The tree property at $\aleph_{\omega+2}$ with a finite gap

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Abstract: Let n be a natural number, $2 \leq n < \omega$. We show that it is consistent to have a model of set theory where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+n}$, and the tree property holds at $\aleph_{\omega+2}$; we use a hypermeasurable cardinal of an appropriate degree for the result and a variant of the Mitchell forcing followed by the Prikry forcing with collapses.

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1 Introduction

Let κ be a regular cardinal. We say that the tree property holds at κ if every κ-tree has a cofinal branch. The tree property is a compactness property which can hold at successor cardinals low in the set-theoretical hierarchy: Mitchell first showed in [14] that it is equiconsistent with the existence of a weakly compact cardinal that $\aleph_2$ has the tree property (his argument readily generalises to any κ^{++} for an infinite regular cardinal κ).

The situation is more complex when we wish to get the tree property at the double successor of a singular strong limit cardinal κ. First, since one needs to have $2^\kappa > \kappa^+$ (and thus the failure of SCH, the singular cardinal hypothesis), it is known that a measurable cardinal of high Mitchell order is required. Second, a new idea is required which connects the Mitchell construction and the known ways of obtaining the failure of SCH. This was first achieved by Cummings and Foreman [4] who proved that it is consistent to have a singular strong limit cardinal κ of countable cofinality with the tree property at κ^{++}. The cardinal κ in [4] was supercompact in the ground model; without a proof, [4] claimed that κ can be collapsed to $\aleph_\omega$ using a similar argument. However, for some time no such proof had been found.

The first argument which yields a model where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2}$, and the tree property holds at $\aleph_{\omega+2}$, was given by Friedman and Halilović in [5]. The argument started with a much weaker hypothesis than [4] (an $H(\lambda)$-hypermeasurable κ for a weakly compact $\lambda > \kappa$), and used an iteration of the κ-Sacks forcing followed by the Prikry forcing and therefore was quite different in spirit from the construction in [4]. Also, the use of the κ-Sacks forcing restricts the value of $2^{\aleph_\omega}$ to $\aleph_{\omega+2}$ (due to its support of size κ).

In the present paper, we show a generalization of these results starting with modest large cardinal assumptions: We construct a model where $\aleph_\omega$ is strong limit, violates SCH with a prescribed finite gap at $\aleph_\omega$, and the tree property holds at $\aleph_{\omega+2}$. Let us summarize the basic steps of the proof with a fixed $n < \omega$, where $2^{\aleph_\omega} = \aleph_{\omega+2+n}$ holds in the final model with the tree property at $\aleph_{\omega+2}$:

- We start with a suitably large hypermeasurable cardinal κ and a weakly compact $\lambda > \kappa$ and carry out a preparation which ensures that the largeness of κ is indestructible under the iteration of the Mitchell forcing below and at κ. Let us denote this preparation by Q^n and the iteration of the Mitchell forcing below κ by P^n_κ.
- After the forcing $Q^n * P^n_\kappa$, λ is no longer strong limit but retains enough of weak-compactness to continue with the argument.

- In the generic extension by $Q^n * P^n_\kappa$, we use a variant of the Mitchell forcing at κ to force $2^\kappa = \kappa^{+2+n}$ for the prescribed n, followed by the Prikry forcing with collapses to turn κ into $\aleph_\omega$: if $\mathbb{M}(\kappa, \lambda, \lambda^{+n})$ denotes the Mitchell forcing and $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ the Prikry forcing with collapses, we can write the whole iteration as

$$\mathbb{P}^n = Q^n * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n}) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g). \tag{1.1}$$

See Section 4.3 for more details regarding the outline of the proof for $n = 0$ and Section 5 for $n \geq 1$.

Let us briefly comment on how our proof compares with other approaches in literature. In [4], instead of putting the Prikry forcing after the Mitchell forcing, as we do, the authors integrated the Prikry part with the Mitchell forcing (let us call this forcing a "Prikry-ised Mitchell forcing"). We used the "Prikry-ised Mitchell" method in [6] to obtain a large value of 2^κ with the tree property at κ^{++}, κ strong-limit with countable cofinality. However, when the collapsing is involved, it is easy to see that the Prikry forcing with collapses must come after the Mitchell forcing, and cannot be integrated into the Mitchell part. In an earlier version of the present paper, we attempted to find a generalization of the product-analysis of the "Prikry-ised Mitchell forcing" in [4] and apply it to the different setting of Mitchell followed by Prikry: Roughly speaking, we attempted to find in the relevant model a projection to the Mitchell forcing followed by the Prikry forcing of a dense part of the product of two forcings $P_1 \times P_2$, with P_1 having nice closure (κ^+-closure) and P_2 having nice chain condition (the square of P_2 should be κ^+-cc). However, we encountered numerous obstacles and finally decided to abandon this approach: instead, we use a "hands-on" approach as in [5] with a direct analysis of the quotient $\mathbb{N}$ which we show does not add cofinal branches to λ-trees (see Section 4.5).

Let us briefly comment on the structure of the paper. Recall that $n < \omega$ is the parameter of the construction and determines the gap at $\aleph_\omega$ in the final model, i.e. $2^{\aleph_\omega}$ is equal to $\aleph_{\omega+2+n}$ in the final model. In Section 2 we review the forcings which we will use in the rest of the paper. In Section 3, we argue that it is possible to start with a hypermeasurable cardinal κ of a suitable degree and prepare the ground model so that an iteration of the Mitchell forcing below and at κ does not destroy the measurability of κ.[1] In Section 4 we deal with case $n = 0$: We show that over the prepared ground model, a variant of the Mitchell forcing followed by the Prikry forcing with collapses forces that $\kappa = \aleph_\omega$ is a strong limit cardinal, $2^{\aleph_\omega} = \aleph_{\omega+2}$, and the tree property holds at $\aleph_{\omega+2}$. The case $n = 0$ captures the heart of the argument for the tree property. When $n > 0$, some new obstacles must be overcome, but the main structure of the proof is the same: see Section 5. In Section 6 we mention some open questions.

2 Preliminaries

2.1 A variant of Mitchell forcing

We will use a variant of the standard Mitchell forcing as presented in [1].

[1] With more work, we can also preserve the initial degree of hypermeasurability; see Remark 3.6.

If κ is a regular infinite cardinal and α is an ordinal greater than 0, we identify the Cohen forcing for adding α-many subsets of κ, $\mathrm{Add}(\kappa, \alpha)$, with a collection of functions p from a subset of $\kappa \times \alpha$ of size $< \kappa$ into $\{0, 1\}$. The ordering is by reverse inclusion.

Let $\kappa < \lambda$ be regular cardinals, and assume λ is inaccessible. Let $\mu \geq \lambda$ be an ordinal. We define a variant of the Mitchell forcing, $\mathbb{M}(\kappa, \lambda, \mu)$, as follows: Conditions are pairs (p, q) such that p is in $\mathrm{Add}(\kappa, \mu)$, and q is a function whose domain is a subset of λ of size at most κ such that for every $\xi \in \mathrm{dom}(q)$, $q(\xi)$ is an $\mathrm{Add}(\kappa, \xi)$-name, and $\emptyset \Vdash_{\mathrm{Add}(\kappa,\xi)} q(\xi) \in \mathrm{Add}(\kappa^+, 1)$. The ordering is as in the standard Mitchell forcing, i.e.: $(p', q') \leq (p, q)$ if and only if p' is stronger than p in the Cohen forcing, the domain of q' contains the domain of q and if ξ is in the domain of q, then p' restricted to ξ forces $q'(\xi)$ extends $q(\xi)$.

If $\mu = \lambda$, we write simply $\mathbb{M}(\kappa, \lambda)$.

Remark 2.1 Notice that $\mathbb{M}(\kappa, \lambda, \mu)$ is equivalent to $\mathbb{M}(\kappa, \lambda) \times \mathrm{Add}(\kappa, \mu)$ for $\mu > \lambda$.

Lemma 2.2 *Assume GCH.*

(i) $\mathbb{M}(\kappa, \lambda, \mu)$ *is λ-Knaster.*
(ii) In $V[\mathbb{M}(\kappa, \lambda, \mu)]$, $2^\kappa = |\mu|$, *and the cardinals in the open interval* (κ^+, λ) *are collapsed (and no other cardinals are collapsed).*

Proof. The proof is standard (using a Δ-system argument for Knasterness). □

The following follows as in [1]:

Lemma 2.3 *(i)* $\mathbb{M}(\kappa, \lambda, \mu)$ *is a projection of* $\mathrm{Add}(\kappa, \mu) \times \mathbb{T}$, *where* $\mathbb{T}$ *is a* κ^+*-closed term forcing defined by* $\mathbb{T} = \{(\emptyset, q) \mid (\emptyset, q) \in \mathbb{M}(\kappa, \lambda, \mu)\}$.
(ii) $\mathbb{M}(\kappa, \lambda, \mu)$ *is equivalent to* $\mathrm{Add}(\kappa, \mu) * \dot{R}$, *where* $\dot{R}$ *is forced to be* κ^+*-distributive.*

Proof. The proof is as in [1]. □

As will be apparent from the arguments in Section 4, it is also the case that if λ is weakly compact, then the tree property holds at $\lambda = \kappa^{++}$ in $V[\mathbb{M}(\kappa, \lambda, \mu)]$.

2.2 Prikry forcing with collapses

We use the forcing as it is described in Gitik's paper [8].

Here we give just a quick review to fix the notation. Let κ be a measurable cardinal, U a normal measure at κ, and $j_U : V \to M$ the ultrapower embedding generated by U. The Prikry forcing with collapses, which we denote $\mathrm{PrkCol}(U, G^g)$, is determined by U and a guiding generic G^g. G^g is a $\mathrm{Coll}(\kappa^{+n}, < j(\kappa))^M$-generic filter over M, where n typically satisfies $2 < n < \omega$ (Coll denotes the Levy collapse).

A condition r in $\mathrm{PrkCol}(U, G^g)$ has a lower part ("stem") which is a finite increasing sequence of cardinals below κ with information about collapses between the cardinals (thus the stem is an element of V_κ), and an upper part which is composed of sets A and H, where

A is in U, and H is a function defined on A such that $[H]_U$, the equivalence class of H in M, belongs to G^g.

Definition 2.4 *If r is in* $\mathrm{PrkCol}(U, G^g)$*, let us write $s(r)$ for the stem of r, and* $\mathrm{up}(r)$ *for the upper part of r.*

If all is set up correctly in V, the forcing $\mathrm{PrkCol}(U, G^g)$ turns κ into $\aleph_\omega$ while preserving all cardinals above κ.

3 Preserving measurability by Mitchell forcing

In [4], the construction which yields the tree property at the double successor of a singular strong limit κ with countable cofinality starts by assuming that κ is supercompact. The reason is that we can then invoke Laver's indestructibility result [12], and assume that adding any number of Cohen subsets of κ will preserve the measurability of κ. Such an assumption tends to simplify the subsequent constructions because one can avoid the work of lifting a weaker embedding using a surgery argument, or some other methods.

A natural question is whether a "Laver-like" indestructibility is available also for smaller large cardinals. In this Section, we use a modification of the idea of Cummings and Woodin (see [2]) to argue that it is possible to have a limited indestructibility for μ-tall cardinals κ, $\mu > \kappa$ regular, where κ is μ-tall if there is an embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \mu$ and M is closed under κ-sequences. Let us note that with more work, one can also preserve the hypermeasurability of κ (see Remark 3.6).

