The tree property at $\aleph_{\omega+2}$ with a finite gap[1]

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Abstract: Let n be a natural number, $2 \leq n < \omega$. We show that it is consistent to have a model of set theory where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+n}$, and the tree property holds at $\aleph_{\omega+2}$; we use a weakly compact hypermeasurable cardinal for the result. This generalises the known result with $n = 2$. We note that this is the first exposition of the tree property at $\aleph_{\omega+2}$ with $\aleph_\omega$ strong limit which uses a projection-of-product analysis of the Mitchell forcing followed by Prikry forcing with collapses reminiscent of the analysis in Abraham [1].

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1 Introduction

Let κ be a regular cardinal. We say that the tree property holds at κ if every κ-tree has a cofinal branch. The tree property is a compactness property which can hold at successor cardinals low in the set-theoretical hierarchy: Mitchell first showed in [13] that it is equiconsistent with the existence of a weakly compact cardinal that $\aleph_2$ has the tree property (his argument readily generalises to any κ^{++} for an infinite regular cardinal κ).

The situation is more complex when we wish to get the tree property at the double successor of a singular strong limit cardinal κ. First, since one needs to have $2^\kappa > \kappa^+$ (and thus the failure of SCH, the singular cardinal hypothesis), it is known that a measurable cardinal of high Mitchell order is required. Second, a new idea is required which connects the Mitchell construction and the known ways of obtaining the failure of SCH. This was first achieved by Cummings and Foreman [4] who proved that it is consistent to have a singular strong limit cardinal κ of countable cofinality with the tree property at κ^{++}. The cardinal κ in [4] was supercompact in the ground model; without a proof, [4] claimed that κ can be collapsed to $\aleph_\omega$ using a similar argument. However, for some time no such proof had been found.

The first argument which yields a model where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2}$, and the tree property holds at $\aleph_{\omega+2}$, was given by Friedman and Halilović in [6]. The argument started with a much weaker hypothesis than [4] (an $H(\lambda)$-hypermeasurable κ for a weakly

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compact $\lambda > \kappa$), and used an iteration of the κ-Sacks forcing followed by the Prikry forcing and therefore was quite different in spirit from the construction in [4].

Recently, the authors of [5] found another construction which yields the same configuration (in particular with $2^{\aleph_\omega} = \aleph_{\omega+2}$). The construction is based on the Mitchell forcing followed by the Prikry forcing with collapses, and starts with a supercompact cardinal. It is of interest to note that the proof that the tree property holds at $\aleph_{\omega+2}$ proceeds directly without using a projection-of-product analysis (which is at the heart of the presentations of Abraham [1] and crucial in the analysis in [4]).

In the present paper, we formulate yet another approach to constructing a model where $\aleph_\omega$ is strong limit, violates SCH, and the tree property holds at $\aleph_{\omega+2}$. The main features of our construction are as follows:

- We start with a suitably large hypermeasurable cardinal;
- We are able to ensure that $2^{\aleph_\omega}$ is equal to $\aleph_{\omega+n}$ for any fixed $2 \leq n < \omega$;
- We use a variant of the Mitchell forcing (to ensure the large value of $2^{\aleph_\omega}$) followed by the Prikry forcing with collapses. Our proof relies on a projection-of-product analysis and is therefore similar to the methods of [4].[2]

The paper is structured as follows.

In Section 1.1 we review the forcings which we will use: in particular, we define a variant of the Mitchell forcing which ensures the large value of $2^{\aleph_\omega}$, and provide a product analysis of the Mitchell forcing followed by the Prikry forcing with collapses which is reminiscent of the analysis in [1] and [4] (Section 1.1.3).

In Section 2, we argue that it is possible to start with a hypermeasurable cardinal κ of a suitable degree and prepare the ground model so that a further forcing with the Cohen product at κ (of a prescribed length) does not destroy the measurability of κ.[3]

In Section 3, we show that over the prepared ground model, the standard Mitchell forcing followed by the Prikry forcing with collapses forces that $\kappa = \aleph_\omega$ is a strong limit cardinal, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and the tree property holds at $\aleph_{\omega+2}$.

In Section 4 we generalise the construction in Section 3 to any finite gap $2 \leq n < \omega$.

Finally, in Section 5 we mention some open questions.

1.1 Preliminaries

1.1.1 A variant of Mitchell forcing

We will use a variant of the standard Mitchell forcing as presented in [1].

[2]There is a fine difference in the proofs, though: in [4], a "Prikry-ised Mitchell forcing" is used, i.e. the Prikry part is integrated into the Mitchell forcing; we used a similar method in [7] to obtain a large value of 2^κ with the tree property at κ^{++}, κ strong-limit with countable cofinality. When the collapsing is involved, it is easy to see that the Prikry forcing with collapses must come after the Mitchell forcing, and cannot be integrated into the Mitchell part. Surprisingly, the latter method seems easier and more universal (it can be used to reprove the result of [4]).

[3]With more work, we can also preserve the initial degree of hypermeasurability; see Remark 2.6.

