu} X_i$ are contained in β_0. Repeat this procedure κ^+-many times, building an increasing chain of ordinals and let $\alpha = \sup\{\beta_k \,|\, k < \kappa^+\}$, $\mathrm{cf}(\alpha) = \kappa^+$. Now, if $\dot{x}$ is a $\mathbb{P}|\alpha$-name for a subset of κ, then there is some $\alpha' < \alpha$ such that all coordinates mentioned by $\dot{x}$ are below α'; it follows that $\dot{x}$ was considered in the construction, together with a maximal antichain X in $\mathbb{P}$ of conditions deciding the statement $\dot{x} \in \dot{U}$. Using these $\dot{x}$'s and X's, one can build a $\mathbb{P}|\alpha$-name $\dot{U}_\alpha$ such that for every nice $\mathbb{P}|\alpha$-name $\dot{x}$ for a subset of κ:

$$\dot{x}^{G|\alpha} = \dot{x}^G \in \dot{U}^G \Leftrightarrow \dot{x}^{G|\alpha} \in \dot{U}_\alpha^{G|\alpha}.$$

It is clear that if A is defined to be the set of $\alpha < \lambda^+$ constructed as above and α with cofinality $> \kappa$ is a limit point of A, then $\alpha \in A$.[4] □

[4] We mention the closure of A because the current proof is directly applicable to Lemma 2.5 with a set B, where the closure is relevant to ensure $B^* \subseteq B$ for a certain set B^* defined in Section 2.2, 2nd paragraph. The closure will not be used for A, though.

Fix temporarily a $\mathbb{P}$-generic filter G. Denote $\dot{U}^G = U$. For any $\alpha \in A$ such that $\lambda < \alpha < \lambda^+$ there is a $\mathbb{P}_\alpha$-name, which we denote by $\dot{U}_\alpha$, such that

$$(\dot{U}_\alpha)^{G|\alpha} = U \cap V[G|\alpha]. \tag{2.10}$$

Let us write U_α for $U \cap V[G|\alpha]$. Let us fix $\beta \in A, \lambda < \beta$. This β is going to be fixed for the remainder of the proof.

For $\alpha \leq \lambda$, let $\mathrm{Even}(\alpha)$ denote the set of even ordinals below α. For $\alpha \leq \lambda$, let us write $\mathbb{P}|\mathrm{Even}(\alpha)$ to denote the Cohen forcing $\mathrm{Add}(\kappa, \mathrm{Even}(\alpha))$ which only mentions coordinates indexed by even ordinals. Let π be a bijection between β and $\mathrm{Even}(\lambda)$; π naturally generates an isomorphism between $\mathbb{P}|\beta$ and $\mathbb{P}|\mathrm{Even}(\lambda)$ which we also denote π. Let us further extend the domain of π to all $\mathbb{P}|\beta$-names, and also to $\mathbb{P}|\beta$-generic filters, in the obvious way.

Since $1_{\mathbb{P}|\beta} \Vdash \dot{U}_\beta$ is a measure, we have $1_{\mathbb{P}|\mathrm{Even}(\lambda)} \Vdash \pi(\dot{U}_\beta)$ is a measure.

Remark 2.4 Note that π generates a $\mathbb{P}|\mathrm{Even}(\lambda)$-generic filter $\pi(G|\beta)$ such that $V[G|\beta] = V[\pi(G|\beta)]$, and

$$U_\beta = (\dot{U}_\beta)^{G|\beta} = \pi(\dot{U}_\beta)^{\pi(G|\beta)}. \tag{2.11}$$

However, it is not true that $\pi(\dot{U}_\beta)^{G|\mathrm{Even}(\lambda)} = U_\beta$, where $G|\mathrm{Even}(\lambda)$ is the $\mathbb{P}|\mathrm{Even}(\lambda)$-generic filter composed of the Cohen generics on the even coordinates of G below λ. The reason is that $V[G|\mathrm{Even}(\lambda)]$ is a proper submodel of $V[\pi(G|\beta)] = V[G|\beta]$.

The proof of the following lemma is the same as for Lemma 2.3.

Lemma 2.5 *There is a set B of unboundedly many $\alpha < \lambda$ containing its limit points of cofinality $> \kappa$ such that for every $\alpha \in B$ and every* $\mathbb{P}|\mathrm{Even}(\lambda)$*-generic filter H,*

$$\pi(\dot{U}_\beta)^H \cap V[H|\mathrm{Even}(\alpha)] \in V[H|\mathrm{Even}(\alpha)],$$

where $H|\mathrm{Even}(\alpha)$ is the restriction of H to $\mathbb{P}|\mathrm{Even}(\alpha)$.

Let us write $\dot{U}^\pi_\alpha$ for the natural (i.e. obtained from the construction in the proof of Lemma 2.5) $\mathbb{P}|\text{Even}(\alpha)$-name for the measure $\pi(\dot{U}_\beta)^H \cap V[H|\text{Even}(\alpha)]$.

For concreteness, let us review the definition of Prikry forcing.

Definition 2.6 *Assume κ is measurable and U is a normal measure at κ. Prikry forcing at κ with the measure U, which we will denote $\mathbb{Q}$ (with indexes to distinguish different κ's and U's), is a collection of pairs (s, A) where s is a finite subset of κ, A is in U, and $A \cap \max(s)+1 = \emptyset$. (s, A) is stronger than (t, B) if s extends t (i.e. $t = s \cap \alpha$ for some α), $A \subseteq B$ and $s \setminus t \subseteq B$.*

We fix the following notation: Denote $\hat{A} = (A \cap [\beta, \lambda^+)) \cup \{\lambda^+\}$. For every $\gamma \in \hat{A}$, let $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$ denote the Cohen forcing $\text{Add}(\kappa, \gamma)$ followed by the Prikry forcing $\mathbb{Q}_\gamma$ defined with respect to the measure $\dot{U}_\gamma$ (where we identify $\dot{U}_{\lambda^+}$ with $\dot{U}$ and $\mathbb{P}|\lambda^+ * \mathbb{Q}_{\lambda^+}$ with $\mathbb{P} * \mathbb{Q}$). For $\alpha \in B$, where B is as in Lemma 2.5, let $\mathbb{Q}^\pi_\alpha$ be a $\mathbb{P}|\text{Even}(\alpha)$-name for the Prikry forcing defined with the $\mathbb{P}|\text{Even}(\alpha)$-name $\dot{U}^\pi_\alpha$. Let us also define $\mathbb{Q}^\pi_\lambda$ as the Prikry forcing with the measure $\pi(\dot{U}_\beta)$ in $\mathbb{P}|\text{Even}(\lambda)$.