3.1 Stage 1

Assume GCH. Fix $n < \omega$, and let $\kappa < \lambda \le \lambda^{+n} = \mu$ be cardinals with λ being weakly compact and κ being $H(\mu)$-hypermeasurable. Let us further assume that the $H(\mu)$-hypermeasurability of κ is witnessed by an extender embedding $j : V \to M$ so that

$$M = \{j(f)(\alpha) \mid f : \kappa \to V \;\&\; \alpha < \mu\}. \tag{3.2}$$

In particular, $H(\mu)$ is included in M and M is closed under κ-sequences in V. If $n = 0$, we make the extra assumption that λ is the least weakly compact cardinal in M above κ.[2]

Let U be the normal measure derived from j, and let $i : V \to N$ be the ultrapower embedding generated by U. Let $k : N \to M$ be elementary so that $j = k \circ i$. Note that κ is the critical point of j, i and j, i have width κ, i.e. every element of M and N is of the form $j(f)(\alpha)$, or $i(f)(\kappa)$ respectively, for some f with domain κ. In contrast, the critical point of k is $(\kappa^{++})^N$ and k has width μ_κ, where μ_κ is the n-th successor of the least weakly compact cardinal above κ in N, i.e. every element of M can be written as $k(f)(\alpha)$ for some

[2] This assumption simplifies the presentation of the argument but has a slightly greater consistency strength than $H(\lambda)$-hypermeasurability; it is sufficient to assume there is some $f : \kappa \to \kappa$ such that $j(f)(\kappa) = \lambda$ (which can be obtained by forcing just from the assumption of $H(\lambda)$-hypermeasurability).

f in N with domain μ_κ; in particular $(\kappa^{++})^N < \mu_\kappa < i(\kappa) < \kappa^{++}$. See [3] for more details regarding the lifting of embeddings and the notion of width.

Let P denote the forcing $\mathrm{Add}(\kappa, \mu)$ in V, $Q = i(P)$, and let g be a Q-generic filter over V. The following theorem is an analogue of a related theorem in [2], formulated to fit our purposes.

Theorem 3.1 *Assume GCH. Forcing with Q preserves cofinalities and the following hold in $V[g]$:*

(i) j lifts to $j^1 : V[g] \to M[j^1(g)]$, where j^1 restricted to V is the original j.
(ii) i lifts to $i^1 : V[g] \to N[i^1(g)]$, where i^1 restricted to V is the original i. $N[i^1(g)]$ is the measure ultrapower obtained from j^1.
(iii) k lifts to $k^1 : N[i^1(g)] \to M[j^1(g)]$, where k^1 restricted to N is the original k.
(iv) g is Q-generic over $N[i^1(g)]$.
(v) There is $\tilde{g}$ in $V[g]$ such that $\tilde{g}$ is $k(Q) = j(P)$-generic over $M[j^1(g)]$.

Proof. We show that Q is κ^+-closed and κ^{++}-cc in V. Closure is obvious by the fact that N is closed under κ-sequences in V. Regarding the chain condition, notice that every element of Q can be identified with the equivalence class of some function $f : \kappa \to \mathrm{Add}(\kappa, \mu)$. For $f, f' : \kappa \to \mathrm{Add}(\kappa, \mu)$, set $f \le f'$ if for all $i < \kappa$, $f(i) \le f'(i)$; it suffices to check that the ordering $\le$ on these f's is κ^{++}-cc. Let A be a maximal antichain in this ordering; take an elementary substructure $\bar{M}$ in some large enough $H(\theta)$ of V which contains all relevant data, has size κ^+ and is closed under κ-sequences. Then it is not hard to check that $A \cap \bar{M}$ is maximal in the ordering (and so $A \subseteq \bar{M}$), and therefore has size at most κ^+.

(i) and (ii). These follow by κ^+-distributivity of Q in V and the fact that j, i have width κ: the pointwise image of g generates a generic for $j(Q)$ and $i(Q)$, respectively.

(iii). $i(Q)$ is $i(\kappa^+)$-closed in N, and since $\mu_\kappa < i(\kappa^+)$, we use the distributivity of $i(Q)$ and the fact that k has width μ_κ to argue that the pointwise image $k"(i^1(g))$ generates a generic filter which is equal to the generic filter generated by $j"g$ by commutativity of j, i, k.

(iv). Q is $i(\kappa^+)$-cc in N and $i(Q)$ is $i(\kappa^+)$-closed in N. Therefore g and $i^1(g)$ are mutually generic over N by Easton's lemma.

(v). Q is $i(\kappa)$-closed in $N[i^1(g)]$ since the generic $i^1(g)$ does not add new sequences of length $i(\kappa)$; it follows as in (iii) that $k^1"g$ generates a $j(P)$-generic filter $\tilde{g}$ over $M[j^1(g)]$. □

Remark 3.2 Notice that g is not present in $M[j^1(g)]$. However, if so desired, we can ensure that κ is still $H(\mu)$-hypermeasurable after the generic object $\tilde{g}$ is added. This is not required for the present proof, but may be useful if more complicated forcings are to be defined over $V[g]$ (such as the Radin forcing). See Remark 3.6 for more details.

3.2 Stage 2

Let us work in the model $V[g] = V^1$ and let us use the notation j^1, V^1, M^1 to denote the resulting models and embeddings in Theorem 3.1. Recall that by Remark 3.2, j^1 is just

μ-tall (but the initial $H(\mu)$-hypermeasurability of j still implies that the cardinals in the interval $[\kappa,\mu]$ coincide between V^1 and M^1). Note that λ is no longer strong limit in V^1, but we will argue in Section 4.4 that it retains enough of weak compactness in V^1 for further arguments.

Define P_κ to be the following Easton-supported iteration:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \,|\, \alpha < \kappa \text{ is measurable}\rangle, \tag{3.3}$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathbb{M}(\alpha,\lambda_\alpha,\mu_\alpha)^{V^1[P_\alpha]}$, where λ_α is the least weakly compact cardinal above α, and $\mu_\alpha = (\lambda_\alpha)^{+n}$.

Theorem 3.3 *Let $\mathbb{M}(\kappa,\lambda,\mu)$ be a name for the Mitchell forcing as defined in $V^1[P_\kappa]$. Then the following hold:*

(i) In $V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\mu)]$, $\lambda=\kappa^{++}$, $2^\kappa=\kappa^{+n}=\mu$, and κ is measurable.
(ii) The measurability of κ is witnessed by a lifting of j^1, which we call j^2,

$$j^2: V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\mu)] \to M^2 = M^1[j^2(P_\kappa * \mathbb{M}(\kappa,\lambda,\mu))].$$

Moreover, j^2 is the normal measure embedding derived from j^2, and M^2 satisfies $\lambda=\kappa^{++}$ and $2^\kappa=\kappa^{+n}=\mu$.

Proof. Let $G_\kappa * H$ be $P_\kappa * \mathbb{M}(\kappa,\lambda,\mu)$-generic over V^1.

(i). We follow closely the argument in Cummings [2] but with the important simplification that we use the factoring through k only in stage 1 (Theorem 3.1), and use directly the generic object $\tilde{g}$ (Theorem 3.1) to lift only the embedding j^1 (we do not lift k^1 and i^1).[3]

Using standard methods, lift j^1 to

$$j^2: V^1[G_\kappa] \to M^1[G_\kappa][H][h],$$

where h is constructed using the extender representation of M^1: the dense open sets in the forcing $j^1(P_\kappa)$ in the interval $(\kappa, j^1(\kappa))$ can be grouped into κ^+-many groups each of size μ in $M^1[G_\kappa][H]$; these groups are of the form $\{j^1(f)(\alpha)\,|\,\alpha<\mu\}$, where f is a function from κ to $H(\kappa)$. The intersection of each group is a dense set because the forcing $j^1(P_\kappa)$ in the interval $(\kappa, j^1(\kappa))$ is μ^+-closed in $M^1[G_\kappa][H]$. Since there are only κ^+-many of these groups, a generic h can be constructed in $V^1[G_\kappa][H]$ which meets them all, using κ^+-closure.

It remains to find a generic filter for the j^2-image of $\mathbb{M}(\kappa,\lambda,\mu)$. Using the fact that the Mitchell forcing decomposes over $V^1[G_\kappa]$ into $\mathrm{Add}(\kappa,\mu)^{V^1[G_\kappa]} * \dot{R}$ for some $\dot{R}$ which is forced to be κ^+-distributive by $\mathrm{Add}(\kappa,\mu)^{V^1[G_\kappa]}$ (see Section 2.1), it suffices first to lift $\mathrm{Add}(\kappa,\mu)^{V^1[G_\kappa]}$, and then (easily) lift the distributive part $\dot{R}$. Let us write $H = g_\kappa * h_\kappa$ where g_κ is $\mathrm{Add}(\kappa,\mu)^{V^1[G_\kappa]}$-generic and h_κ is $\dot{R}$-generic.

[3]Lifting through k^1 is problematic at stage κ where we deal with the forcing $\mathbb{M}(\kappa,\lambda,\mu)$ in the sense of the ultrapower (the forcing is non-trivially moved by k^1 – a fact innocuous for the Cohen forcing at κ, but problematic for the Mitchell forcing).

In order to lift $\mathrm{Add}(\kappa,\mu)^{V^1[G_\kappa]}$, we use the generic object $\tilde{g}$ which we prepared in V^1. Notice that $\tilde{g}$ is not generic for the right forcing: it is $j^1(\mathrm{Add}(\kappa,\mu)^{V^1})$-generic over M^1, but we need a generic object for $j^2(\mathrm{Add}(\kappa,\mu)^{V^1[G_\kappa]})$ over $M^1[G_\kappa][H][h]$. We use the following fact to overcome this problem (it appears as Fact 2 in [2]). Recall that Q_μ – mentioned in Fact 3.4 – is the term forcing defined as follows: the elements of Q_μ are names τ such that τ is an S-name and it is forced by 1_S to be in $\mathrm{Add}(\kappa,\mu)$ of $V[S]$. The ordering is $\tau \leq \sigma \leftrightarrow 1_S \Vdash \tau \leq \sigma$.

Fact 3.4 *Let S be a κ-cc forcing notion of cardinality κ, $\kappa^{<\kappa} = \kappa$. Then for any μ, the term forcing $Q_\mu = \mathrm{Add}(\kappa,\mu)^{V[S]}/S$ is isomorphic to $\mathrm{Add}(\kappa,\mu)$.*

By elementarity, Fact 3.4 implies that in $V^1[G_\kappa][H]$, $\tilde{g}$ yields a generic object g^* over $M^1[G_\kappa][H][h]$ for $j^2(\mathrm{Add}(\kappa,\mu)^{V^1[G_\kappa]})$ (note that $j^1(P_\kappa)$ has size $j^1(\kappa)$ in M^1 and is $j^1(\kappa)$-cc). g^* is still not good enough to lift j^2 because it may not contain the pointwise image $j^2{}''g_\kappa$. Using the method of surgery (see [2]), we modify g^* to g^{**} which is still $j^2(\mathrm{Add}(\kappa,\mu)^{V^1[G_\kappa]})$-generic, but in addition contains the pointwise image $j^2{}''g_\kappa$. It follows we can lift to

$$j^2 : V^1[G_\kappa][g_\kappa] \to M^1[G_\kappa][H][h][g^{**}],$$

and then finally to $V^1[G_\kappa][g_\kappa][h_\kappa] = V^1[G_\kappa][H]$:

$$j^2 : V^1[G_\kappa][H] \to M^2 = M^1[G_\kappa][H][h][g^{**}][h^*],$$

where h^* is generated from $j^2{}''h_\kappa$. The last lifting shows that κ remains measurable as desired.