If κ is a regular infinite cardinal and α is an ordinal greater than 0, we identify the Cohen forcing for adding α-many subsets of κ, $\mathrm{Add}(\kappa,\alpha)$, with a collection of functions p from a subset of $\kappa\times\alpha$ of size $<\kappa$ into $\{0,1\}$. The ordering is by reverse inclusion.

Let $\kappa<\lambda$ be regular cardinals, and assume λ is inaccessible. Let $\mu>\lambda$ be an ordinal. We define a variant of the Mitchell forcing, $\mathbb{M}(\kappa,\lambda,\mu)$, as follows: Conditions are pairs (p,q) such that p is in $\mathrm{Add}(\kappa,\mu)$, and q is a function whose domain is a subset of λ of size at most κ such that for every $\xi\in\mathrm{dom}(q)$, $q(\xi)$ is an $\mathrm{Add}(\kappa,\xi)$-name, and $\emptyset\Vdash_{\mathrm{Add}(\kappa,\xi)} q(\xi)\in\mathrm{Add}(\kappa^+,1)$. The ordering is as in the standard Mitchell forcing, i.e.: $(p',q')\le(p,q)$ if and only if p' is stronger than p in the Cohen forcing, the domain of q' contains the domain of q and if ξ is in the domain of q, then p' restricted to ξ forces $q'(\xi)$ extends $q(\xi)$.

Lemma 1.1 *Assume GCH.*

(i) $\mathbb{M}(\kappa,\lambda,\mu)$ *is* λ*-Knaster.*
(ii) In $V[\mathbb{M}(\kappa,\lambda,\mu)]$, $2^\kappa=|\mu|$, *and the cardinals in the open interval* (κ^+,λ) *are collapsed (and no other cardinals are collapsed).*

Proof. The proof is standard (using a Δ-system argument for Knasterness). □

The following follows as in [1]:

Lemma 1.2 *(i)* $\mathbb{M}(\kappa,\lambda,\mu)$ *is a projection of* $\mathrm{Add}(\kappa,\mu)\times\mathbb{T}$, *where* $\mathbb{T}$ *is a* κ^+*-closed term forcing defined by* $\mathbb{T}=\{(\emptyset,q)\mid(\emptyset,q)\in\mathbb{M}(\kappa,\lambda,\mu)\}$.
(ii) $\mathbb{M}(\kappa,\lambda,\mu)$ *is equivalent to* $\mathrm{Add}(\kappa,\mu)*\dot{R}$, *where* $\dot{R}$ *is forced to be* κ^+*-distributive.*

Proof. The proof is as in [1]. □

As will be apparent from the arguments in Section 3, it is also the case that if λ is weakly compact, then the tree property holds at $\lambda=\kappa^{++}$ in $V[\mathbb{M}(\kappa,\lambda,\mu)]$.

1.1.2 Prikry forcing with collapses

We use the forcing as it is described in Gitik's paper [8].

Here we give just a quick review to fix the notation. Let κ be a measurable cardinal, U a normal measure at κ, and $j_U:V\to M$ the ultrapower embedding generated by U. The Prikry forcing with collapses, which we denote $\mathrm{PrkCol}(U,G^g)$, is determined by U and a guiding generic G^g. G^g is a $\mathrm{Coll}(\kappa^{+n},<j(\kappa))^M$-generic filter over M, where n typically satisfies $2<n<\omega$ (Coll denotes the Levy collapse).

A condition r in $\mathrm{PrkCol}(U,G^g)$ has a lower part ("stem") which is a finite increasing sequence of cardinals below κ with information about collapses between the cardinals (thus the stem is an element of V_κ), and an upper part which is composed of sets A and H, where A is in U, and H is a function defined on A such that $[H]_U$, the equivalence class of H in M, belongs to G^g.

If all is set up correctly in V, the forcing $\mathrm{PrkCol}(U,G^g)$ collapses κ to $\aleph_\omega$ while preserving all cardinals above κ.

1.1.3 Mitchell followed by Prikry forcing with collapses

Assume $\kappa < \lambda < \mu$ are as above, $\mathbb{M} = \mathbb{M}(\kappa, \lambda, \mu)$ is the Mitchell forcing, and $\dot{U}$ and $\dot{G}^g$ are $\mathbb{M}$-names such that the weakest condition in $\mathbb{M}$ forces that $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ is the Prikry forcing with collapses defined with respect to $\dot{U}$ and $\dot{G}^g$.