The following lemma defines certain projections which will be used later on.

Lemma 2.7 *(i) For every $\gamma < \delta$ in $\hat{A}$, there is a projection*

$$\sigma^\delta_\gamma : \mathbb{P}|\delta * \mathbb{Q}_\delta \to \text{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma). \tag{2.12}$$

(ii) For every γ in $\hat{A}$ and every $\alpha \in B$, there is a projection

$$\sigma^\gamma_\alpha : \mathbb{P}|\gamma * \mathbb{Q}_\gamma \to \text{RO}^+(\mathbb{P}|\text{Even}(\alpha) * \mathbb{Q}^\pi_\alpha). \tag{2.13}$$

(iii) For $\gamma \in A \cap (\beta, \lambda^+)$ and $\alpha \in B$, let $\hat{\sigma}^\gamma_\alpha$ be the extension of σ^γ_α to the Boolean completion of $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$ obtained according to Lemma 1.2(ii)(b):

$$\hat{\sigma}^\gamma_\alpha : \text{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma) \to \text{RO}^+(\mathbb{P}|\text{Even}(\alpha) * \mathbb{Q}^\pi_\alpha). \tag{2.14}$$

Then the projections commute:

$$\sigma^{\lambda^+}_\alpha = \hat{\sigma}^\gamma_\alpha \circ \sigma^{\lambda^+}_\gamma. \tag{2.15}$$

PROOF. (i). Let $G * x$ be a $\mathbb{P}|\delta * \mathbb{Q}_\delta$-generic filter,[5] where x is an ω-sequence cofinal in κ. By the geometric condition for Prikry genericity,[6] and the fact that $\dot{U}_\gamma$ is the restriction of $\dot{U}_\delta$, it is clear that $G|\gamma * x$ is $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$-generic. The result follows by Lemma 1.1.

(ii). Let $G * x$ be a $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$-generic filter, where x is an ω-sequence cofinal in κ. By (2.11) and the geometric condition for the generic filters for Prikry forcings,

$$\pi(G|\beta) * x \text{ is } \mathbb{P}|\mathrm{Even}(\lambda) * \mathbb{Q}^\pi_\lambda\text{-generic.}$$

Substituting $H = \pi(G|\beta)$ in Lemma 2.5, for every $\alpha \in B$, $\mathbb{Q}^\pi_\alpha$ is a forcing in $V[H|\mathrm{Even}(\alpha)]$ defined with respect to the restriction of the measure U; it follows that $H|\mathrm{Even}(\alpha) * x$ is a generic filter for $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha$ existing in $V[G * x]$. The result again follows by Lemma 1.1.

(iii). σ^γ_α is correctly defined by Lemma 1.2(ii)(b). Let us fix $(p, (s, \dot{A}))$ in $\mathbb{P} * \mathbb{Q}$ and let us denote

$$b_\alpha = \bigwedge\{b \in \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha) \mid (p, (s, \dot{A})) \Vdash b \in \dot{G}_\alpha\},$$

$$b_\gamma = \bigwedge\{b \in \mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma) \mid (p, (s, \dot{A})) \Vdash b \in \dot{G}_\gamma\},$$

and

$$b^\gamma_\alpha = \bigwedge\{b \in \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha) \mid b_\gamma \Vdash b \in \dot{G}_\alpha\},$$

where $\dot{G}_\gamma$ and $\dot{G}_\alpha$ are the canonical names for the generic filters. The intuition is that the Boolean value b_α (and similarly b_γ and b^γ_α) corresponds to a condition $(\pi(p|\beta)|\alpha, (s, \dot{C}))$ for some $\dot{C}$ which is the intersection of all elements in $\dot{U}_\alpha$ in $V^{\mathbb{P}|\mathrm{Even}(\alpha)*\mathbb{Q}^\pi_\alpha}$ which contain $\dot{A}$; the problem is that this condition in general may not exist in $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha$, and it is necessary to use the more abstract Boolean names.

We show that $b_a = b^\gamma_\alpha$.

[5]We abuse notation here and identify $G * x$ with the generic filter which it determines.

[6]The geometric condition characterises the genericity for Prikry forcing: a cofinal ω-sequence in κ determines a generic filter if and only if it is eventually contained in every element of the measure used to define the forcing.

To argue for $b_\alpha^\gamma \leq b_\alpha$, notice that we can identify every element of $\mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}_\alpha^\pi)$ with an element b of $\mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma)$ by virtue of the projection $\hat{\sigma}_\alpha^\gamma$; now if $(p,(s,\dot{A}))$ forces b into $\dot{G}_\alpha$, then clearly $(p,(s,\dot{A}))$ forces b into $\dot{G}_\gamma$. In particular b_γ forces b into $\dot{G}_\alpha$, and so $b_\alpha^\gamma \leq b_\alpha$.

Conversely, b_γ can be identified with an element of $\mathrm{RO}^+(\mathbb{P} * \mathbb{Q})$, and under this identification $(p,(s,\dot{A})) \leq b_\gamma$. It follows that if b_γ forces $b \in \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}_\alpha^\pi)$ into $\dot{G}_\alpha$, so does $(p,(s,\dot{A}))$, and hence $b_\alpha \leq b_\alpha^\gamma$. □

We are now ready to define the main forcing $\mathbb{R}$.

Definition 2.8 *Conditions in $\mathbb{R}$ are triples (p,q,r) which satisfy the following (where B is as in Lemma 2.5):*

(i) (p,q) *is a condition in* $\mathbb{P} * \mathbb{Q}$.
(ii) r *is a function with* $\mathrm{dom}(r) \subseteq B$ *and* $|\mathrm{dom}(r)| \leq \kappa$ *such that for every* $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ *is a nice* $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}_\alpha^\pi$*-name and:*

$$\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}_\alpha^\pi \Vdash r(\alpha) \in \mathrm{Add}(\kappa^+, 1).$$

The ordering is defined as follows: $(p',q',r') \leq (p,q,r)$ *if the following hold:*

(i) $(p',q') \leq (p,q)$ *in* $\mathbb{P} * \mathbb{Q}$.
(ii) $\mathrm{dom}(r) \subseteq \mathrm{dom}(r')$ *and for every* $\alpha \in \mathrm{dom}(r)$,

$$\sigma_\alpha^{\lambda^+}(p',q') \Vdash_{\mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha)*\mathbb{Q}_\alpha^\pi)} r'(\alpha) \leq r(\alpha).$$

The following lemmas identify the basic properties of $\mathbb{R}$.