(ii). It remains to show that j^2 is a measure ultrapower embedding. Let N^* be the normal measure ultrapower via the measure U generated from j^2 with the associated embedding $i_U : V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\mu)] \to N^*$, and let $j^2 = k^* \circ i_U$ be the commutative triangle with $k^* : N^* \to M^2$. First note that k^* is the identity on μ since its critical point must be a regular cardinal in N^* and N^* computes κ^{+n} $(= \mu)$ correctly. Then the claim follows since k^* must be onto (and therefore the identity) using the extender representation of M^2 and elementarity: any element of M^2 is of the form $j^2(f)(\alpha)$ for some $\alpha < \mu$, and as k^* is the identity on α, $j^2(f)(\alpha) = k^*(i_U(f))(\alpha) = k^*(i_U(f))(k^*(\alpha)) = k^*(i_U(f)(\alpha))$, and thus $j^2(f)(\alpha)$ is in the range of k^*. □

Remark 3.5 It can also be shown that the tree property holds at $\kappa^{++} = \lambda$ in the model $V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\mu)]$. This is implicit in the proof of Theorem 4.3.

Remark 3.6 Notice that in constructing M^1 in Theorem 3.1 we lost the $H(\mu)$-hypermeasurability of j. By a more complicated construction in Theorem 3.1, this can be retained (and then automatically retained by the further construction in Theorem 3.3). See [10] for more details and generalizations of the construction.

Here is a brief description of a construction to preserve $H(\mu)$-hypermeasurability of j:

In the first step, argue exactly as in the proof of Theorem 3.1; in particular there exists in $V^1 = V[g]$, where g is Q-generic over V, a generic object $\tilde{g}$ for $j(\mathrm{Add}(\kappa,\mu))$ over M^1. However, M^1 does not contain g, which we are going to repair now.

Let $\mathcal{M}$ be the set of all measurable cardinals $\alpha < \kappa$ in V. Define in V an Easton product $P^1_\kappa = \prod_{\alpha \in \mathcal{M}} Q^1_\alpha$ of length κ such that at every $\alpha \in \mathcal{M}$, Q^1_α is chosen by a lottery among all forcings R in V which satisfy the following condition $(*)$:

$(*)$ There is a measure W on α in V such that the embedding i_W generated by W is of the form $i_W : V \to N_W$ for some N_W and R is equal to $i_W(\mathrm{Add}(\alpha, \mu_\alpha))$,

where μ_α is the local version of μ in V with respect to α, i.e. it is the n-th successor of the least weakly compact cardinal λ_α above α.

For $\alpha \in \mathcal{M}$, let us write P^1_α for the product P^1_κ below α, and $P^1_{>\alpha}$ for the product indexed above α so that $P^1_\kappa = P^1_\alpha \times Q^1_\alpha \times P^1_{>\alpha}$. We state some facts concerning P^1_κ. For $\alpha \in \mathcal{M}$, let G^1_α, $G^1_{>\alpha}$ and g^1_α be generic filters for $P^1_\alpha, P^1_{>\alpha}$, and Q^1_α over V, respectively. Let G^1_κ be P^1_κ-generic.

(i) For $\alpha \in \mathcal{M}$, let α^* denote the next element of $\mathcal{M}$ above α. For every $\alpha \in \mathcal{M}$, Q^1_α has size less than α^*.
(ii) For every $\alpha \in \mathcal{M}$, G^1_α, g^1_α, and $G^1_{>\alpha}$ are mutually generic. Also, G^1_κ and g are mutually generic.
(iii) In $V[g][G^1_\alpha]$, forcing with Q^1_α does not collapse cardinals. More generally, forcing with $P^1_\kappa \times Q$ does not collapse cardinals.

For (i) note that there are at most α^{++}-many measures W at stage α and each R in the lottery has size less than α^*. Regarding (ii) note that for $\alpha \in \mathcal{M}$, P^1_α is α-cc, Q^1_α is α^+-closed and has size less than α^*, and $P^1_{>\alpha}$ is $(\alpha^*)^+$-closed. The same facts apply to Q and P^1_κ. For (iii), it suffices to show that R chosen by the lottery does not collapse cardinals over $V[g][G^1_\alpha]$. We argue by a variant of Theorem 3.1: R is α^+-closed in V, and therefore also in $V[g]$, and since P^1_α is α-cc, it remains α^+-distributive in $V[g][G^1_\alpha]$; R is α^{++}-cc in V, and hence also in $V[g]$, and since P^1_α has size just α, it forces that R is still α^{++}-cc (if there were an antichain in $V[g][G^1_\alpha]$ of size α^{++}, a single condition in P^1_α would determine a cofinal part of it, which would yield an antichain in $V[g]$ of size α^{++}). Using the fact that Q^1_α does not collapse cardinals and has size less than α^*, standard methods can be used to argue that the whole forcing $P^1_\kappa \times Q$ does not collapse cardinals.

Let G^1_κ be P^1_κ-generic over $V[g]$. As we argued in (ii), Q and P^1_κ are mutually generic since Q is κ^+-closed and P^1_κ is κ-cc. In $V[g][G^1_\kappa] = V[G^1_\kappa][g]$ we lift j to

$$j^* : V[G^1_\kappa] \to M[G^1_\kappa][g][h],$$

choosing by the lottery at stage κ of $j(P^1_\kappa)$ the forcing Q which is available here. The generic object h is constructed using the extender representation of M and the fact that $j(P^1_\kappa)$ at the interval $(\kappa, j(\kappa))$ is κ^+-closed in V, and more than μ^+-closed in the sense of M; by mutual genericity argued in (ii), h constructed in V as generic over M is actually generic over $M[G^1_\kappa][g]$. Since g is added by a κ^+-distributive forcing over $V[G^1_\kappa]$, it lifts easily, and so we get

$$j^{**} : V[G^1_\kappa][g] \to M[G^1_\kappa][g][h][h^*],$$

where h^* is generated by $j^{*}{}''g$.

Let us now look at $M[G^1_\kappa][g][h][h^*] = M^{**}$. We know from Theorem 3.1 that the object $\tilde{g}$ is $j(\mathrm{Add}(\kappa, \mu))$-generic over $M[h^*] = M^1$. The forcing $j(\mathrm{Add}(\kappa, \mu))$ remains $j(\kappa)$-closed in M^1 since h^* does not add new $j(\kappa)$-sequences. Since the forcing $j(P^1_\kappa)$ is $j(\kappa)$-cc in M^1, $\tilde{g}$ is mutually generic with $G^1_\kappa * g * h$, and therefore $\tilde{g}$ is $j(\mathrm{Add}(\kappa,\mu))$-generic over M^{**}. Finally we apply over M^1 Fact 3.4 arguing that there is $\tilde{g}^*$ in $V[G^1_\kappa][g]$ which is $j^{**}(\mathrm{Add}(\kappa,\mu)^{V[G^1_\kappa]})$-generic over M^{**} (in more detail, the forcing $j(P^1_\kappa)$ over M^1 has size $j(\kappa)$ and is $j(\kappa)$-cc and therefore $\tilde{g}$ yields the required $\tilde{g}^*$). Thus

$$j^{**} : V[G^1_\kappa][g] \to M^{**}$$

satisfies the assumptions necessary for the proof of Theorem 3.3 (with $\tilde{g}^*$ now being the required generic), with M^{**} now containing $H(\mu)$ of $V[G^1_\kappa][g]$.

Renaming $V^1 = V[G^1_\kappa][g]$, $M^1 = M^{**}$, and $j^{**} = j^1$, arguments in this paper using these objects can be carried out with the additional assumption that M^1 contains $H(\mu)$ of V^1. This ends Remark 3.6.

4 The tree property with gap 2

In this section we present the argument for gap 2, i.e. having $2^{\aleph_\omega} = \aleph_{\omega+2}$ in the final model, because it shows the heart of the argument without the extra ballast which is required to deal with larger gaps.

4.1 Definition of the forcing

Let us work with the model $V^1[P_\kappa][\mathbb{M}(\kappa,\lambda)]$. By Theorem 3.3 with $n = 0$, κ is measurable here. In order to analyse this model, let us introduce notation for the generic filters: let $G_\kappa * H$ be a generic filter over V^1 for $P_\kappa * \mathbb{M}(\kappa,\lambda)$. As we showed in Theorem 3.3, the lifted extender embedding j^2 in Theorem 3.3,

$$j^2 : V^1[G_\kappa * H] \to M^1[j^2(G_\kappa * H)] \tag{4.4}$$

becomes a measure ultrapower embedding i_U in $V^1[G_\kappa * H]$, generated by the normal measure U derived from j^2. Let us rename j^2 to j for simplicity. Note that by (4.4), $M^1[j(G_\kappa * H)]$ is closed under κ-sequences from $V^1[G_\kappa * H]$ because it is the measure ultrapower via U.

In particular, we can define the Prikry forcing with collapses $\mathrm{PrkCol}(U, G^g)$ using this U and a suitable guiding generic G^g which we construct in Lemma 4.1 (the small g stands for "guiding"). See Section 2.2 for more details and references for this forcing.

Let Coll denote the forcing $\mathrm{Coll}((\kappa^{+3}), < j(\kappa))^{M^1[j(G_\kappa * H)]}$.

Lemma 4.1 *In $V^1[G_\kappa * H]$, there exists an $M^1[j(G_\kappa * H)]$-generic filter for* Coll.

Proof. Consider the extender representation $j^1 : V^1 \to M^1$ ensured by the arguments in Section 3.1, where

$$M^1 = \{j^1(f)(\alpha) \mid f \in V^1 \ \&\ f : \kappa \to V^1 \ \&\ \alpha < \lambda\}. \tag{4.5}$$

Now notice that every maximal antichain of Coll in $M^1[j(G_\kappa * H)]$ has a name of the form $j^1(f)(\alpha)$ for some $f : \kappa \to H(\kappa)^{V^1}$ and $\alpha < \lambda$, with the range of f being composed of P_κ-names. There are only κ^+-many such f's, and since Coll is κ^{+3}-closed in $M^1[j(G_\kappa * H)]$, we can built a Coll-generic filter G^g in $V^1[G_\kappa * H]$ over $M^1[j(G_\kappa * H)]$ by the standard method of grouping the antichains into κ^+ many blocks each of size at most λ, where λ is equal to κ^{+2} in $M^1[j(G_\kappa * H)]$. □

Let us define in V:

$$\mathbb{P} = Q * P_\kappa * \mathbb{M}(\kappa, \lambda) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g), \tag{4.6}$$

where the parameter μ in Q and P_κ is equal to λ, and $\dot{G}^g$ is a name for a guiding generic which we know exists by Lemma 4.1.

Lemma 4.2 *$\mathbb{P}$ is λ-cc.*

Proof. This is a standard argument using Theorem 3.1 for Q. □

We plan to show that $V[\mathbb{P}]$ is the desired model.