Lemma 1.3 *$\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ is λ-Knaster.*

Proof. This follows by a Δ-system argument applied to $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ (the compatibility of the Prikry component is determined by the compatibility of the stems, and there are only κ-many of these). □

Lemma 1.4 *In $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, conditions $((p, q), r)$, where r in the Prikry forcing depends only on the Cohen information of the Mitchell forcing and its stem is a checked name, are dense.*

Proof. This is because all conditions in the Prikry forcing exist already in the extension by the Cohen part of the Mitchell forcing (in contrast, the definition of $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ itself may require the whole $\mathbb{M}$ in order to refer to $\dot{U}$ and $\dot{G}^g$; this will be the case in our argument in Section 3.2.3). Given $((p, q), r)$ we can extend (p, q) to some (p', q') such that (p', q') forces that r is equal to some r' in the generic extension by the Cohen part of the Mitchell forcing with its stem being a ground model object (since the Cohen forcing at κ does not add bounded subsets of V_κ). □

Using Lemma 1.4, we can formulate a projection-of-product analysis of the forcing $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ reminiscent of Abraham's analysis of the Mitchell forcing in [1]. Let us define:

$$\mathbb{C} = \{((p, \emptyset), r) \mid ((p, \emptyset), r) \in \mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)\}, \tag{1.1}$$

where we require that r depends only on the Cohen information of the Mitchell forcing and its stem is a checked name.[4] Let us also define:

$$\mathbb{T} = \{(\emptyset, q) \mid (\emptyset, q) \in \mathbb{M}\}. \tag{1.2}$$

Define a function τ from $\mathbb{C} \times \mathbb{T}$ to $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ as follows: τ applied to the pair composed of $((p, \emptyset), r)$ and $(\emptyset, q)$ is equal to the condition $((p, q), r)$.

Lemma 1.5 *(i) τ is a projection from $\mathbb{C} \times \mathbb{T}$ onto a dense part of $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$.*
(ii) $\mathbb{T}$ is κ^+-closed in V.
(iii) $\mathbb{C}$ is κ^+-cc in V.

Proof. (i). If $((p', \emptyset), r') \leq ((p, \emptyset), r)$ and $(\emptyset, q') \leq (\emptyset, q)$, then $((p, q'), r') \leq ((p, q), r)$, so τ is order-preserving.

[4] The requirement that the stem of r is a checked name is not important for Lemma 1.5, but will be useful in Section 3.2.3 when a similar analysis is performed.

Now, suppose we are given $((p', q'), r') \leq ((p, q), r)$, where

$$((p, q), r) = \tau(((p, \emptyset), r), (\emptyset, q))$$

and r' depends only on the Cohen information of the Mitchell forcing and its stem is a checked name (by Lemma 1.4 such conditions are dense). We will find q^* and r^* such that

(a) $(\emptyset, q^*) \leq (\emptyset, q)$ in $\mathbb{T}$,
(b) $((p', \emptyset), r^*) \leq ((p, \emptyset), r)$ in $\mathbb{C}$,
(c) $\tau(((p', \emptyset), r^*), (\emptyset, q^*)) = ((p', q^*), r^*) \leq ((p', q'), r')$.

In order to get (a)–(b), first define q^* so that it interprets as q' below p', and as q below conditions incompatible with p' (ensuring (a)). Since we assume that r and r' depend only on the Cohen forcing (and have checked names for their stems), we can take $r^* = r'$ (ensuring (b)). (c) is clear by the definition of q^* and r^*.

Items (ii) and (iii) are obvious. □

The existence of the projection τ in Lemma 1.5 will be useful (in a quotient setting) in Section 3.2.3.

2 Preserving measurability by Mitchell forcing

In [4], the construction which yields the tree property at the double successor of a singular strong limit κ with countable cofinality starts by assuming that κ is supercompact. The reason is that we can then invoke Laver's indestructibility result [11], and assume that adding any number of Cohen subsets of κ will preserve the measurability of κ. Such an assumption tends to simplify the subsequent constructions because one can avoid the work of lifting a weaker embedding using a surgery argument, or some other methods.

A natural question is whether a "Laver-like" indestructibility is available also for smaller large cardinals. In this Section, we use an idea of Cummings and Woodin (see [2]) to argue that it is possible to have a limited indestructibility for μ-tall cardinals κ,[5] where $\mu > \kappa$ is a regular cardinal.[6]

2.1 Stage 1

Assume GCH and suppose that $\mu > \kappa$ is the successor[7] of the least weakly compact cardinal λ above κ and $j : V \to M$ is an $H(\mu)$-hypermeasurable embedding with the extender representation:

$$M = \{j(f)(\alpha) \mid f : \kappa \to V \ \& \ \alpha < \mu\}.$$

In particular, $H(\mu)$ is included in M and M is closed under κ-sequences in V. Let U be the normal measure derived from j, and let $i : V \to N$ be the ultrapower embedding generated

[5] κ is μ-tall if there is an embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \mu$ and M is closed under κ-sequences.
[6] With more work, one can also preserve the hypermeasurability of κ; see Remark 2.6.
[7] Or more generally the n-th successor for some finite $n > 1$.

by U. Let $k : N \to M$ be elementary so that $j = k \circ i$. Note that κ is the critical point of j, i and j, i have width κ, i.e. every element of M and N is of the form $j(f)(\alpha)$, or $i(f)(\kappa)$ respectively, for some f with domain κ. In contrast, the critical point of k is $(\kappa^{++})^N$ and k has width μ_κ, i.e. every element of M can be written as $k(f)(\alpha)$ for some f in N with domain μ_κ, where μ_κ is the successor of the least weakly compact cardinal above κ in N, in particular $(\kappa^{++})^N < \mu_\kappa < i(\kappa) < \kappa^{++}$. See [3] for more details regarding the lifting of embeddings and the notion of width.