Define $\mathbb{U}$ to consist of all elements of $\mathbb{R}$ of the form $(1,1,r)$, with the induced partial ordering. Let $\nu : (\mathbb{P} * \mathbb{Q}) \times \mathbb{U} \to \mathbb{R}$ be given by $\nu((p,q),(1,1,r)) = (p,q,r)$.

Assume μ is a regular uncountable cardinal. We say that a partial order has the *μ-Knaster property* if every family of conditions of size μ contains μ-many pairwise compatible conditions.

Lemma 2.9 *The following hold:*

(i) $\mathbb{P} * \mathbb{Q}$ *has a dense subset which has the* κ^+*-Knaster property.*
(ii) $\mathbb{U}$ *is* κ^+*-closed, i.e. every decreasing sequence of conditions in* $\mathbb{U}$ *of length less than* κ^+ *has a lower bound.*
(iii) ν *is a projection which commutes with the natural projections from* $\mathbb{R}$ *and* $(\mathbb{P} * \mathbb{Q}) \times \mathbb{U}$ *to* $\mathbb{P} * \mathbb{Q}$ *(so that in a natural way* $V[\mathbb{P} * \mathbb{Q}] \subseteq V[\mathbb{R}] \subseteq V[(\mathbb{P} * \mathbb{Q}) \times \mathbb{U}]$*).*
(iv) $V[\mathbb{R}]$ *and* $V[\mathbb{P} * \mathbb{Q}]$ *have the same* κ*-sequences.*

Proof. (i). Let Z contain all conditions of the form $(p, (\check{s}, \dot{A})$; then Z is dense in $\mathbb{P} * \mathbb{Q}$ and has the κ^+-Knaster property. (ii)–(iii) are obvious. Regarding (iv), by (i), (ii) and the Easton lemma, $\mathbb{U}$ is κ^+-distributive over $V[\mathbb{P} * \mathbb{Q}]$; then (iv) follows by (iii). □

Lemma 2.10 *The following hold:*

(i) $\mathbb{R}$ *has the* λ*-Knaster property.*
(ii) $\mathbb{R}$ *collapses cardinals in the interval* (κ^+, λ) *(and no other cardinals), making* κ^{++} *in* $V[\mathbb{R}]$ *equal to* λ*. In* $V[\mathbb{R}]$*,* $2^\kappa = \lambda^+ = \kappa^{+3}$*.*

Proof. (i). Let $Y = \{(p_\alpha, q_\alpha, r_\alpha) \,|\, \alpha < \lambda\}$ be a set of conditions in $\mathbb{R}$ of size λ. We wish to find a subset Y' of size λ which consists of pairwise compatible conditions. By a Δ-system argument there is a cofinal $a \subseteq \lambda$ such that $\{(p_\alpha, q_\alpha) \,|\, \alpha \in a\}$ is a family of pairwise compatible conditions in $\mathbb{P} * \mathbb{Q}$. By another Δ-system argument, there is a cofinal $a' \subseteq a$, and a root $r \subseteq B$ of size $\leq \kappa$, such that for all $\alpha, \beta \in a'$, $\alpha \neq \beta$, $\mathrm{dom}(r_\alpha) \cap \mathrm{dom}(r_\beta) = r$. By the inaccessibility of λ, the number of nice $\mathbb{P}|\mathrm{Even}(\gamma) * \mathbb{Q}^\pi_\gamma$-names, $\gamma \in r$, for conditions in $\mathrm{Add}(\kappa^+, 1)$ is less than λ. Hence there is a cofinal $a'' \subseteq a'$ such that if α, β are in a'', then for all $\gamma \in r$, $r_\alpha(\gamma) = r_\beta(\gamma)$. It follows $Y' = \{(p_\alpha, q_\alpha, r_\alpha) \,|\, \alpha \in a''\}$ is as required.

(ii). Obvious. □

We will need to consider truncations of $\mathbb{R}$, which we define next.

Definition 2.11 *Let* $\gamma \in A$*, and* $\lambda < \beta < \gamma$*. Conditions in* $\mathbb{R}|\gamma$ *are triples* (p, q, r) *which satisfy the following:*

(i) (p, q) *is a condition in* $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$*, where* $\mathbb{Q}_\gamma$ *is the Prikry forcing defined with respect to the measure* $\dot{U}_\gamma$*.*

(ii) r *is a function with* $\mathrm{dom}(r) \subseteq B$ *and* $|\mathrm{dom}(r)| \leq \kappa$ *such that for every* $\alpha \in \mathrm{dom}(r)$*,* $r(\alpha)$ *is a nice* $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha$*-name and:*

$$\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha \Vdash r(\alpha) \in \mathrm{Add}(\kappa^+, 1).$$

The ordering is defined as for $\mathbb{R}$*, but using the projections* σ^γ_α*,* $\alpha \in B$*.*

Lemma 2.12 *Let* γ *be in* A *and* $\beta < \gamma < \lambda^+$*. There is a projection from* $\mathbb{R}$ *to* $\mathrm{RO}^+(\mathbb{R}|\gamma)$*.*

Proof. First notice that $\mathbb{R}|\gamma$ is densely embeddable in $\hat{\mathbb{R}}|\gamma$, which is defined as $\mathbb{R}|\gamma$ but with elements of $\mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}_\gamma)$ instead of $\mathbb{P}|\gamma * \mathbb{Q}_\gamma$, and with the projection $\hat{\sigma}^\gamma_\alpha$. Because of the commutativity $\sigma^{\lambda^+}_\alpha = \hat{\sigma}^\gamma_\alpha \circ \sigma^{\lambda^+}_\gamma$, see Lemma 2.7(iii), it is easy to check that in $V^{\mathbb{R}}$, we can find a generic for $\hat{\mathbb{R}}|\gamma$. □

We now show that if $\mathbb{R}$ adds an Aronszajn tree on λ, then a truncation $\mathbb{R}|\beta^*$ for a certain β^* must add an Aronszajn tree on λ.

Before we give the lemma, let us define some terminology. Let (p, q) be a condition in $\mathbb{P} * \mathbb{Q}$; without loss of generality, q is of the form $(s, \dot{E})$ for some finite subset s of κ and some nice $\mathbb{P}$-name $\dot{E}$ for a subset of κ. We say that a coordinate $\alpha < \lambda^+$ is in the support of (p, q) if α is in the support of p or in the support of some p' which occurs in the nice name $\dot{E}$.