4.2 The theorem

Now we show that the tree property holds with gap 2. See Section 5 for a generalization to any finite gap.

Theorem 4.3 *(GCH). Assume that κ is $H(\lambda)$-hypermeasurable, where $\lambda > \kappa$ is the least weakly compact above κ and that this fact is witnessed by an extender embedding $j : V \to M$ such that λ is the least weakly compact cardinal above κ in M. Then the forcing $\mathbb{P}$ in (4.6) forces $\kappa = \aleph_\omega$, $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2}$, and the tree property holds at $\lambda = \aleph_{\omega+2}$.*

By Lemma 2.2 and standard facts about the Prikry forcing with collapses, it suffices to check that we have the tree property at $\aleph_{\omega+2}$.

4.3 The outline of the proof

Let $V^1[G_\kappa]$ be a generic extension via $Q * P_\kappa$. Notice that Q destroys the strong limitness of λ by adding many subsets of κ^+, so λ is not even strongly inaccessible in $V^1[G_\kappa]$. The analysis in Section 4.4 shows that the key properties of weak compactness do survive in $V^1[G_\kappa]$. Thus, with the help of the observations in Section 4.4, we start the proof by fixing in $V^1[G_\kappa]$ a fragment of a weakly compact embedding with critical point λ,

$$k : \mathscr{M} \to \mathscr{N}, \tag{4.7}$$

which is still strong enough for our purposes as expressed by (4.11) below.

We wish to show that $\mathbb{M}(\kappa, \lambda) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ forces the tree property at $\aleph_{\omega+2}$. We will suppose for contradiction that $\dot{T}$ is forced by the weakest condition in $\mathbb{M}(\kappa, \lambda) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$

to be a λ-Aronszajn tree. We start by fixing generic filters $h * x$ for $\mathbb{M}(\kappa,\lambda) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ and $h^* * x^*$ for $k(\mathbb{M}(\kappa,\lambda) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g))$ so that the embedding k lifts to

$$k : \mathscr{M}[h][x] \to \mathscr{N}[h^*][x^*] \tag{4.8}$$

and $\dot{T}^{h*x} = T$, the λ-Aronszajn tree, is an element of both $\mathscr{M}[h][x]$ and $\mathscr{N}[h^*][x^*]$. Now we follow the usually strategy in which we show that the quotient forcing

$$\mathbb{N} = k(\mathbb{M}(\kappa,\lambda) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g))/h * x \tag{4.9}$$

does not add new cofinal branches to λ-trees over the model $\mathscr{N}[h][x]$. This is the key step of the argument (Lemma 4.14) and uses a "labeled tree" argument as in [5]. The end of the proof is standard: if there were in $V^1[P_\kappa][h][x]$ a λ-Aronszajn tree T, we could arrange it to be already in $\mathscr{N}[h][x]$. T certainly has a cofinal branch in $\mathscr{N}[h^*][x^*]$ (due to the lifted embedding k), but since $\mathbb{N}$ cannot add this branch, it must have been already in $\mathscr{N}[h][x]$. But T was supposed to be a λ-Aronszajn tree in $\mathscr{N}[h][x]$, a contradiction.

4.4 The fragment of weak compactness of λ in $V^1[G_\kappa]$

Suppose for contradiction that $\mathbb{P}$ forces that there is a λ-Aronszajn tree (assume for simplicity the weakest condition forces this, otherwise work below a suitable condition). Let $g * G_\kappa$ be $Q * P_\kappa$-generic, and let $\dot{W}$ be a $Q * P_\kappa$-name for an $\mathbb{M}(\kappa,\lambda) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$-name $\dot{T}$ such that over $V^1[P_\kappa]$, $\mathbb{M}(\kappa,\lambda) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ forces that $\dot{T}$ is a λ-Aronszajn tree.

By Lemma 4.2, we can assume that $\dot{W}$ can be expressed as a nice name for a subset of λ, and that $\dot{T}$ itself is a nice name for a subset of λ in $V^1[G_\kappa]$.

Let β^* be an ordinal between λ and λ^+. For the present case with gap 2, β^* can be taken to be λ, but for larger gaps β^* may be any ordinal between λ and λ^+.[4] Let us fix back in V a weakly compact embedding k with critical point λ,

$$k : \mathscr{M} \to \mathscr{N} \tag{4.10}$$

with the following properties:[5]

(i) $\mathscr{M}$ and $\mathscr{N}$ are transitive models of size λ closed under $< \lambda$-sequences,
(ii) $\mathscr{M} \in \mathscr{N}$, $k \in \mathscr{N}$, $\beta^* < k(\lambda)$, and
(iii) $\mathscr{M}$ contains all relevant information (in particular, β^* and $\dot{W}$ are elements of $\mathscr{M}$).

As λ^+ is a fixed point of the mapping i, Q is equal to $\mathrm{Add}(i(\kappa), \lambda^+)$ of the measure ultrapower N. Let us consider Q restricted to β^* (let us denote it $Q(\beta^*)$), and notice that $Q(\beta^*)$ is an element of $\mathscr{M}$. Let $g(\beta^*)$ be the restriction of g to β^* so that $g(\beta^*) * G_\kappa$ is $Q(\beta^*) * P_\kappa$-generic. Note that $Q(\beta^*) * P_\kappa$ is actually equivalent to $Q(\beta^*) \times P_\kappa$ since $Q(\beta^*)$ does not change V_κ where P_κ lives. By standard arguments, k lifts to $\mathscr{M}[G_\kappa] \to \mathscr{N}[G_\kappa]$ since $k(P_\kappa) = P_\kappa$, and both the models are still closed under $< \lambda$-sequences in $V[G_\kappa]$.

[4] Think of this β^* as some ordinal greater or equal to β which appears in Lemma 5.2 below.

[5] See [3] Theorem 16.1. To ensure $\beta^* < k(\lambda)$, define E in the proof of Theorem 16.1 so that it also codes a well-ordering of β^* of type λ: then $\mathscr{N} \models |\beta^*| = \lambda$ and therefore $k(\lambda) > \beta^*$ since by elementarity, $k(\lambda)$ is in $\mathscr{N}$ a limit cardinal greater than λ.

By elementarity, $k(Q(\beta^*))$ is Q restricted to $k(\beta^*)$. Let $b : k(\beta^*) \to k(\beta^*)$ be a bijection which swaps γ and $k(\gamma)$ for every $\lambda \leq \gamma < \beta^*$, and is the identity otherwise. b extends to an automorphism on $k(Q(\beta^*))$ by mapping $p \in k(Q(\beta^*))$ to $b(p)$ where the coordinates in $b(p)$ are swapped by b. Note that $b(p)$ is a valid condition in Q since by the elementarity of k, $k(p) = k"p$ is a condition in $k(Q(\beta^*))$ (and hence in Q) for every p in $Q(\beta^*)$. This follows from the fact that the support of p is some set of size less than $i(\kappa)$ in the measure ultrapower N, but certainly less than λ in V: thus the k-image of the support is just its pointwise image.

Let $g(k(\beta^*))$ be the restriction of g to $k(\beta^*)$. The automorphism b generates from $g(k(\beta^*))$ a generic filter g^* on $k(Q(\beta^*))$ which contains the pointwise image $k"g(\beta^*)$. It follows k lifts to

$$k : \mathscr{M}[G_\kappa][g(\beta^*)] \to \mathscr{N}[G_\kappa][g^*]. \tag{4.11}$$

Since Q is κ^+-distributive over P_κ it holds that both the models are still closed under κ-sequences in $V[g * G_\kappa]$ (but note that they are not closed under κ^+-sequences).

Thus for any $\dot{W}$ and β^* as above, we have in $V[g * G_\kappa]$ a fragment of a weakly compact embedding (4.11) such that all the relevant parameters are in $\mathscr{M}$, including the name $\dot{T}$, and the models are closed under κ-sequences in the universe.

Remark 4.4 In the text which follows, we will abuse notation, and write in $V^1[G_\kappa]$ the embedding $k : \mathscr{M}[G_\kappa][g(\beta^*)] \to \mathscr{N}[G_\kappa][g^*]$ simply as $k : \mathscr{M} \to \mathscr{N}$ with the parameters understood from the context.

4.5 The heart of the argument

Let $\mathbb{M}$ denote the forcing $\mathbb{M}(\kappa, \lambda)$. Let us work in $V^1[G_\kappa]$ and let us continue to assume for contradiction that $\dot{T}$ is a name for a λ-Aronszajn tree.

Using the abuse of notation mentioned in Remark 4.4, let us fix in $V^1[G_\kappa]$ an embedding

$$k : \mathscr{M} \to \mathscr{N} \tag{4.12}$$

such that $\mathscr{M}$ and $\mathscr{N}$ are transitive models of size λ closed under κ-sequences, $\mathscr{M} \in \mathscr{N}$ and $\mathscr{M}$ contains all relevant information (in particular $\dot{T}$ are elements of $\mathscr{M}$). Let $\mathbb{M}^*$ denote $k(\mathbb{M}(\kappa, \lambda))$, which is equal to $\mathbb{M}(\kappa, k(\lambda))$.

Let h^* be $\mathbb{M}^*$-generic over $V^1[G_\kappa]$; use h^* to define h which is $\mathbb{M}$-generic over $V^1[G_\kappa]$ and $k"h \subseteq h^*$. Now lift to

$$k : \mathscr{M}[h] \to \mathscr{N}[h^*]. \tag{4.13}$$

Let us write $U = (\dot{U})^h$ and $G^g = (\dot{G}^g)^h$.

In $\mathscr{N}[h^*]$, consider $U^* = k(U)$, and $G^{g*} = k(G^g)$, and the forcing $\mathrm{PrkCol}(U^*, G^{g*})$. Note that by elementarity $U \subseteq U^*$ (since $k(X) = X$ for every $X \in U$), and all functions F whose equivalence class is in G^g appear in the forcing $\mathrm{PrkCol}(U^*, G^{g*})$ (since $k(F) = F$ for every

$F : \kappa \to V_\kappa^{\mathscr{M}[h]}, F \in \mathscr{M}[h]$), and $k(\mathrm{PrkCol}(U, G^g)) = \mathrm{PrkCol}(U^*, G^{g*})$. However, note that the equivalence classes of a fixed F with respect to U and U^* may be different objects (after the transitive collapse).

It follows that k is a regular embedding:

$$k : \mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g) \to \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}), \tag{4.14}$$

as by the λ-cc of $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, if A is a maximal antichain in $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, then $k(A) = k``A$ is a maximal antichain in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$.

Let x^* be $\mathrm{PrkCol}(U^*, G^{g*})$-generic over $V^1[G_\kappa][h^*]$; the pull-back of x^* via k^{-1} is a generic filter x for $\mathrm{PrkCol}(U, G^g)$ such that $k``x \subseteq x^*$. Let us lift k further to

$$k : \mathscr{M}[h][x] \to \mathscr{N}[h^*][x^*]. \tag{4.15}$$

By (4.14) and (4.15), we can define in $\mathscr{N}[h^*][x^*]$ a generic filter $h*x$ for $\mathbb{M}*\mathrm{PrkCol}(\dot{U}, \dot{G}^g) \in \mathscr{N}$ using the inverse of k restricted to $H(\lambda)$.[6] It follows

$$\mathscr{N}[h][x] \subseteq \mathscr{N}[h^*][x^*]. \tag{4.16}$$

Let us denote the quotient determined by k in (4.15) as $\mathbb{N}$:

$$\begin{aligned} \mathbb{N} = \{((p, q), \dot{r}) \in \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \mid \\ ((p, q), \dot{r}) \text{ is compatible with all conditions in } k``(h * x) = h * x\}. \end{aligned} \tag{4.17}$$

It follows that over $\mathscr{N}$,

$$\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \text{ is equivalent to } \mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g) * \dot{\mathbb{N}}. \tag{4.18}$$

Our task is to show that $\mathbb{N}$ does not add new branches to λ-trees over $\mathscr{N}[h][x]$.