Let P denote the forcing $\mathrm{Add}(\kappa, \mu)$ in V, $Q = i(P)$, and let g be a Q-generic filter over V. Then the following hold:

Theorem 2.1 *Assume GCH. Forcing with Q preserves cofinalities and the following hold in $V[g]$:*

(i) j lifts to $j^1 : V[g] \to M[j^1(g)]$, where j^1 restricted to V is the original j.
(ii) i lifts to $i^1 : V[g] \to N[i^1(g)]$, where i^1 restricted to V is the original i. $N[i^1(g)]$ is the measure ultrapower obtained from j^1.
(iii) k lifts to $k^1 : N[i^1(g)] \to M[j^1(g)]$, where k^1 restricted to N is the original k.
(iv) g is Q-generic over $N[i^1(g)]$.
(v) There is $\tilde{g}$ in $V[g]$ such that $\tilde{g}$ is $k(Q) = j(P)$-generic over $M[j^1(g)]$.

Proof. We show that Q is κ^+-closed and κ^{++}-cc in V. Closure is obvious by the fact that N is closed under κ-sequences in V. Regarding the chain condition, notice that every element of Q can be identified with the equivalence class of some function $f : \kappa \to \mathrm{Add}(\kappa, \mu)$. For $f, f' : \kappa \to \mathrm{Add}(\kappa, \mu)$, set $f \le f'$ if for all $i < \kappa$, $f(i) \le f'(i)$; it suffices to check that the ordering $\le$ on these f's is κ^{++}-cc. Let A be a maximal antichain in this ordering; take an elementary substructure $\bar{M}$ in some large enough $H(\theta)$ of V which contains all relevant data, has size κ^+ and is closed under κ-sequences. Then it is not hard to check that $A \cap \bar{M}$ is maximal in the ordering (and so $A \subseteq \bar{M}$), and therefore has size at most κ^+.

(i) and (ii). These follow by κ^+-distributivity of Q in V and the fact that j, i have width κ: the pointwise image of g generates a generic for $j(Q)$ and $i(Q)$, respectively.

(iii). $i(Q)$ is $i(\kappa^+)$-closed in N, and since $\mu_\kappa < i(\kappa^+)$, we use the distributivity of $i(Q)$ and the fact that k has width μ_κ to argue that the pointwise image $k"(i^1(g))$ generates a generic filter which is equal to the generic filter generated by $j"g$ by commutativity of j, i, k.

(iv). Q is $i(\kappa^+)$-cc in N and $i(Q)$ is $i(\kappa^+)$-closed in N. Therefore g and $i^1(g)$ are mutually generic over N by Easton's lemma.

(v). Q is $i(\kappa)$-closed in $N[i^1(g)]$ since the generic $i^1(g)$ does not add new sequences of length $i(\kappa)$; it follows as in (iii) that $k^1"g$ generates a $j(P)$-generic filter $\tilde{g}$ over $M[j^1(g)]$. □

Remark 2.2 Notice that g is not present in M^1. However, if so desired,[8] we can ensure that κ is still $H(\mu)$-hypermeasurable after the generic object $\tilde{g}$ is added; see Remark 2.6 for more details.

[8] This is not required for the present proof, but may be useful if more complicated forcings are to be defined over V^1 (such as the Radin forcing).

2.2 Stage 2

Let us work in the model $V[g] = V^1$ and let us use the notation j^1, V^1, M^1 to denote the resulting models and embeddings in Theorem 2.1. Recall that by Remark 2.2, j^1 is just μ-tall (but the initial $H(\mu)$-hypermeasurability of j still implies that the cardinals in the interval $[\kappa, \mu]$ coincide between V^1 and M^1). Note that λ is no longer strong limit in V^1, but we will argue in Section 3.2.1 that it retains enough of weak compactness in V^1 for further arguments.

Define P_κ to be the following Easton-supported iteration:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa \text{ is measurable} \rangle, \tag{2.3}$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathbb{M}(\alpha, \lambda_\alpha, \mu_\alpha)$, where λ_α is the least weakly compact cardinal above α, and $\mu_\alpha = (\lambda_\alpha)^+$.

Theorem 2.3 *The following hold:*

(i) In $V^1[P_\kappa][\mathbb{M}(\kappa, \lambda, \mu)]$, $\lambda = \kappa^{++}$, $2^\kappa = \kappa^{+3} = \mu$, and κ is measurable.
(ii) The measurability of κ is witnessed by a lifting of j^1, which we call j^2,

$$j^2 : V^1[P_\kappa][\mathbb{M}(\kappa, \lambda, \mu)] \to M^2 = M^1[j^2(P_\kappa * \mathbb{M}(\kappa, \lambda, \mu))].$$

Moreover, j^2 is the normal measure embedding derived from j^2, and M^2 satisfies $\lambda = \kappa^{++}$ and $2^\kappa = \kappa^{+3} = \mu$.

Proof. Let $G_\kappa * H$ be $P_\kappa * \mathbb{M}(\kappa, \lambda, \mu)$-generic over V^1.