Lemma 2.13 *Suppose* $\mathbb{R}$ *forces that there is an Aronszajn tree on* λ*. Then for some* β^* *in* A*,* $\beta < \beta^*$*,* $\mathbb{R}|\beta^*$ *forces there is an Aronszajn tree on* λ*.*

Proof. Let $\dot{T}$ be a nice name for a subset of λ which in some natural way corresponds to an Aronszajn tree on λ, which we assume exists in $V^{\mathbb{R}}$. $\dot{T}$ is of the form $\bigcup\{\{\alpha\} \times K_\alpha \,|\, \alpha < \lambda\}$, where K_α for $\alpha < \lambda$ is an antichain in $\mathbb{R}$. By the λ-Knaster property, $|K_\alpha| < \lambda$ for every $\alpha < \lambda$. It follows there are at most λ many coordinates $\alpha < \lambda^+$ which are in the support of (p,q) such that for some r, $(p,q,r) \in \bigcup_{\alpha<\lambda} K_\alpha$ (we say that α is *in the support of* $\dot{T}$). Hence we can choose β^* in A such that $\beta < \beta^*$, and $\mathbb{R}|\beta^*$ forces that $\dot{T}'$ is an Aronszajn tree on λ, for some name $\dot{T}'$ which is naturally obtained from $\dot{T}$. □

Suppose now that $\mathbb{R}$ does force that there is an Aronszajn tree on λ and let us fix β^* as above (we will later show that the assumption that $\mathbb{R}$ adds an Aronszajn tree on λ leads to a contradiction).

Let π^* be an isomorphism between $\mathbb{P}|\beta^*$ and $\mathbb{P}|\lambda$; choose π^* so that it extends π (the fixed isomorphism between $\mathbb{P}|\beta$ and $\mathbb{P}|\mathrm{Even}(\lambda)$). This implies $\pi(\dot{U}_\beta) = \pi^*(\dot{U}_\beta)$, and therefore the measure $\pi^*(\dot{U}_{\beta^*})$ is forced to extend the measure $\pi(\dot{U}_\beta)$. More precisely, if $G|\beta^*$ is $\mathbb{P}|\beta^*$-generic, then the following hold:

- (i) $\bar{G} = \pi^*(G|\beta^*)$ is $\mathbb{P}|\lambda$-generic and its restriction to its even coordinates, to be denoted as $\bar{G}|\mathrm{Even}(\lambda)$, is equal to $\pi(G|\beta)$ (and $\bar{G}|\mathrm{Even}(\lambda)$ is $\mathbb{P}|\mathrm{Even}(\lambda)$-generic).
- (ii) The measure $\pi(\dot{U}_\beta)^{\bar{G}|\mathrm{Even}(\lambda)}$ in $V[\bar{G}|\mathrm{Even}(\lambda)]$ is extended by the measure $\pi^*(\dot{U}_{\beta^*})^{\bar{G}}$ in $V[\bar{G}]$.

Define $\mathbb{Q}^{\pi^*}_\lambda$ as the Prikry forcing in $\mathbb{P}|\lambda$ with the measure $\pi^*(\dot{U}_{\beta^*})$.

Lemma 2.14 *(i) π^* extends to an isomorphism from $\mathbb{P}|\beta^* * \mathbb{Q}_{\beta^*}$ onto $\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_\lambda$.*

(ii) For every $\alpha \in B$, $\sigma^\lambda_\alpha = \sigma^{\beta^*}_\alpha {}^\frown (\pi^*)^{-1}$ is a projection

$$\sigma^\lambda_\alpha : \mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_\lambda \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha). \tag{2.16}$$

Proof. (i). Let us view $\mathbb{Q}_{\beta^*}$ as a collection of conditions $(p,(s,\dot{A}))$, where $\dot{A}$ is a nice name. It is clear that we can naturally extend π^* so that $\pi^*(\dot{A})$ is a nice name in $\mathbb{P}|\lambda$. Moreover, since π^* is an isomorphism, $\mathbb{P}|\beta^*$ forces that $\dot{A}$ is in $\dot{U}_{\beta^*}$ if and only if $\pi(\dot{A})$ is in $\pi^*(\dot{U}_{\beta^*})$.

(ii). This is clear because $(\pi^*)^{-1}$ is an isomorphism. □

Let us define the following variant of $\mathbb{R}$, and call it $\mathbb{R}^*$:

Definition 2.15 *Conditions in $\mathbb{R}^*$ are triples (p,q,r) which satisfy the following:*

(i) (p,q) *is a condition in* $\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_\lambda$.

(ii) r *is a function with* $\mathrm{dom}(r) \subseteq B$ *and* $|\mathrm{dom}(r)| \leq \kappa$ *such that for every* $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ *is a nice* $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha$*-name and:*

$$\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha \Vdash r(\alpha) \in \mathrm{Add}(\kappa^+, 1).$$

The ordering is defined by means of the projections σ^λ_α, $\alpha \in B$.

Lemma 2.16 $\mathbb{R}|\beta^*$ *and* $\mathbb{R}^*$ *are isomorphic.*

Proof. Define $f : \mathbb{R}|\beta^* \to \mathbb{R}^*$ by assigning to $(p, (s, \dot{A}), r)$ the condition $(\pi^*(p), (s, \pi^*(\dot{A})), r)$, where $(s, \dot{A})$ is a condition in $\mathbb{Q}_{\beta^*}$. Since σ^λ_α is determined by π^* and $\sigma^{\beta^*}_\alpha$ (2.16), it is easy to check that f is an isomorphism. □

By Lemma 2.13, it follows that if $\mathbb{R}$ adds an Aronszajn tree on λ, $\mathbb{R}^*$ adds an Aronszajn tree on λ. In Stage 2, we show that this cannot happen.

2.2 Stage 2

We verify that the method of [1] can be applied in our case to verify that $\mathbb{R}^*$ does not add an Aronszajn tree at λ. In the argument, we use ideas from [6] to fill some gaps in [1].