4.5.1 A quotient analysis

For simplicity of notation let $\mathbb{R}$ denote the forcing $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, and $\mathbb{R}^*$ the forcing $\mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$.

Let us start by listing some general facts which give some conditions regarding when a condition in $\mathbb{M} * \mathbb{R}$ forces a condition in $\mathbb{M}^* * \mathbb{R}^*$ into $\dot{\mathbb{N}}$. For clarity of notation, let us write conditions in $\mathbb{M} * \mathbb{R}$ (and similarly for $\mathbb{M}^* * \mathbb{R}^*$) as

$$(p, q, s(\dot{r}), \mathrm{up}(\dot{r})), \tag{4.19}$$

where (p, q) is a condition in $\mathbb{M}$, and $(s(\dot{r}), \mathrm{up}(\dot{r}))$ is equal to $\dot{r}$, i.e. $s(\dot{r})$ and $\mathrm{up}(\dot{r})$ denote the stem and the upper part of the Prikry condition $\dot{r}$, respectively.

(4.20) We adopt the following notational conventions to further simplify notation:

[6] For the present case of gap 2, the relevant part of k is the identity and is therefore present in $\mathscr{N}$ automatically, but for the general case we will need to assume that k is an element of $\mathscr{N}$ to proceed with this step.

1. If the stem of $\dot{r}$ is a checked name for a stem s, we often simply write $(s, \mathrm{up}(\dot{r}))$, with the understanding that $\dot{r} = (\check{s}, \mathrm{up}(\dot{r}))$. On occasion, as in Lemma 4.9, we denote a checked stem of $\dot{r}$ by $\check{s}(\dot{r})$.

2. If $(p, q) \in \mathbb{M}$ and $(p', q') \in \mathbb{M}^*$, we say that they are k-compatible if $(k(p), q)$ is compatible with (p', q').

3. If $(p, q) \in \mathbb{M}$ and $(p', q') \in \mathbb{M}^*$, we write $(p, q) \leq_k (p', q')$ instead of
$$(p, q) \leq (k^{-1}{}''(p'), q'|\lambda).$$

4. If s, t are two stems in $\mathbb{R}$, we write $s \leq t$ to indicate that s extends t (the Prikry points in s end-extend the Prikry points in t, and the collapsing information in s extends the collapsing information in t).

5. If $r = (t, \mathrm{up}(r))$ is in $\mathbb{R}$ and $s \leq t$, we say that s is incompatible with $\mathrm{up}(r)$ if $(s, \mathrm{up}(r^*))$ does not extend $(t, \mathrm{up}(r))$ for any r^*; we write $s \perp \mathrm{up}(r)$. We will say that s and $\mathrm{up}(r)$ are compatibile, and write $s || \mathrm{up}(r)$, if there is r^* such that $(s, \mathrm{up}(r^*))$ extends $(t, \mathrm{up}(r))$.

Remark 4.5 Notice that in the present case of gap 2, $k(p) = p$, and $k^{-1}{}''(p')$ is just the restriction of p' to λ. The notation will be relevant for larger gaps in which the forcing is longer, and therefore k applied to the conditions is no longer the identity.

The following facts are standard (see the analogous Facts in [6] for more details).

Fact 4.6 *Assume $(p, q, s, \mathrm{up}(\dot{r}))$ is a condition in $\mathbb{M} * \mathbb{R}$ and $(p', q', t, \mathrm{up}(\dot{r}'))$ is a condition in $\mathbb{M}^* * \mathbb{R}^*$ (where s and t are checked stems of the Prikry conditions). Then the first condition forces that the second condition is not a condition in $\dot{\mathbb{N}}$ if and only if one of the following conditions hold:*

(i) *(p, q) is k-incompatible with (p', q'),*
(ii) *(p, q) is k-compatible with (p', q'), $s \not\leq t$, and $t \not\leq s$,*
(iii) *(p, q) is k-compatible with (p', q'), $s \leq t$ and the greatest lower bound of $(k(p), q)$ and (p', q') forces $s \perp \mathrm{up}(\dot{r}')$.*
(iv) *(p, q) is k-compatible with (p', q'), $t \leq s$ and the greatest lower bound of $(k(p), q)$ and (p', q') restricted to $\mathbb{M}$ forces in $\mathbb{M}$ that $t \perp \mathrm{up}(\dot{r})$.*

Remark 4.7 Notice that depending on the context, Fact 4.6 applies to the quotient forcing $\mathbb{M}^*/h$ (given by the regular embedding $k : \mathbb{M} \to \mathbb{M}^*$) over a model where we already have a generic filter h for $\mathbb{M}$: if $(p, q, s, \mathrm{up}(\dot{r}))$ is in $\mathbb{M} * \mathbb{R}$, (p, q) is in h and $(p', q', t, \mathrm{up}(\dot{r}'))$ is in $\mathbb{M}^* * \mathbb{R}^*$ and (p', q') is in $\mathbb{M}^*/h$, then Fact 4.6 translates into statements about the first condition forcing the second condition out of the quotient $\dot{\mathbb{N}}$ over the model $\mathscr{N}[h]$ (i.e. $\dot{\mathbb{N}}$ is composed of conditions $(p', q', t, \mathrm{up}(\dot{r}'))$ with (p', q') being in $\mathbb{M}^*/h$).

Fact 4.8 *Assume $(p, q, s, \mathrm{up}(\dot{r}))$ is a condition in $\mathbb{M} * \mathbb{R}$ and $(p', q', t, \mathrm{up}(\dot{r}'))$ is a condition in $\mathbb{M}^* * \mathbb{R}^*$. Moreover assume that*

(i) $(p, q) \leq_k (p', q')$,
(ii) $s \leq t$,
(iii) (p', q') *forces* $s||\mathrm{up}(\dot{r}')$.

Then there is a direct extension of $(p, q, s, \mathrm{up}(\dot{r}))$ *(i.e. a condition which does not change* (p, q) *and is a direct extension of* $(s, \mathrm{up}(\dot{r}))$*) which forces the condition* $(p', q', t, \mathrm{up}(\dot{r}'))$ *into* $\dot{\mathbb{N}}$.

Proof. We give a sketch of proof. Using the Prikry property of $\mathbb{R}$, (p, q) forces that there is a direct extension $(s^*, \mathrm{up}(\dot{r}^*))$ of $(s, \mathrm{up}(\dot{r}))$ which decides the statement $\varphi :=$ "$(p', q', t, \mathrm{up}(\dot{r}')) \in \dot{\mathbb{N}}$" (for details see [8]). We argue that this decision is positive, i.e. (p, q) forces that $(s^*, \mathrm{up}(\dot{r}^*))$ forces φ. Suppose for contradiction this is not the case and (p, q) forces that $(s^*, \mathrm{up}(\dot{r}^*))$ refutes φ; then by Fact 4.6, it must be the case that the greatest lower bound of $(k(p), q)$ and (p', q') forces that s^* is not compatible with $\mathrm{up}(\dot{r}')$. But this contradicts the fact that (p', q') forces the compatibility of s with $\mathrm{up}(\dot{r}')$ (note that any direct extension $s^* \leq s$ stays compatible with $\mathrm{up}(\dot{r}')$ because it has the same length and may only extend the collapsing information). It follows that $(p, q, s^*, \mathrm{up}(\dot{r}^*))$ is the desired condition. □

The conditions in the forcing $\mathbb{R} * \dot{\mathbb{N}}$ are in general of the form $(r, ((\dot{p}, \dot{q}), \dot{r}))$, where $\dot{p}$ and $\dot{q}$ are $\mathbb{R}$-names, and $\dot{r}$ is an $\mathbb{R}$-name for an $\mathbb{M}^*$-name. It will be crucial for us that we can consider conditions in which $(\dot{p}, \dot{q})$ are determined objects in $\mathscr{N}[h]$, and r and $\dot{r}$ have the same (determined) stem.

To simplify notation, we will remove brackets and write conditions in $\mathbb{R} * \dot{\mathbb{N}}$ as follows:

$$(s(r), \mathrm{up}(r), \dot{p}, \dot{q}, s(\dot{r}), \mathrm{up}(\dot{r})). \tag{4.21}$$

For the following lemma, please recall the notational conventions as stated in (4.20).

Lemma 4.9 *Let us work in* $\mathscr{N}[h]$. *Then the following conditions are dense in* $\mathbb{R} * \dot{\mathbb{N}}$:

$$D = \{(s(r), \mathrm{up}(r), \check{p}, \check{q}, \check{s}(\dot{r}), \mathrm{up}(\dot{r})) \mid (s(r), \mathrm{up}(r), \check{p}, \check{q}, \check{s}(\dot{r}), \mathrm{up}(\dot{r}))) \in \mathbb{R} * \dot{\mathbb{N}} \text{ and } \check{s}(\dot{r}) \text{ is a checked name for } s(r)\}, \tag{4.22}$$

i.e. for some stem s, $r = (s, \mathrm{up}(r))$ *and* $\dot{r} = (\check{s}, \mathrm{up}(\dot{r}))$. *We will write such conditions simply as* $(s, \mathrm{up}(r), p, q, s, \mathrm{up}(\dot{r}))$.

Proof. First notice that a condition r in $\mathbb{R}$ forces a condition $(p', q', t, \mathrm{up}(\dot{r}))$ into $\dot{\mathbb{N}}$ if and only if for some condition (p_0, q_0) in h, (p_0, q_0, r) forces $(p', q', t, \mathrm{up}(\dot{r}))$ into $\dot{\mathbb{N}}$. In particular, if r forces $(p', q', t, \mathrm{up}(\dot{r}))$ into $\dot{\mathbb{N}}$, then (p', q') is compatible with all conditions in $k"h$.

Let $(s, \mathrm{up}(r), \dot{p}, \dot{q}, \dot{r})$ be a condition in $\mathbb{R} * \dot{\mathbb{N}}$. We can extend r to $r' = (s', \mathrm{up}(r'))$ which forces that $(\dot{p}, \dot{q}, \dot{r})$ is equal to $(p', q', t, \mathrm{up}(\dot{r}))$ for some p', q', t, and we can assume $s' \leq t$. As r' forces $(p', q', t, \mathrm{up}(\dot{r}))$ into $\dot{\mathbb{N}}$, we apply Fact 4.6(iii) to choose a condition (p'', q'') below (p', q') which forces $s'||\mathrm{up}(\dot{r})$; we can choose (p'', q'') in $\mathbb{M}^*/h$ by Remark 4.7 (otherwise (p', q') would force in the quotient forcing $\mathbb{M}^*/h$ that s' is incompatible with $\mathrm{up}(\dot{r})$). It follows that there is $\dot{r}'$ such that $(p'', q'', s', \mathrm{up}(\dot{r}'))$ extends $(p', q', t, \mathrm{up}(\dot{r}))$ in $\mathbb{M}^* * \mathbb{R}^*$.