(i). We follow closely the argument in Cummings [2] but with the important simplification that we use the factoring through k only in stage 1 (Theorem 2.1), and use directly the generic object $\tilde{g}$ (Theorem 2.1) to lift only the embedding j^1 (we do not lift k^1 and i^1).[9]

Using standard methods, lift j^1 to

$$j^2 : V^1[G_\kappa] \to M^1[G_\kappa][H][h],$$

where h is constructed using the extender representation of M^1: the dense open sets in the forcing $j^1(P_\kappa)$ in the interval $(\kappa, j^1(\kappa))$ can be grouped into κ^+-many groups each of size μ in $M^1[G_\kappa][H]$; these groups are of the form $\{j^1(f)(\alpha) \mid \alpha < \mu\}$, where f is a function from κ to $H(\kappa)$. The intersection of each group is a dense set because the forcing $j^1(P_\kappa)$ in the interval $(\kappa, j^1(\kappa))$ is μ^+-closed in $M^1[G_\kappa][H]$. Since there are only κ^+-many of these groups, a generic h can be constructed in $V^1[G_\kappa][H]$ which meets them all.

It remains to find a generic filter for the j^2-image of $\mathbb{M}(\kappa, \lambda, \mu)$. Using the fact that the Mitchell forcing decomposes into $\mathrm{Add}(\kappa, \mu) * \dot{R}$ for some $\dot{R}$ which is forced to be κ^+-distributive by $\mathrm{Add}(\kappa, \mu)$ (see Section 1.1.1), it suffices first to lift $\mathrm{Add}(\kappa, \mu)$, and then

[9] Lifting through k^1 is problematic at stage κ where we deal with the forcing $\mathbb{M}(\kappa, \lambda, \mu)$ in the sense of the ultrapower (the forcing is non-trivially moved by k^1 – a fact innocuous for the Cohen forcing at κ, but problematic for the Mitchell forcing).

(easily) lift the distributive part $\dot{R}$. Let us write $H = g_\kappa * h_\kappa$ where g_κ is Cohen generic and h_κ is $\dot{R}$-generic.

In order to lift $\mathrm{Add}(\kappa,\mu)$, we use the generic object $\tilde{g}$ which we prepared in V^1. Notice that $\tilde{g}$ is generic for the wrong forcing: it is $j^1(\mathrm{Add}(\kappa,\mu))$-generic over M^1, but we need a generic object for $j^2(\mathrm{Add}(\kappa,\mu))$ over $M^1[G_\kappa][H][h]$. We use the following fact to overcome this problem.[10]

Fact 2.4 *Let S be a κ-cc forcing notion of cardinality κ, $\kappa^{<\kappa} = \kappa$. Then for any μ, the term forcing $Q_\mu = \mathrm{Add}(\kappa,\mu)^{V[S]}/S$ is isomorphic to $\mathrm{Add}(\kappa,\mu)$.*

By elementarity, Fact 2.4 implies that in $V^1[G_\kappa][H]$, $\tilde{g}$ yields a generic object g^* over $M^1[G_\kappa][H][h]$ for $j^2(\mathrm{Add}(\kappa,\mu))$ (note that $j^1(P_\kappa)$ has size $j^1(\kappa)$ in M^1 and is $j^1(\kappa)$-cc). g^* is still not good enough to lift j^2 because it may not contain the pointwise image $j^{2}{}''g_\kappa$. Using the method of surgery (see [2]), we modify g^* to g^{**} which is still $j^2(\mathrm{Add}(\kappa,\mu))$-generic, but in addition contains the pointwise image $j^{2}{}''g_\kappa$. It follows we can lift to

$$j^2 : V^1[G_\kappa][g_\kappa] \to M^1[G_\kappa][H][h][g^{**}],$$

and then finally to $V^1[G_\kappa][g_\kappa][h_\kappa] = V^1[G_\kappa][H]$:

$$j^2 : V^1[G_\kappa][H] \to M^2 = M^1[G_\kappa][H][h][g^{**}][h^*],$$

where h^* is generated from $j^{2}{}''h_\kappa$. The last lifting shows that κ remains measurable as desired.

(ii). It remains to show that j^2 is a measure ultrapower embedding. Let N^* be the normal measure ultrapower via the measure U generated from j^2 with the associated embedding $i_U : V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\mu)] \to N^*$, and let $j^2 = k^* \circ i_U$ be the commutative triangle with $k^* : N^* \to M^2$. First note that k^* is the identity on μ since its critical point must be a regular cardinal in N^* and N^* computes κ^{+3} $(= \mu)$ correctly. Then the claim follows since k^* must be onto (and therefore the identity) using the extender representation of M^2 and elementarity: any element of M^2 is of the form $j^2(f)(\alpha)$ for some $\alpha < \mu$, and if k^* is the identity on α, then $j^2(f)(\alpha) = k^*(i_U(f))(\alpha) = k^*(i_U(f))(k^*(\alpha)) = k^*(i_U(f)(\alpha))$, and thus $j^2(f)(\alpha)$ is in the range of k^*. □

Remark 2.5 It can also be shown that the tree property holds at $\kappa^{++} = \lambda$ in the model $V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\mu)]$. This is implicit in the proof of Theorem 3.3.