In order to carry out the analysis of $\mathbb{R}^*$, we need to be able to define truncations $\mathbb{R}^*|\alpha$ for a large set $B^* \subseteq B$ below λ. First we apply the construction in Lemma 2.5 to the measure $\pi^*(\dot{U}_{\beta^*})$ in $\mathbb{P}|\lambda$, and obtain an unbounded set B^* below λ where the measure $\pi^*(\dot{U}_{\beta^*})$ reflects. Using the closure at points of cofinality $> \kappa$, one can in fact refine to get $B^* \subseteq B$. For $\alpha \in B^*$, define $\mathbb{Q}^{\pi^*}_\alpha$ as the Prikry forcing defined with respect to the restriction of the measure $\pi^*(\dot{U}_{\beta^*})$. Denote $\hat{B}^* = B^* \cup \{\lambda\}$. We now proceed as in Lemma 2.7, and in particular using Lemma 1.1, to obtain for every $\alpha < \gamma$ in $\hat{B}^*$ projections:

$$\varrho^\gamma_\alpha : \mathbb{P}|\gamma * \mathbb{Q}^{\pi^*}_\gamma \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha) \tag{2.17}$$

and

$$\hat{\varrho}^\gamma_\alpha : \mathrm{RO}^+(\mathbb{P}|\gamma * \mathbb{Q}^{\pi^*}_\gamma) \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha), \tag{2.18}$$

which moreover satisfy:

$$\varrho^\lambda_\alpha = \hat{\varrho}^\gamma_\alpha \circ \varrho^\lambda_\gamma. \tag{2.19}$$

Recall we used projections σ^λ_α, $\alpha \in B$, to define the forcing $\mathbb{R}^*$. We show that σ^λ_α is the same projection as ϱ^λ_α for $\alpha \in B^*$, and therefore we can view $\mathbb{R}^*$ as being defined with the projections ϱ^λ_α, $\alpha \in B^*$.

Lemma 2.17 *For $\alpha \in B^*$, $\sigma^\lambda_\alpha = \varrho^\lambda_\alpha$.*

Proof. Let us fix $\alpha \in B^*$ and a condition (p,q) in $\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_\lambda$, and let us temporarily denote $\mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha)$ by $B^{\mathrm{Even}(\alpha)}$. Let F be a $\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_\lambda$-generic filter, F^* a $\mathbb{P}|\beta^* * \mathbb{Q}_{\beta^*}$-generic filter, and let $\dot{F}|\mathrm{Even}(\alpha)$ and $\dot{F}^*|\mathrm{Even}(\alpha)$ be the canonical names for the $B^{\mathrm{Even}(\alpha)}$-generic filters existing in $V[F]$ and $V[F^*]$, respectively. Since π^* extends π, it is clear that for every $b \in B^{\mathrm{Even}(\alpha)}$,

$$(p,q) \Vdash b \in \dot{F}|\mathrm{Even}(\alpha) \Leftrightarrow (\pi^*)^{-1}(p,q) \Vdash b \in \dot{F}^*|\mathrm{Even}(\alpha),$$

and therefore

$$\varrho^\lambda_\alpha(p,q) = \bigwedge\{b \in B^{\mathrm{Even}(\alpha)} \mid (p,q) \Vdash b \in \dot{F}|\mathrm{Even}(\alpha)\} = \\ \bigwedge\{b \in B^{\mathrm{Even}(\alpha)} \mid (\pi^*)^{-1}(p,q) \Vdash b \in \dot{F}^*|\mathrm{Even}(\alpha)\} = \sigma^\lambda_\alpha(p,q),$$

as desired. $\square$

Now we can define truncations $\mathbb{R}|\gamma$ for $\gamma \in B^*$:

Definition 2.18 *For $\gamma \in B^*$, define $\mathbb{R}^*|\gamma$ as follows. Conditions in $\mathbb{R}^*|\alpha$ are triples (p,q,r):*

(i) (p,q) is a condition in $\mathbb{P}|\gamma * \mathbb{Q}^{\pi^*}_\gamma$.

(ii) r is a function with $\mathrm{dom}(r) \subseteq B^* \cap \gamma$ and $|\mathrm{dom}(r)| \leq \kappa$ such that for every $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a nice $\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha$-name and:

$$\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha \Vdash r(\alpha) \in \mathrm{Add}(\kappa^+, 1).$$

The ordering is defined as follows: $(p', q', r') \leq (p, q, r)$ *if the following hold:*

(i) $(p', q') \leq (p, q)$ *in* $\mathbb{P} * \mathbb{Q}$.

(ii) $\mathrm{dom}(r) \subseteq \mathrm{dom}(r')$ *and for every* $\alpha \in \mathrm{dom}(r)$,

$$\varrho^{\gamma}_{\alpha}(p', q') \Vdash_{\mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^{\pi}_{\alpha})} r'(\alpha) \leq r(\alpha).$$

Lemma 2.19 *For every* $\gamma \in B^*$ *there exists a projection from* $\mathbb{R}^*$ *to* $\mathrm{RO}^+(\mathbb{R}^*|\gamma)$.

Proof. It follows as in Lemma 2.12, using the fact that $\sigma^{\lambda}_{\alpha} = \varrho^{\lambda}_{\alpha}$, $\alpha \in B^*$ (see Lemma 2.17). □

The analysis in Lemma 2.9 can be applied to $\mathbb{R}^*$ straightforwardly. Let $\mathbb{U}^*$ denote the κ^+-closed forcing such that there is a projection from $(\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_{\lambda}) \times \mathbb{U}^*$ to $\mathbb{R}^*$. By arguments similar to Lemma 2.9 and 2.10, for an inaccessible $\alpha \in \hat{B}^*$, $\mathbb{R}^*|\alpha$ preserves all cardinals except in the interval (κ^+, α) and forces $2^{\kappa} = \alpha$. Moreover, $V[\mathbb{R}^*|\alpha]$ is a submodel of $V[\mathbb{R}^*]$ and every bounded subset of λ in $V[\mathbb{R}^*]$ appears in $V[\mathbb{R}^*|\alpha]$, for some $\alpha \in B^*$.

The existence of $\mathbb{U}^*$ generalizes to the truncations $\mathbb{R}^*|\alpha$, $\alpha \in B^*$.

Lemma 2.20 *Let* α *be in* B^*. *Then* $\mathbb{R}/(\mathbb{R}|\alpha)$ *is in* $V[\mathbb{R}|\alpha]$ *a projection of* $(\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_{\lambda}/\mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_{\alpha}) \times \mathbb{U}^*_{\alpha}$ *for some* κ^+*-closed forcing* $\mathbb{U}^*_{\alpha}$ *in* $\mathbb{R}|\alpha$.

Proof. Obvious. □

Following [1, Lemma 6.5], and the correction in [6], the proof is finished by showing that for every $\alpha \in B^*$, the product $(\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_{\lambda}/\mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_{\alpha}) \times (\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_{\lambda}/\mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_{\alpha})$ ("the square of $(\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_{\lambda}/\mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_{\alpha})$") is κ^+-cc in $V[\mathbb{R}^*|\alpha]$ (this result is stated as Lemma 2.27). The importance of the square of the forcing being κ^+-cc follows from the following Lemma due to Spencer [6] which we give with a proof for the convenience of the reader (our proof is possibly a bit simpler).