By Fact 4.6, $r' = (s', \mathrm{up}(r'))$ cannot force $(p'', q'', s', \mathrm{up}(\dot{r}'))$ out of $\dot{\mathbb{N}}$; so by Fact 4.8 there is a direct extension $(s', \mathrm{up}(r''))$ of r' which forces $(p'', q'', s', \mathrm{up}(\dot{r}'))$ into the quotient $\dot{\mathbb{N}}$. Therefore

$$(s', \mathrm{up}(r''), p'', q'', s', \mathrm{up}(\dot{r}')) \tag{4.23}$$

is a condition in D below $(s, \mathrm{up}(r), \dot{p}, \dot{q}, \dot{r})$. □

Let us work in the set D from Lemma 4.9. The following is easy to verify:

Fact 4.10 *The following are equivalent:*

(i) $(p, q, \dot{r})$ *in* $\mathbb{M} * \mathbb{R}$ *forces* $(p', q', \dot{r}')$ *into* $\dot{\mathbb{N}}$.

(ii) Every generic filter $h * x$ *containing* $(p, q, \dot{r})$ *can be extended to a generic filter* $h^* * x^*$ *for* $\mathbb{M}^* * \mathbb{R}^*$ *containing* $(p', q', \dot{r}')$.

(iii) Whenever $h * x$ *is a generic filter containing* $(p, q, \dot{r})$*, then* $(p', q', \dot{r}')$ *is compatible with every condition in* $k``(h * x)$.

(iv) There is no condition $(p'', q'', \dot{r}'')$ *below* $(p, q, \dot{r})$ *such that* $(p', q', \dot{r}')$ *is incompatible with* $k((p'', q'', \dot{r}''))$.

Recall that by Abraham's observation in [1], we can define in $\mathscr{N}[h]$ the term-part of $\mathbb{M}^*/h$, which we denote $\mathbb{T}$, as follows:

$$\mathbb{T} = \{(\emptyset, q) \in \mathbb{M}^* \mid (\emptyset, q) \text{ is compatible with every condition in } k``h\}, \tag{4.24}$$

and show that $\mathbb{T}$ is κ^+-closed in $\mathscr{N}[h]$ in the induced ordering $\leq_{\mathbb{T}}$ of $\mathbb{M}^*/h$. The ordering $\leq_{\mathbb{T}}$ is not κ^+-closed in $\mathscr{N}[h][x]$, and – because we wish to use the κ^+-closure of $\leq_{\mathbb{T}}$ – we will work in the model $\mathscr{N}[h]$ in what follows.

Remark 4.11 We will often abuse notation and say that q is in $\mathbb{T}$ to indicate that $(\emptyset, q)$ is in $\mathbb{T}$; we will also write $q \leq_{\mathbb{T}} q'$ instead of $(\emptyset, q) \leq_{\mathbb{T}} (\emptyset, q')$.

Let us state some facts which suggest that we should be careful in checking that certain mild extensions of conditions in $\dot{\mathbb{N}}$ are still forced into $\dot{\mathbb{N}}$ (cf. (1) in Fact 4.12).

Fact 4.12 *Suppose* $(p, q, s, \mathrm{up}(\dot{r}))$ *in* $\mathbb{M} * \mathbb{R}$ *forces* $(p', q', t, \mathrm{up}(\dot{r}'))$ *into* $\dot{\mathbb{N}}$ *and suppose* $s \leq t$.

(i) Assume $q'' \leq_{\mathbb{T}} q'$*; then it is* not *ensured that there is a condition below* $(p, q, \dot{r})$ *which forces* $(p', q'', \dot{r}')$ *into* $\dot{\mathbb{N}}$.

(ii) There are $q^* \leq_{\mathbb{T}} q'$ *and* $p^* \leq p'$ *such that a direct-extension of* $(p, q, \dot{r})$ *forces* $(p^*, q^*, \dot{r}')$ *into* $\dot{\mathbb{N}}$*, and moreover whenever* $q^{**} \leq_{\mathbb{T}} q^*$*, then a direct-extension of* $(p, q, \dot{r})$ *forces* $(p^*, q^{**}, \dot{r}')$ *into* $\dot{\mathbb{N}}$.

Remark 4.13 The fragility of the quotient analysis, with Fact 4.12(i) being the most critical, may often lead to erroneous claims. For instance it is very tempting to think that if $(p, q, s, \mathrm{up}(\dot{r}))$ forces both $(p, q, \emptyset, \mathrm{up}(\dot{r}_0))$ and $(p, q, \emptyset, \mathrm{up}(\dot{r}_1))$ into $\dot{\mathbb{N}}$ for some conditions of this form (the last two conditions have the empty stem), then $(p, q, s, \mathrm{up}(\dot{r}))$ forces the greatest lower bound of $(p, q, \emptyset, \mathrm{up}(\dot{r}_0))$ and $(p, q, \emptyset, \mathrm{up}(\dot{r}_1))$ into $\dot{\mathbb{N}}$. But this may fail quite easily because the names $\mathrm{up}(\dot{r}_0)$ and $\mathrm{up}(\dot{r}_1)$ may be built out of incompatible conditions preventing the stem s being compatibile with the natural "intersection" of $\mathrm{up}(\dot{r}_0)$ and $\mathrm{up}(\dot{r}_1)$ (while allowing compatibility of s with $\mathrm{up}(\dot{r}_0)$ and $\mathrm{up}(\dot{r}_1)$ separately). For this reason we are forced to work with the dense set D from Lemma 4.9, in which the Prikry forcing $\mathbb{R}$ and $\mathbb{R}^*$ have the same stems.

4.5.2 The labeled tree of conditions $\mathscr{T}$

Recall that we assume that $\dot{T}$ is an $\mathbb{R}$-name in $\mathscr{N}[h]$ and $1_{\mathbb{R}}$ forces in $\mathscr{N}[h]$ that $\dot{T}$ is a λ-tree (actually we assume it is forced to be a λ-Aronszajn tree for contradiction). We identify $\dot{T}^x$ with a subset of $\kappa^+ \times \lambda$ with ordering $<_{\dot{T}^x}$ in the natural way: if $(\beta, \gamma) \in \dot{T}^x$ then β is in $\dot{T}^x_\gamma$. We will write β_γ instead of (β, γ).

We wish to argue that $\mathbb{N}$ does not add new cofinal branches to $\dot{T}^x$. This will be proved in Lemma 4.14 as in [5] for which we now do some preparatory work.

Let us assume that $\dot{b}$ is an $\mathbb{R} * \dot{\mathbb{N}}$-name and $1_{\mathbb{R}*\dot{\mathbb{N}}}$ forces that $\dot{b}$ is a new branch in $\dot{T}$. Let $\langle s_i \mid i < \kappa \rangle$ be some enumeration of all stems in $\mathbb{R}$.

Let us work in $\mathscr{N}[h]$. Build a labeled full binary tree $\mathscr{T}$ in $2^{<\kappa}$ such that for every $h \in 2^{<\kappa}$, we define sets

$$q_h, \gamma_h \text{ and } Y^i_h, i < \kappa, \tag{4.25}$$

where q_h is a condition in $\mathbb{T}$, γ_h is an ordinal below λ, and Y^i_h is an antichain in $\mathrm{Add}(\kappa, k(\lambda))^2$, for $i < \kappa$. It is routine to build the tree $\mathscr{T}$ to satisfy the following properties:

Property 1. There is a single ordinal $\gamma < \lambda$, such that for every $y \in 2^\kappa$, the supremum of $\langle \gamma_{y|\alpha} \mid \alpha < \kappa \rangle$ is equal to γ.

Property 2. For every $y \in 2^\kappa$, the conditions $\langle q_{y|\alpha} \mid \alpha < \kappa \rangle$ are decreasing in $\leq_{\mathbb{T}}$ so that a lower bound q_y exists.

Property 3. For every $y_0 \neq y_1$ in 2^κ and any stem s, whenever

$$(s, \mathrm{up}(r), p_0, q_0, s, \mathrm{up}(\dot{r})) \tag{4.26}$$

and

$$(s, \mathrm{up}(r), p_1, q_1, s, \mathrm{up}(\dot{r}')) \tag{4.27}$$

in $\mathbb{R} * \dot{\mathbb{N}}$ have the property that $q_0 \leq_{\mathbb{T}} q_{y_0}$, $q_1 \leq_{\mathbb{T}} q_{y_1}$, and the conditions (4.26) and (4.27) decide $\dot{b}(\gamma')$ for some $\gamma' \geq \gamma$ incompatibly (i.e. they decide $\dot{b}(\gamma')$ to be two distinct ordinals), then if $h = y_0 \cap y_1$, then there are $(p, p') \in Y^i_h$ (where $s_i = s$) compatible with (p_0, p_1) in the ordering $\mathrm{Add}(\kappa, k(\lambda))^2$ and $\gamma'' < \gamma$ (with $\gamma'' < \gamma_{h^\frown 0}$ and $\gamma'' < \gamma_{h^\frown 1}$) such that for some $r^* = (s, \mathrm{up}(r^*))$ and $\dot{r}^* = (s, \mathrm{up}(\dot{r}^*))$,

$$(s, \mathrm{up}(r^*), p, q_{h^\frown 0}, s, \mathrm{up}(\dot{r}^*)) \tag{4.28}$$

and

$$(s, \mathrm{up}(r^*), p', q_{h^\frown 1}, s, \mathrm{up}(\dot{r}^*)) \tag{4.29}$$

decide $\dot{b}(\gamma'')$ incompatibly.

The construction of $\mathscr{T}$ uses the standard method of diagonalizing over antichains in the square $\mathrm{Add}(\kappa, k(\lambda))^2$ (antichains maximal with respect to Property 3) while taking the lower bounds for the conditions on the second coordinate of $\mathbb{M}^*$ (the q-part), using the κ^+-closure of $\leq_{\mathbb{T}}$ in $\mathscr{N}[h]$. Note that it is possible to deal with all the κ-many stems s at every node of the tree.

4.5.3 The argument

We conclude the proof of Theorem 4.3 by proving the following Lemma.