Remark 2.6 Notice that in constructing M^1 in Theorem 2.1 we lost the $H(\mu)$-hyper-measurability of j. By a more complicated construction in Theorem 2.1, this can be retained (and then automatically retained by the further construction in Theorem 2.3):

In the first step, argue exactly as in the proof of Theorem 2.1; in particular there exists in $V^1 = V[g]$, where g is Q-generic over V, a generic object $\tilde{g}$ for $j(\mathrm{Add}(\kappa,\mu))$ over M^1. However, M^1 does not contain g, which we are going to repair now.

[10] This appears as Fact 2 in [2]. Recall that Q_μ – mentioned in Fact 2.4 – is the term forcing defined as follows: the elements of Q_μ are names τ such that τ is an S-name and it is forced by 1_S to be in $\mathrm{Add}(\kappa,\mu)$ of $V[S]$. The ordering is $\tau \le \sigma \leftrightarrow 1_S \Vdash \tau \le \sigma$.

Let $\mathcal{M}$ be the set of all measurable cardinals $\alpha < \kappa$ in V. Define in V an Easton product $P^1_\kappa = \prod_{\alpha\in\mathcal{M}} Q^1_\alpha$ of length κ such that at every $\alpha \in \mathcal{M}$, Q^1_α is chosen by a lottery among all forcings R in V which satisfy the following condition $(*)$:

$(*)$ There is a measure W on α in V such that the embedding i_W generated by W is of the form $i_W : V \to N_W$ for some N_W and R is equal to $i_W(\mathrm{Add}(\alpha, \mu_\alpha))$,

where μ_α is the local version of μ in V with respect to α, i.e. it is the successor of the least weakly compact cardinal λ_α above α.

For $\alpha \in \mathcal{M}$, let us write P^1_α for the product P^1_κ below α, and $P^1_{>\alpha}$ for the product indexed above α so that $P^1_\kappa = P^1_\alpha \times Q^1_\alpha \times P^1_{>\alpha}$. We state some facts concerning P^1_κ. For $\alpha \in \mathcal{M}$, let G^1_α, $G^1_{>\alpha}$ and g^1_α be generic filters for $P^1_\alpha, P^1_{>\alpha}$, and Q^1_α over V, respectively. Let G^1_κ be P^1_κ-generic.

(i) For $\alpha \in \mathcal{M}$, let α^* denote the next element of $\mathcal{M}$ above α. For every $\alpha \in \mathcal{M}$, Q^1_α has size less than α^*.
(ii) For every $\alpha \in \mathcal{M}$, G^1_α, g^1_α, and $G^1_{>\alpha}$ are mutually generic. Also, G^1_κ and g are mutually generic.
(iii) In $V[g][G^1_\alpha]$, forcing with Q^1_α does not collapse cardinals. More generally, forcing with $P^1_\kappa \times Q$ does not collapse cardinals.

For (i) note that there are at most α^{++}-many measures W at stage α and each R in the lottery has size less than α^*. Regarding (ii) note that for $\alpha \in \mathcal{M}$, P^1_α is α-cc, Q^1_α is α^+-closed and has size less than α^*, and $P^1_{>\alpha}$ is $(\alpha^*)^+$-closed. The same facts apply to Q and P^1_κ. For (iii), it suffices to show that R chosen by the lottery does not collapse cardinals over $V[g][G^1_\alpha]$. We argue by a variant of Theorem 2.1: R is α^+-closed in V, and therefore also in $V[g]$, and since P^1_α is α-cc, it remains α^+-distributive in $V[g][G^1_\alpha]$; R is α^{++}-cc in V, and hence also in $V[g]$, and since P^1_α has size just α, it forces that R is still α^{++}-cc (if there were an antichain in $V[g][G^1_\alpha]$ of size α^{++}, a single condition in P^1_α would determine a cofinal part of it, which would yield an antichain in $V[g]$ of size α^{++}). Using the fact that Q^1_α does not collapse cardinals and has size less than α^*, standard methods can be used to argue that the whole forcing $P^1_\kappa \times Q$ does not collapse cardinals.

Let G^1_κ be P^1_κ-generic over $V[g]$. As we argued in (ii), Q and P^1_κ are mutually generic since Q is κ^+-closed and P^1_κ is κ-cc. In $V[g][G^1_\kappa] = V[G^1_\kappa][g]$ we lift j to

$$j^* : V[G^1_\kappa] \to M[G^1_\kappa][g][h],$$

choosing by the lottery at stage κ of $j(P^1_\kappa)$ the forcing Q which is available here. The generic object h is constructed using the extender representation of M and the fact that $j(P^1_\kappa)$ at the interval $(\kappa, j(\kappa))$ is κ^+-closed in V, and more than μ^+-closed in the sense of M; by mutual genericity argued in (ii), h constructed in V as generic over M is actually generic over $M[G^1_\kappa][g]$. Since g is added by a κ^+-distributive forcing over $V[G^1_\kappa]$, it lifts easily, and so we get

$$j^{**} : V[G^1_\kappa][g] \to M[G^1_\kappa][g][h][h^*],$$

where h^* is generated by $j^{*}{}''g$.