Lemma 2.21 *Suppose* γ *is regular and a forcing* P *adds a subset* x *of* γ *such that* x *is not in* V *but* $x \cap \alpha$ *is in* V *for all* $\alpha < \gamma$. *Then* $P \times P$ *is not* γ*-cc.*

PROOF. It suffices to show that if G is P-generic then P is not γ-cc in $V[G]$. In $V[G]$ let $x = (\dot{x})^G$ be a subset of γ as in the hypothesis and choose a sequence $\langle p_i \,|\, i < \gamma \rangle$ of conditions in G and an increasing sequence of ordinals $\langle \alpha_i \,|\, i < \gamma \rangle$ less than γ such that p_i fixes $\dot{x} \cap \alpha_i$ (i.e. forces it to equal a specific element of V) but does not fix $\dot{x} \cap \alpha_{i+1}$. This is possible as $x \cap \alpha$ is fixed by some condition in G for each $\alpha < \gamma$ but x itself is fixed by no condition in G. Now choose q_{i+1} extending p_i to disagree with p_{i+1} about $\dot{x} \cap \alpha_{i+1}$. This is possible as p_i does not fix $\dot{x} \cap \alpha_{i+1}$. But then the q_{i+1}'s form an antichain as any condition extending q_{i+1} disagrees with p_{i+1} (and therefore with p_j for all $j > i$) about $\dot{x}$ and therefore cannot extend q_{j+1} for $j > i$, as q_{j+1} extends p_j. □

Since the argument in [6] is stated for a different forcing, we provide a self-contained proof of Lemma 2.27. For the proof of Lemma 2.27, we need to prove some preliminary facts (Lemma 2.22 – Lemma 2.26).

Lemma 2.22 *Assume $(p, (s, \dot{A})) \in \mathbb{P}|\alpha * \mathbb{Q}_\alpha^{\pi^*}$ and $(q, (t, \dot{B})) \in \mathbb{P}|\lambda * \mathbb{Q}_\lambda^{\pi^*}$ are arbitrary conditions. Then $(p, (s, \dot{A}))$ forces that $(q, (t, \dot{B}))$ is not a condition in $\mathbb{P}|\lambda * \mathbb{Q}_\lambda^{\pi^*} / \mathbb{P}|\alpha * \mathbb{Q}_\alpha^{\pi^*}$ if and only if one of the following conditions holds:*

(i) *$q \restriction \alpha$ is incompatible with p,*

(ii) *$q \restriction \alpha$ is compatible with p, s does not extend t and t does not extend s,*

(iii) *$q \restriction \alpha$ is compatible with p, s extends t and $q \cup p \Vdash s \setminus t \nsubseteq \dot{B}$,*

(iv) *$q \restriction \alpha$ is compatible with p, t extends s and $(q \restriction \alpha) \cup p \Vdash t \setminus s \nsubseteq \dot{A}$.*

PROOF. $(p, (s, \dot{A})) \Vdash (q, (t, \dot{B})) \notin \mathbb{P}|\lambda * \mathbb{Q}_\lambda^{\pi^*} / \mathbb{P}|\alpha * \mathbb{Q}_\alpha^{\pi^*}$ if and only if there is no generic filter $G * x$ such that $(q, (t, \dot{B})) \in G * x$ and $(p, (s, \dot{A})) \in G \restriction \alpha * x$.

First, it is easy to see that each of the conditions above rules out the existence of such a generic filter $G * x$.

Second, assume that all conditions above fail. Then p is compatible with q and it has to hold that either s extends t or t extends s. If s extends t, then $q \cup p \nVdash s \setminus t \nsubseteq \dot{B}$. This means that there is r below $q \cup p$ such that $r \Vdash s \setminus t \subseteq \dot{B}$. Consider the condition $(r, (s, \dot{A} \cap \dot{B}))$ and let $G * x$ be generic filter such that $(r, (s, \dot{A} \cap \dot{B})) \in G * x$. It is easy to verify that $(q, (t, \dot{B})) \in G * x$ and $(p, (s, \dot{A})) \in G \restriction \alpha * x$. The second case, if t extends s, is similar. □

We have just characterised the case when a condition in $\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_{\lambda}$ is forced out of the quotient. Now, we focus on the case when a condition is forced into the quotient. First we prove an auxiliary lemma.

Lemma 2.23 *Assume $(p, (s, \dot{A})) \in \mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_{\alpha}$ and $(q, (t, \dot{B})) \in \mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_{\lambda}$ are arbitrary conditions such that $p \leq q \restriction \alpha$. Then there is a $\mathbb{P}|\alpha$-name $\dot{C}$ such that p forces that $\dot{C}$ is in $\dot{U}^{\pi^*}_{\alpha}$ and for each finite set x in $\dot{C}$ such that $s \cup x$ is a stem,[7] $q \nVdash^{V[\mathbb{P}|\alpha]}_{\mathbb{P}|\lambda/\mathbb{P}|\alpha} x \nsubseteq \dot{B}$.*

Proof. Assume that G is a $\mathbb{P}|\alpha$-generic filter such that $p \in G$. We define a colouring of $[\kappa]^{<\omega}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } s \cup x \text{ is a stem and } q \nVdash^{V[G]}_{\mathbb{P}|\lambda/\mathbb{P}|\alpha} x \nsubseteq \dot{B}; \\ 1 & \text{if } s \cup x \text{ is a stem and } q \Vdash^{V[G]}_{\mathbb{P}|\lambda/\mathbb{P}|\alpha} x \nsubseteq \dot{B}; \\ 2 & \text{if } s \cup x \text{ is not a stem.} \end{cases}$$

By Rowbottom's theorem, there is a set C in $U^{\pi^*}_{\alpha}$ homogeneous for f. Let $\dot{C}$ be a $\mathbb{P}|\alpha$-name for C.

Assume for contradiction that the Lemma does not hold for $\dot{C}$. Then there is some $r \leq p$ and a finite set x in $\dot{C}$ such that r forces that $q \Vdash^{V[\mathbb{P}|\alpha]}_{\mathbb{P}|\lambda/\mathbb{P}|\alpha} x \nsubseteq \dot{B}$. Let n be the size of x. Since $\dot{C}$ is a homogeneous set for f, we know that for each set y of size n, $f(y) = 1$. As we assume $p \leq q \restriction \alpha$, $r \cup q$ is a condition in $\mathbb{P}|\lambda$. Let H be a $\mathbb{P}|\lambda$-generic filter which contains $r \cup q$. Then $\dot{B}^{V[H]} \cap \dot{C}^{V[H]} = \emptyset$ in $V[H]$. This is a contradiction since $\dot{B}^{V[H]}$ and $\dot{C}^{V[H]}$ are sets in $(\dot{U}^{\pi^*}_{\lambda})^{V[H]}$. □

[7] $s \cup x$ is a stem if $s \cup x$ extends s.