Lemma 4.14 *Suppose $\mathscr{N}[h][x]$, $\mathbb{N}$ and $\dot{T}^x$ are as above. The forcing $\mathbb{N}$ does not add over $\mathscr{N}[h][x]$ new cofinal branches to the tree $\dot{T}^x$.*

Proof. Suppose $\mathscr{T}$ is the labeled tree constructed in the previous section. For every $h \in 2^\kappa$, let q_h be some lower bound of $\langle q_{h|\beta} \mid \beta < \kappa\rangle$; notice that q_h exists as the ordering $\leq_{\mathbb{T}}$ is κ^+-closed. Choose for each h a condition $(s_h, \mathrm{up}(r_h), p_h, q'_h, s_h, \mathrm{up}(\dot{r}_h)) \in D$ from Lemma 4.9 which decides the value of $\dot{b}(\gamma)$. Since $2^\kappa = \lambda$ in $\mathscr{N}[h]$, there are two distinct g and h in 2^κ such that their corresponding conditions force the same value for $\dot{b}$ at level γ; let us denote this common value β_γ. Moreover we can assume that these conditions have the same Prikry stem s. Thus we have

$$\sigma_0 = (s, \mathrm{up}(r_g), p_g, q'_g, s, \mathrm{up}(\dot{r}_g)) \Vdash \dot{b}(\gamma) = \beta_\gamma \tag{4.30}$$

and

$$\sigma_1 = (s, \mathrm{up}(r_h), p_h, q'_h, s, \mathrm{up}(\dot{r}_h)) \Vdash \dot{b}(\gamma) = \beta_\gamma. \tag{4.31}$$

Now, we will briefly work in $\mathscr{N}[h][x]$ to show that there are $\gamma' > \gamma$ and conditions τ'_0 and τ'_1 in D with the same stem s', with $\tau'_0 \leq \sigma_0$ and $\tau'_1 \leq \sigma_1$, and these conditions decide $\dot{b}(\gamma')$ differently. We argue as follows: As D is dense in $\mathbb{R} * \dot{\mathbb{N}}$, the following set is dense in $\mathbb{N}$:

$$D_0 = \{(p, q, s, \mathrm{up}(\dot{r})) \mid \exists (s, \mathrm{up}(r)) \in x \ (s, \mathrm{up}(r), p, q, s, \mathrm{up}(\dot{r})) \in D\}. \tag{4.32}$$

As $\dot{b}$ is forced to be a new branch, there are $\gamma' > \gamma$ and conditions $\tilde{\tau}_0$ and $\tilde{\tau}^*_0$ in D_0 below $(p_g, q'_g, s, \mathrm{up}(\dot{r}_g))$ which decide $\dot{b}(\gamma')$ differently. Moreover, there is a condition $\tilde{\tau}_1$ below

$(p_h, q'_h, s, \mathrm{up}(\dot{r}_h))$ which decides the value of $\dot{b}(\gamma')$ and as $\tilde{\tau}_0$ and $\tilde{\tau}_0^*$ force different values, $\tilde{\tau}_1$ has to disagree with at least one of the two conditions $\tilde{\tau}_0$ or $\tilde{\tau}_0^*$. Assume without loss of generality that $\tilde{\tau}_0$ and $\tilde{\tau}_1$ disagree.[7] Now, as both these conditions are in D_0, their stems must be compatible, and therefore we can take their greatest lower bound if necessary to assume that these stems are equal. In some detail, assume $\tilde{\tau}_0 = (p_0, q_0, s_0, \mathrm{up}(\dot{r}_0))$, $\tilde{\tau}_1 = (p_1, q_1, s_1, \mathrm{up}(\dot{r}_1))$ and s_0 properly extends s_1 with regard to the Prikry points (and s_0 and s_1 are compatible regarding the collapsing information); then there is an extension (p'_1, q'_1) of (p_1, q_1) such that $(p'_1, q'_1, s_0^*, \mathrm{up}(\dot{r}_1))$ is a condition in $\mathbb{N}$ which extends $(p_1, q_1, s_1, \mathrm{up}(\dot{r}_1))$, where s_0^* has the same Prikry points as s_0 and the collapsing information is the greatest lower bound of the information in s_0 and s_1.

Let us denote this common stem s' and τ_0 and τ_1 the conditions with the common stem s' (note that τ_0 and τ_1 are still in D_0 after this modification by an argument similar to the proof of Lemma 4.9, in particular Remark 4.7).

The following sums up the situation (and fixes notation for τ_0 and τ_1):

$$\tau_0 = (p'_g, q''_g, s', \mathrm{up}(\dot{r}'_g)) \Vdash \dot{b}(\gamma') = \beta^0_{\gamma'} \tag{4.33}$$

and

$$\tau_1 = (p'_h, q''_h, s', \mathrm{up}(\dot{r}'_h)) \Vdash \dot{b}(\gamma') = \beta^1_{\gamma'} \tag{4.34}$$

for some $\beta^0_{\gamma'} \neq \beta^1_{\gamma'}$. Let i be such that $s_i = s'$.

As τ_0 is in D_0, there is $(s', \mathrm{up}(r'_g)) \in x$ such that the condition $(s', \mathrm{up}(r'_g), p'_g, q''_g, s', \mathrm{up}(\dot{r}'_g))$ is in D. The same holds also for τ_1: there is $(s', \mathrm{up}(r'_h)) \in x$ such that the condition $(s', \mathrm{up}(r'_h), p'_h, q''_h, s', \mathrm{up}(\dot{r}'_h))$ is in D. Moreover we can assume that $(s', \mathrm{up}(r'_g))$ extends $(s, \mathrm{up}(r_g))$ and $(s', \mathrm{up}(r'_h))$ extends $(s, \mathrm{up}(r_h))$. Let us denote these conditions by τ'_0 and τ'_1, respectively. Thus we obtain:

$$\tau'_0 = (s', \mathrm{up}(r'_g), p'_g, q''_g, s', \mathrm{up}(\dot{r}'_g)) \Vdash \dot{b}(\gamma') = \beta^0_{\gamma'} \tag{4.35}$$

and

$$\tau'_1 = (s', \mathrm{up}(r'_h), p'_h, q''_h, s', \mathrm{up}(\dot{r}'_h)) \Vdash \dot{b}(\gamma') = \beta^1_{\gamma'} \tag{4.36}$$

Note that as τ'_0 extends σ_0 and τ'_1 extends σ_1, they both force that $\dot{b}(\gamma) = \beta_\gamma$.

Denote $h^* = g \cap h$ and assume without loss of generality that g extends $h^* {}^\frown 0$ and h extends $h^* {}^\frown 1$. By the construction of the tree $\mathscr{T}$ (Property 3) there are $(p_0, p_1) \in Y_i^{h^*}$ compatible with (p'_g, p'_h), some $\gamma'' < \gamma$, and conditions ρ_0 and ρ_1 such that

$$\rho_0 = (s', \mathrm{up}(r^0), p_0, q_{h^* {}^\frown 0}, s', \mathrm{up}(\dot{r}^0)) \Vdash \dot{b}(\gamma'') = \beta^0_{\gamma''} \tag{4.37}$$

and

$$\rho_1 = (s', \mathrm{up}(r^1), p_1, q_{h^* {}^\frown 1}, s', \mathrm{up}(\dot{r}^1)) \Vdash \dot{b}(\gamma'') = \beta^1_{\gamma''} \tag{4.38}$$

[7] Put differently, some such $\tilde{\tau}_0$ and $\tilde{\tau}_1$ must exist otherwise the branch $\dot{b}$ would already be in the model $\mathscr{N}[h][x]$: for any generic $K_0 \times K_1$ for $\mathbb{N} \times \mathbb{N}$ below the pair composed of $(p_g, q'_g, s, \mathrm{up}(\dot{r}_g))$ and $(p_h, q'_h, s, \mathrm{up}(\dot{r}_h))$, we would have $\dot{b}^{K_0} = \dot{b}^{K_1}$.

with $\beta^0_{\gamma''} \neq \beta^1_{\gamma''}$.

We proceed to argue that τ'_0 is compatible with ρ_0 and τ'_1 is compatible with ρ_1.

Claim 4.15 *τ'_0 is compatible with ρ_0 and τ'_1 is compatible with ρ_1.*

Proof. We will show that τ'_0 is compatible with ρ_0; the argument for τ'_1 and ρ_1 is the same. Let us recall the two conditions:

$$\tau'_0 = (s', \mathrm{up}(r'_g), p'_g, q''_g, s', \mathrm{up}(\dot{r}'_g)), \tag{4.39}$$

$$\rho_0 = (s', \mathrm{up}(r^0), p_0, q_{h^* \frown 0}, s', \mathrm{up}(\dot{r}^0)). \tag{4.40}$$

Note that q''_g extends $q_{h^* \frown 0}$ in $\leq_{\mathbb{T}}$ and p_0 is compatible with p'_g. Let

$$(s', \mathrm{up}(r^*_0)) \in \mathbb{R} \tag{4.41}$$

be the greatest lower bounds of the conditions $(s', \mathrm{up}(r'_g))$ and $(s', \mathrm{up}(r^0))$, and

$$(p'_g \cup p_0, q''_g, s', \mathrm{up}(\dot{r}^*_0)) \in \mathbb{M}^* * \mathbb{R}^* \tag{4.42}$$

the greatest lower bound of $(p'_g, q''_g, s', \mathrm{up}(\dot{r}'_g))$ and $(p_0, q_{h^* \frown 0}, s', \mathrm{up}(\dot{r}^0))$.

By Fact 4.8 there is a direct extension $(s', \mathrm{up}(r_0))$ of $(s', \mathrm{up}(r^*_0))$ which forces the condition $(p'_g \cup p_0, q''_g, s', \mathrm{up}(\dot{r}^*_0))$ into $\dot{\mathbb{N}}$. Therefore the condition

$$\tau''_0 = (s', \mathrm{up}(r_0), p'_g \cup p_0, q''_g, s', \mathrm{up}(\dot{r}^*_0)) \tag{4.43}$$

is in $\mathbb{R} * \dot{\mathbb{N}}$ and witnesses that τ'_0 is compatible with ρ_0.

Consider a similar witness τ''_1 for τ'_1 and ρ_1.

$$\tau''_1 = (s', \mathrm{up}(r_1), p'_g \cup p_0, q''_g, s', \mathrm{up}(\dot{r}^*_1)) \tag{4.44}$$

This finishes the proof of the claim. □

Note that we have $\gamma'' < \gamma$ and as τ''_0 is below both τ'_0 and ρ_0 we have

$$\tau''_0 \Vdash \dot{b}(\gamma) = \beta_\gamma \ \&\ \dot{b}(\gamma'') = \beta^0_{\gamma''} \tag{4.45}$$

and similarly for τ'_1 and ρ_1:

$$\tau''_1 \Vdash \dot{b}(\gamma) = \beta_\gamma \ \&\ \dot{b}(\gamma'') = \beta^1_{\gamma''}. \tag{4.46}$$

Let $(s', \mathrm{up}(r))$ be the greatest lower bound of $(s', \mathrm{up}(r_0))$ and $(s', \mathrm{up}(r_1))$. Since $\dot{b}$ is forced to be a branch, (4.45) implies that τ''_0 forces $\beta^0_{\gamma''} <_{\dot{T}} \beta_\gamma$. As $<_{\dot{T}}$ depends only on x (as it is equivalent to an $\mathbb{R}$-name) already $(s', \mathrm{up}(r)) \leq (s', \mathrm{up}(r_0))$ forces $\beta^0_{\gamma''} <_{\dot{T}} \beta_\gamma$. By the same argument with (4.46) we have that $(s', \mathrm{up}(r))$ forces $\beta^1_{\gamma''} <_{\dot{T}} \beta_\gamma$. This is a contradiction because $\beta^0_{\gamma''} \neq \beta^1_{\gamma''}$ contradicts the fact that $\dot{T}$ is forced by $\mathbb{R}$ to be a tree so it cannot have two incomparable nodes below the node β_γ.

This finishes the proof of the lemma and also of Theorem 4.3. □

5 The tree property with a finite gap

In this section we modify the argument in Section 4 and show that it is possible to have a strong limit $\aleph_\omega$ with $2^{\aleph_\omega} = \aleph_{\omega+n+2}$ and the tree property at $\aleph_{\omega+2}$, for any $0 \le n < \omega$.