Let us now look at $M[G^1_\kappa][g][h][h^*] = M^{**}$. We know from Theorem 2.1 that the object $\tilde{g}$ is $j(\mathrm{Add}(\kappa, \mu))$-generic over $M[h^*] = M^1$. The forcing $j(\mathrm{Add}(\kappa, \mu))$ remains $j(\kappa)$-closed in

M^1 since h^* does not add new $j(\kappa)$-sequences. Since the forcing $j(P^1_\kappa)$ is $j(\kappa)$-cc in M^1, $\tilde{g}$ is mutually generic with $G^1_\kappa * g * h$, and therefore $\tilde{g}$ is $j(\mathrm{Add}(\kappa,\mu))$-generic over M^{**}. Finally we apply over M^1 Fact 2.4 arguing that there is $\tilde{g}^*$ in $V[G^1_\kappa][g]$ which is $j^{**}(\mathrm{Add}(\kappa,\mu))$-generic over M^{**} (in more detail, the forcing $j(P^1_\kappa)$ over M^1 has size $j(\kappa)$ and is $j(\kappa)$-cc and therefore $\tilde{g}$ yields the required $\tilde{g}^*$). Thus

$$j^{**} : V[G^1_\kappa][g] \to M^{**}$$

satisfies the assumptions necessary for the proof of Theorem 2.3 (with $\tilde{g}^*$ now being the required generic), with M^{**} now containing $H(\mu)$ of $V[G^1_\kappa][g]$.

Renaming $V^1 = V[G^1_\kappa][g]$, $M^1 = M^{**}$, and $j^{**} = j^1$, arguments in this paper using these objects can be carried out with the additional assumption that M^1 contains $H(\mu)$ of V^1. This ends Remark 2.6.

3 The tree property with gap 3

In this section we will prove that it is consistent to have a model where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and the tree property holds at $\aleph_{\omega+2}$. It is relatively straightforward to generalise this construction to get a finite gap: $2^{\aleph_\omega} = \aleph_{\omega+n}$, $3 \leq n < \omega$ (see Section 4).

3.1 Definition of the forcing

Let us work with the model $V^1[P_\kappa][\mathbb{M}(\kappa,\lambda,\lambda^+)]$. As we showed in Theorem 2.3, κ is measurable in here. In order to analyse this model, let us introduce notation for the generic filters: let $G_\kappa * H$ be a generic filter over V^1 for $P_\kappa * \mathbb{M}(\kappa,\lambda,\lambda^+)$. As we showed in Theorem 2.3, the lifted extender embedding j^2 in Theorem 2.3 becomes a measure ultrapower embedding i_U in $V^1[G_\kappa * H]$, generated by the normal measure U derived from j^2. Let us rename j^2 to j for simplicity.

In particular, we can define the Prikry forcing with collapses $\mathrm{PrkCol}(U, G^g)$ using this U and a suitable guiding generic G^g which we construct in Lemma 3.1 (the small g stands for "guiding").[11]

Let Coll denote the forcing $\mathrm{Coll}((\kappa^{+4}), < j(\kappa))^{M^1[j(G_\kappa * H)]}$.

Lemma 3.1 *In $V^1[G_\kappa * H]$, there exists an $M^1[j(G_\kappa * H)]$-generic filter for* Coll.

Proof. Consider the extender representation $j^1 : V^1 \to M^1$ ensured by the arguments in Section 2.1, where

$$M^1 = \{j^1(f)(\alpha) \mid f \in V^1 \ \& \ f : \kappa \to V^1 \ \& \ \alpha < \lambda^+\}. \tag{3.4}$$

Now notice that every maximal antichain of Coll in $M^1[j(G_\kappa * H)]$ has a name of the form $j^1(f)(\alpha)$ for some $f : \kappa \to H(\kappa)^{V^1}$ and $\alpha < \lambda^+$, with the range of f being composed of

[11] See Section 1.1.2 for more details about this forcing.

P_κ-names. There are only κ^+-many such f's, and since Coll is κ^{+4}-closed in $M^1[j(G_\kappa * H)]$, we can built a Coll-generic filter G^g in $V^1[G_\kappa * H]$ over $M^1[j(G_\kappa * H)]$ by the standard method of grouping the antichains into κ^+ many blocks each of size at most λ^+, where λ^+ is equal to κ^{+3} in $M^1[j(G_\kappa * H)]$. □

Let us define in V:

$$\mathbb{P} = Q * P_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g), \tag{3.5}$$

where Q is the forcing from Theorem 2.1, and $\dot{G}^g$ is a name for a guiding generic which we know exists by Lemma 3.1.

Lemma 3.2 *$\mathbb{P}$ is λ-cc.*

Proof. This is a standard argument using Theorem 2.1 for Q and Lemma 1.3 for the forcing $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$. □

We plan to show that $V[\mathbb{P}]$ is the desired model.