Now we provide a sufficient condition for a condition in $\mathbb{P}|\lambda * \mathbb{Q}_\lambda^{\pi *}$ to be forced into the quotient.

Lemma 2.24 *Assume $(p,(s,\dot{A})) \in \mathbb{P}|\alpha * \mathbb{Q}_\alpha^{\pi^*}$ and $(q,(t,\dot{B})) \in \mathbb{P}|\lambda * \mathbb{Q}_\lambda^{\pi^*}$ are arbitrary conditions. If they satisfy the following conditions*

(i) s extends t,

(ii) $p \leq q \restriction \alpha$ and

(iii) $q \cup p \Vdash s \setminus t \subseteq \dot{B}$,

then for the set $\dot{C}$ from Lemma 2.23, it holds that $(p,(s,\dot{A} \cap \dot{C}))$ forces $(q,(t,\dot{B}))$ into the quotient $\mathbb{P}|\lambda * \mathbb{Q}_\lambda^{\pi^*}/\mathbb{P}|\alpha * \mathbb{Q}_\alpha^{\pi^*}$.

Proof. For contradiction assume that there is $(p',(s',\dot{A}')) \leq (p,(s,\dot{A} \cap \dot{C}))$ such that $(p',(s',\dot{A}')) \Vdash (q,(t,\dot{B})) \notin \mathbb{P}|\lambda * \mathbb{Q}_\lambda^{\pi^*}/\mathbb{P}|\alpha * \mathbb{Q}_\alpha^{\pi^*}$. It means that one of the conditions in Lemma 2.22 is true. It is easy to see that conditions (i), (ii) and (iv) are false, hence the condition (iii) has to be true. Thus $p' \cup q \Vdash s' \setminus t \nsubseteq \dot{B}$. By the assumptions (i) and (iii) of the present lemma, s extends t and $p \cup q \Vdash s \setminus t \subseteq \dot{B}$. Therefore $p' \cup q \Vdash s' \setminus s \nsubseteq \dot{B}$.

Since $(p',(s',\dot{A}')) \leq (p,(s,\dot{A} \cap \dot{C}))$, $p' \Vdash s' \setminus s \subseteq \dot{A} \cap \dot{C}$. Hence $p' \Vdash s' \setminus s \subseteq \dot{C}$. By Lemma 2.23, we know that p' forces $q \nVdash^{V[\mathbb{P}|\alpha]}_{\mathbb{P}|\lambda/\mathbb{P}|\alpha} s' \setminus s \nsubseteq \dot{B}$. Therefore $p' \cup q \nVdash s' \setminus s \nsubseteq \dot{B}$. This is in contradiction with the result of the previous paragraph. □

Lemma 2.25 *Assume $(p,(s,\dot{A}))$ is a condition in $\mathbb{P}|\alpha * \mathbb{Q}_\alpha^{\pi^*}$, and $\dot{r}_i$ for $i < 2$, are conditions forced by the weakest condition of $\mathbb{P}|\alpha * \mathbb{Q}_\alpha^{\pi^*}$ into the quotient $\mathbb{P}|\lambda * \mathbb{Q}_\lambda^{\pi^*}/\mathbb{P}|\alpha * \mathbb{Q}_\alpha^{\pi^*}$. Then there are $(p',(s',\dot{A}')) \leq (p,(s,\dot{A}))$, $(q_i,(t_i,\dot{B}_i))$ and $\bar{q}_i \leq q_i$, $i < 2$, such that for $i < 2$:*

(i) $(p,(s,\dot{A}))$ decides $\dot{r}_i$ to be $(q_i,(t_i,\dot{B}_i))$,

(ii) $(p,(s,\dot{A}))$ and $(\bar{q}_i,(t_i,\dot{B}_i))$ satisfy the assumptions (i)–(iii) of Lemma 2.24.

Proof. Let $(p', (s', \dot{A}')) \leq (p, (s, \dot{A}))$ be such that it decides the value of $\dot{r}_i$ to be $(q_i, (t_i, \dot{B}_i))$ for $i < 2$. We may assume that s' extends t_i and $p' \leq q_i \restriction \alpha$ for $i < 2$. Since $(p', (s', \dot{A}'))$ forces that $\dot{r}_0$ is in $\mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_\alpha$, the condition (iii) in Lemma 2.22 has to fail, hence there is $\bar{q}_0 \leq p' \cup q_0$ such that $\bar{q}_0$ forces $s' \setminus t_0 \subseteq \dot{B}_0$. Now, if it is necessary we can extend p' such that $p' \leq \bar{q}_0 \restriction \alpha$.

Now, we need to deal with $\dot{r}_1 = (q_1, (t_1, \dot{B}_1))$. Since $(p', (s', \dot{A}'))$ forces that $\dot{r}_1$ is in $\mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_\alpha$, the condition (iii) in Lemma 2.22 has to fail. Therefore there is $\bar{q}_1 \leq p' \cup q_1$ such that $\bar{q}_1$ forces $s' \setminus t_1 \subseteq \dot{B}_1$. Again, if it is necessary we can extend p' so that $p' \leq \bar{q}_1 \restriction \alpha$. □

Lemma 2.26 *$(\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_\lambda)^2 \times (\mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_\alpha)$ is κ^+-cc.*

Proof. Obvious. □

Finally we can prove the desired lemma which finishes the proof of Theorem 2.1.

Lemma 2.27 *For every $\alpha \in B^*$, the square of $\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_\alpha$ is κ^+-cc in $V[\mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_\alpha]$.*

Proof. For contradiction assume that $\{(\dot{r}^0_\beta, \dot{r}^1_\beta) \mid \beta < \kappa^+\}$ is a $\mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_\alpha$-name for an antichain in $\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_\lambda / \mathbb{P}|\alpha * \mathbb{Q}^{\pi^*}_\alpha$. By Lemma 2.25, we can find for each $\beta < \kappa^+$ and $i < 2$ conditions $(p_\beta, (s_\beta, \dot{A}_\beta))$, $(q^i_\beta(t^i_\beta, \dot{B}^i_\beta))$ and extensions $\bar{q}^i_\beta \leq q^i_\beta$ which satisfy items (i) and (ii) in Lemma 2.25.