Let $1 \le n < \omega$ be fixed. We need to modify the forcing used for gap 2 as follows:

- Using the argument in Section 3.1, we prepare the universe with $\mu = \lambda^{+n}$. Let us denote this forcing Q^n.
- The definition of P_κ in (3.3) is to be modified as follows:

  (5.47) $$P^n_\kappa = \langle (P^n_\alpha, \dot{Q}^n_\alpha) \,|\, \alpha < \kappa \text{ is measurable}\rangle,$$

  where $\dot{Q}^n_\alpha$ denotes the forcing $\mathbb{M}(\alpha, \lambda_\alpha, \lambda_\alpha^{+n})$.
- Let $G_\kappa * H$ be a generic filter for $P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n})$, and let $j : V^1[G_\kappa * H] \to M^1(j(G_\kappa * H))$ be the lifting of j as in Theorem 3.3.
- Let Coll^n denote the forcing $\mathrm{Coll}((\kappa^{+3+n}, < j(\kappa)))^{M^1[j(G_\kappa * H)]}$. As in Lemma 4.1, we can fix a guiding generic G^g for Coll^n over $M^1[j(G_\kappa * H)]$.
- The main forcing is defined as follows:

  (5.48) $$\mathbb{P}^n = Q^n * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n}) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g),$$

  where $\dot{U}$ is a name for a normal measure and $\dot{G}^g$ is a name for a guiding generic (defined with respect to $\dot{U}$).

Now we get the following generalisation of Theorem 4.3:

Theorem 5.1 *(GCH) Let $1 < n < \omega$ be fixed and assume that κ is $H(\lambda^{+n})$-hypermeasurable, where $\lambda > \kappa$ is the least weakly compact cardinal above κ. The forcing $\mathbb{P}^n$ in (5.48) forces $\kappa = \aleph_\omega$, $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2+n}$, and the tree property holds at $\lambda = \aleph_{\omega+2}$.*

Proof. The basic strategy of the proof is to reduce the general case to a configuration essentially identical to the argument for the gap 2.

Recall that the whole forcing in V looks as follows:

(5.49) $$\mathbb{P}^n = Q^n * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n}) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g),$$

where Q^n is the preparation $i(\mathrm{Add}(\kappa, \lambda^{+n}))^N = (\mathrm{Add}(i(\kappa), \lambda^{+n})^N$ (i is the normal measure embedding derived from j which witnesses the $H(\lambda^{+n})$-hypermeasurability of κ). Let us denote by Q^n_β the natural truncation of Q^n to length $\beta < \lambda^{+n}$. Note that the forcing (5.49) is λ-cc.

Suppose for contradiction that the forcing in (5.49) adds a λ-Aronszajn tree $\dot{T}$ (and assume for simplicity that the weakest condition forces it).

Let $\mathscr{A}$ be an elementary substructure of large enough $H(\theta)^V$ which has size λ, is closed under $< \lambda$-sequences, and contains the name $\dot{T}$ and other relevant data. Let $c : \mathscr{A} \to \bar{\mathscr{A}}$ be the transitive collapse. Then the following hold:

(i) $c(\lambda^{+n})$ is an ordinal between λ and λ^+, let us denote this ordinal as β.
(ii) $c(Q^n)$ is isomorphic to Q^n_β.
(iii) The name $c(P^n_\kappa)$ interprets in $V[Q^n_\beta]$ as P^n_κ does in $V[Q^n]$.
(iv) The name $c(\mathbb{M}(\kappa, \lambda, \lambda^{+n}))$ interprets in $V[Q^n_\beta * P^n_\kappa]$ as a forcing equivalent to $\mathbb{M}(\kappa, \lambda, \beta)$ as interpreted in $V[Q^n * P^n_\kappa]$.
(v) The name $c(\dot{U})$ interprets in $V[Q^n_\beta * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \beta)]$ as a normal ultrafilter on κ generating some guiding generic $c(\dot{G}^g)$, and therefore $Q^n_\kappa * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \beta)$ forces that $\mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ is a Prikry forcing with collapses.
(vi) $c(\dot{T})$ is forced (over $\bar{\mathscr{A}}$) by

$$Q^n_\beta * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$$

to be a λ-Aronszajn tree.

Let us denote $Q^n_\beta * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ by $c(\mathbb{P}^n)$. By the λ-cc of the forcing, the elementary embedding c^{-1} lifts to an elementary embedding

$$c^{-1} : \bar{\mathscr{A}}[G(c(\mathbb{P}^n))] \to H(\theta)[G(\mathbb{P}^n)], \tag{5.50}$$

where $G(\mathbb{P}^n)$ is $\mathbb{P}^n$-generic over V and $G(c(P^n)) = c''G(\mathbb{P}^n)$ is $c(\mathbb{P}^n)$-generic over V. Since c^{-1} applied to $T = c(\dot{T})^{G(c(\mathbb{P}^n))}$ is the identity, T is by elementarity a λ-Aronszajn tree in $H(\theta)[G(\mathbb{P}^n)]$, and also in $V[G(\mathbb{P}^n)]$ (because $H(\theta)[G(\mathbb{P}^n)]$ contains any cofinal branch through T existing in $V[G(\mathbb{P}^n)]$). Since the model $V[G(\mathbb{P}^n)]$ contains the model $V[G(c(\mathbb{P}^n))]$, T is a λ-Aronszajn tree not only in $\bar{\mathscr{A}}[G(c(\mathbb{P}^n))]$, but also in $V[G(c(\mathbb{P}^n))]$. It follows that if $\mathbb{P}^n$ adds a λ-Aronszajn tree, so does $c(\mathbb{P}^n)$.

We finish the proof by arguing that $c(\mathbb{P}^n)$ cannot add a λ-Aronszajn tree.

Lemma 5.2 *Let $\lambda \le \beta < \lambda^+$ be as in the previous paragraph. Then*

$$c(\mathbb{P}^n) = Q^n_\beta * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g)) \tag{5.51}$$

does not add a λ-Aronszajn tree.

Proof. Let us work in $V^1[G_\kappa]$, which is a generic extension of V by the forcing $Q^n * P^n_\kappa$. Let $\mathbb{M}$ denote the forcing $\mathbb{M}(\kappa, \lambda, \beta)$ and let us assume for contradiction that $\dot{T}$ is a name for a λ-Aronszajn tree.

Following Remark 4.4, let us fix in $V^1[G_\kappa]$ an embedding

$$k : \mathscr{M} \to \mathscr{N} \tag{5.52}$$

such that $\mathscr{M}$ and $\mathscr{N}$ are transitive models of size λ closed under κ-sequences, $\mathscr{M} \in \mathscr{N}$, $k \in \mathscr{N}$, $\beta < k(\lambda)$, and $\mathscr{M}$ contains all relevant information (in particular, β, $\mathbb{M} * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$, $\dot{U}_\beta$, $\dot{G}^g_\beta$, and $\dot{T}$ are elements of $\mathscr{M}$). Choose β^* in Section 4.4 high enough to ensure $\beta < k(\lambda)$. Let $\mathbb{M}^*$ denote $k(\mathbb{M}(\kappa, \lambda, \beta))$, which is equal to $\mathbb{M}(\kappa, k(\lambda), k(\beta))$.

Let h^* be $\mathbb{M}^*$-generic over $V^1[G_\kappa]$; use h^* to define h which is $\mathbb{M}$-generic over $V^1[G_\kappa]$ and $k``h \subseteq h^*$. Now lift to

$$k : \mathscr{M}[h] \to \mathscr{N}[h^*]. \tag{5.53}$$

Let us write $U = (\dot{U}_\beta)^h$ and $G^g = (\dot{G}^g_\beta)^h$ instead of U_β and G^g_β.

As in Section 4.5, we can argue that k is a regular embedding:

$$k : \mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g) \to \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}), \tag{5.54}$$

as by the λ-cc of $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, if A is a maximal antichain in $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, then $k(A) = k``A$ is a maximal antichain in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$.

Let x^* be $\mathrm{PrkCol}(U^*, G^{g*})$-generic over $V^1[G_\kappa][h^*]$; the pull-back of x^* via k^{-1} is a generic filter x for $\mathrm{PrkCol}(U, G^g)$ such that $k``x \subseteq x^*$. Let us lift k further to

$$k : \mathscr{M}[h][x] \to \mathscr{N}[h^*][x^*]. \tag{5.55}$$

By (5.54) and (5.55), we can define in $\mathscr{N}[h^*][x^*]$ a generic filter $h*x$ for $\mathbb{M}*\mathrm{PrkCol}(\dot{U}, \dot{G}^g) \in \mathscr{N}$ using the inverse of k (by our assumptions in (5.52), k is an element of $\mathscr{N}$); thus

$$\mathscr{N}[h][x] \subseteq \mathscr{N}[h^*][x^*]. \tag{5.56}$$

Let us denote the quotient determined by k in (5.54) as $\mathbb{N}$:

$$\mathbb{N} = \{((p,q),\dot{r}) \in \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \mid ((p,q),\dot{r}) \text{ is compatible with all conditions in } k``(h * x)\}. \tag{5.57}$$

It follows that over $\mathscr{N}$,

$$\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \text{ is equivalent to } \mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g) * \dot{\mathbb{N}}. \tag{5.58}$$

We finish the proof by arguing that $\mathbb{N}$ does not add new branches to λ-trees over $\mathscr{N}[h][x]$. We proceed as in the case of gap 2: the facts in Section 4.5.1 are readily applicable to our case (with the only difference that k is no longer the identity on the conditions of the forcing) and so is the argument with the labeled tree of conditions in Sections 4.5.2 and 4.5.3. □

This finishes the proof of the theorem. □

Remark 5.3 Strictly speaking, the forcing $\mathbb{P}^n$ is not of the type considered in Theorem 4.3: Instead of P_κ, we now have P^n_κ, and the guiding generic $\dot{G}^g$ is generic for the forcing $\mathrm{Coll}(\kappa^{+2+n}, < j(\kappa))$ of the measure ultrapower, and not for $\mathrm{Coll}(\kappa^{+3}, < j(\kappa))$ as in Theorem 4.3. However, it is easy to check that the argument for the tree property at $\aleph_{\omega+2}$ only uses the chain condition and closure properties of the relevant forcings, and these are not affected by these modifications.

6 Open questions

The following questions are not solved by the methods of this paper:

Q1. Can we obtain an infinite gap at $2^{\aleph_\omega}$? More precisely, given an $\omega \leq \alpha < \omega_1$, is there a model where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+\alpha+1}$, and the tree property holds at $\aleph_{\omega+2}$?

It seems that an entirely different method is required for this configuration (perhaps based on the methods of Magidor [13] and Shelah [15]).

Q2. Can we obtain a similar result for $\aleph_{\omega_1}$? Or in general, for a strong limit singular cardinal κ of any uncountable cofinality? Notice that the results in [11] show this for κ which is singularized to a prescribed cofinality with Magidor forcing without collapsing cardinals below κ (with 2^κ being arbitrarily large), but it is open for Prikry-like forcings with interleaved collapses (i.e. cases such as $\aleph_{\omega_1}$ or $\aleph_{\omega_2}$).

Q3. It is of interest to consider the tree property at $\aleph_{\omega+2}$ with different patterns of the continuum function below $\aleph_\omega$ (such as GCH below $\aleph_\omega$ or other more general behaviours). The papers [9] and [7] are relevant for this question. Regarding the paper [7], it is unclear whether Gitik's short extender forcing – which gives the tree property at $\aleph_{\omega+2}$ from the optimal hypothesis (with gap 2) – can be used for larger gaps.

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