By Lemma 2.26, there are $\beta < \beta' < \kappa^+$ such that p_β is compatible with $p_{\beta'}$ and $\bar{q}^i_\beta$ is compatible with $\bar{q}^i_{\beta'}$ for $i < 2$. This means that $p_\beta \cup p_{\beta'}$ and $\bar{q}^i_\beta \cup \bar{q}^i_{\beta'}$ for $i < 2$ are conditions in $\mathbb{P}|\lambda$. Moreover we may assume that $t^i = t^i_\beta = t^i_{\beta'}$ and $s = s_\beta = s_{\beta'}$ for $i < 2$.

For $i < 2$, the conditions $(p_\beta \cup p_{\beta'}, (s, \dot{A}_\beta \cap \dot{A}_{\beta'}))$ and $(\bar{q}^i_\beta \cup \bar{q}^i_{\beta'}, (t^i, \dot{B}^i_\beta \cap \dot{B}^i_{\beta'}))$ satisfy the assumptions of Lemma 2.24. Therefore, there is an extension of $(p_\beta \cup p_{\beta'}, (s, \dot{A}_\beta \cap \dot{A}_{\beta'}))$ which forces the compatibility of $(\dot{r}^0_\beta, \dot{r}^1_\beta)$ and $(\dot{r}^0_{\beta'}, \dot{r}^1_{\beta'})$ in the quotient. This is a contradiction. □

This finishes the proof of Theorem 2.1.

3 An arbitrary gap

Theorem 3.1 *Assume GCH and let κ be a Laver-indestructible supercompact cardinal, λ a weakly compact cardinal and μ a cardinal of cofinality greater than κ such that $\kappa < \lambda < \mu$. Then there is a forcing notion $\mathbb{R}$ such that the following hold:*

(i) $\mathbb{R}$ *preserves cardinals* $\leq \kappa^+$ *and* $\geq \lambda$.
(ii) $V[\mathbb{R}] \models (\kappa^{++} = \lambda \,\&\, 2^\kappa = \mu \,\&\, \mathrm{cf}(\kappa) = \omega \,\&\,$ *κ is strong limit*$)$.
(iii) $V[\mathbb{R}] \models \mathrm{TP}(\lambda)$.

We will not give a detailed proof, but instead specify what modifications to the proof of Theorem 2.1 are needed to prove Theorem 3.1. Assume the notation is the same as in the proof of Theorem 2.1 unless said otherwise.

Modify the construction in Stage 1 in Section 2.1 as follows:

(1) In analogy with Lemma 2.3, find a set $A \subseteq [\mu]^\lambda$ which is unbounded in $[\mu]^\lambda$ and closed under unions of increasing chains of cofinality larger than κ which satisfies:

- For every $x \in A$, $\lambda + 1 \subseteq x$.
- For every $x \in A$, there is a name $\dot{U}_x$ such that in $V[\mathbb{P}|x]$, $\dot{U}_x$ interprets as the restriction of the measure $\dot{U}$ on κ. Let us denote by $\mathbb{P}|x * \mathbb{Q}_x$ the Cohen forcing restricted to x followed by the Prikry forcing with the measure $\dot{U}_x$.

(2) Choose an arbitrary $x_0 \in A$ and an isomorphism $\pi : \mathbb{P}|x_0 \to \mathbb{P}|\mathrm{Even}(\lambda)$. Thus $\pi(\dot{U}_{x_0})$ is a measure in $\mathbb{P}|\mathrm{Even}(\lambda)$.

(3) Denote $\hat{A} = \{y \in A \,|\, x_0 \subseteq y\}$. As in Lemma 2.7, and with the notation naturally modified for the current situation, there is an unbounded set $B \subseteq \lambda$ closed under limits of cofinality larger than κ, and commutative projections

$$\sigma^\mu_y : \mathbb{P} * \mathbb{Q} \to \mathrm{RO}^+(\mathbb{P}|y * \mathbb{Q}_y), \text{ for } y \in \hat{A},$$

$$\hat{\sigma}^y_\alpha : \mathrm{RO}^+(\mathbb{P}|y * \mathbb{Q}_y) \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha), \text{ for } y \in \hat{A}, \alpha \in B,$$

and

$$\sigma^\mu_\alpha : \mathbb{P} * \mathbb{Q} \to \mathrm{RO}^+(\mathbb{P}|\mathrm{Even}(\alpha) * \mathbb{Q}^\pi_\alpha), \text{ for } \alpha \in B$$

with

$$\sigma^\mu_\alpha = \hat{\sigma}^y_\alpha \circ \sigma^\mu_y, \text{ for } y \in \hat{A}, \alpha \in B.$$

Note that we denote by $\mathbb{Q}^\pi_\alpha$ the Prikry forcing defined with respect to the restriction of the measure $\pi(\dot{U}_{x_0})$ to $V[\mathbb{P}|\mathrm{Even}(\alpha)]$.

(4) Modify Definition 2.8 of $\mathbb{R}$ to use π and σ^μ_α, in the sense of the previous paragraph. As in Definition 2.11, define the truncations $\mathbb{R}|y$ for $y \in \hat{A}$.

(5) The key step is to show that if $\dot{T}$ is a λ-Aronszajn tree added by $\mathbb{R}$, then for some $y \in \hat{A}$, $\mathbb{R}|y$ adds an Aronszajn tree on λ and importantly, $\mathbb{R}|y$ is isomorphic to $\mathbb{R}^*$ (which is the same forcing as in Definition 2.15). We argue as follows:

By the λ-Knaster property of (a dense subset of) $\mathbb{R}$, there is $y \in \hat{A}$ such that the support of $\dot{T}$ (see the paragraph after Lemma 2.12) is included in y. Choose a bijection π^* extending π, $\pi^* : \mathbb{P}|y \to \mathbb{P}|\lambda$. Denote $\mathbb{Q}^{\pi^*}_\lambda$ the Prikry forcing in $V[\mathbb{P}|\lambda]$ defined with respect to the measure $\pi^*(\dot{U}_y)$. As in Lemma 2.14, π^* extends to an isomorphism between $\mathbb{P}|y * \mathbb{Q}_y$ and $\mathbb{P}|\lambda * \mathbb{Q}^{\pi^*}_\lambda$. Finally, as in Lemma 2.14, $\mathbb{R}|y$ is isomorphic to $\mathbb{R}^*$.

Stage 2 of the argument is exactly the same as in the proof of Theorem 2.1.

4 Open questions

It is natural to ask the following questions:

(1) Can we add collapses to the Prikry forcing to get $\kappa = \aleph_\omega$ in Theorem 3.1?

(2) Can we prove Theorem 3.1 assuming just the hypermeasurability of κ?

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