3.2 Verifying the tree property with gap 3

Now we show that the tree property holds with gap 3. See Section 4 for a generalisation for any $3 \le n < \omega$.

Theorem 3.3 *The forcing $\mathbb{P}$ in (3.5) forces $\kappa = \aleph_\omega$, $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and the tree property holds at $\lambda = \aleph_{\omega+2}$.*

By Lemma 1.1 and standard facts about the Prikry forcing with collapses, it suffices to check that we have the tree property at $\aleph_{\omega+2}$.

The argument starts with an observation (see Section 3.2.1) which allows us to work over $V^1[P_\kappa]$ with a fragment of a weakly compact embedding with critical point λ (but still strong enough for our purposes).[12]

The core argument has two parts and starts over the model $V^1[P_\kappa]$. In Part 1 (Section 3.2.2), we show that if there were in $V^1[P_\kappa]$ an $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$-name $\dot{T}$ for an $\aleph_{\omega+2}$-Aronszajn tree, we could find a suitable β, $\lambda < \beta < \lambda^+$, and define a "truncation" $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ of the original forcing which forces that that there is an $\aleph_{\omega+2}$-Aronszajn tree (witnessed by $\dot{T}$). Then in Part 2 (Section 3.2.3), we show that in fact this cannot be the case, i.e. we show that $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ forces the tree property at $\aleph_{\omega+2}$. This will yield the final contradiction, finishing the proof of Theorem 3.3.

3.2.1 The fragment of weak compactness of λ in $V^1[P_\kappa]$

Suppose for contradiction that $\mathbb{P}$ forces that there is an $\aleph_{\omega+2}$-Aronszajn tree (assume for simplicity the weakest condition forces this, otherwise work below a suitable condition); let

[12] Note that Q destroys the strong limitness of λ by adding many subsets of κ^+.

$\dot{W}$ be a $Q * P_\kappa$-name for an $\mathbb{M}(\kappa,\lambda,\lambda^+) * \mathrm{PrkCol}(\dot{U},\dot{G}^g)$-name $\dot{T}$ such that over $V^1[P_\kappa]$, $\mathbb{M}(\kappa,\lambda,\lambda^+) * \mathrm{PrkCol}(\dot{U},\dot{G}^g)$ forces that $\dot{T}$ is an $\aleph_{\omega+2}$-Aronszajn tree.

By Lemma 3.2,[13] we can assume that $\dot{W}$ can be expressed as a nice name for a subset of λ, and that $\dot{T}$ itself is a nice name for a subset of λ in $V^1[P_\kappa]$.

Let β^* be an ordinal between λ and λ^+ (the construction of β^* is described in Section 3.2.3) large enough so that $\dot{W}$ only uses coordinates below β^* in the sense that we can fix a weakly compact embedding k with critical point λ,

$$k : \mathscr{M} \to \mathscr{N} \tag{3.6}$$

with the following properties:[14]

(i) $\mathscr{M}$ and $\mathscr{N}$ are transitive models of size λ closed under $<\lambda$-sequences,
(ii) $\mathscr{M} \in \mathscr{N}$, $k \in \mathscr{N}$, $\beta^* < k(\lambda)$, and
(iii) $\mathscr{M}$ contains all relevant information (in particular, β^*, $\mathbb{P}$ and $\dot{W}$ are elements of $\mathscr{M}$).

Let $g * G_\kappa$ be $Q * P_\kappa$-generic over V and let us consider Q restricted to β^* (let us denote it $Q(\beta^*)$)[15]; note that $Q(\beta^*)$ is an element of $\mathscr{M}$. Let $g(\beta^*)$ be the restriction of g to β^* so that $g(\beta^*) * G_\kappa$ is $Q(\beta^*) * P_\kappa$-generic. Note that $Q(\beta^*) * P_\kappa$ is actually equivalent to $Q(\beta^*) \times P_\kappa$ since $Q(\beta^*)$ does not change V_κ where P_κ lives. By standard arguments, k lifts to $\mathscr{M}[G_\kappa] \to \mathscr{N}[G_\kappa]$ since $k(P_\kappa) = P_\kappa$, and both the models are still closed under $<\lambda$-sequences in $V[G_\kappa]$.

By elementarity, $k(Q(\beta^*))$ is Q restricted to $k(\beta^*)$. Let $b : k(\beta^*) \to k(\beta^*)$ be a bijection which swaps γ and $k(\gamma)$ for every $\lambda \le \gamma < \beta^*$, and is the identity otherwise. b extends to an automorphism on $k(Q(\beta^*))$ by mapping $p \in k(Q(\beta^*))$ to $b(p)$ where the coordinates in $b(p)$ are swapped by b. Note that $b(p)$ is a valid condition in Q since by the elementarity of k, $k(p) = k\text{"}p$ is a condition in $k(Q(\beta^*))$ (and hence in Q) for every p in $Q(\beta^*)$.[16]

Let $g(k(\beta^*))$ be the restriction of g to $k(\beta^*)$. The automorphism b generates from $g(k(\beta^*))$ a generic filter g^* on $k(Q(\beta^*))$ which contains the pointwise image $k\text{"}g(\beta^*)$. It follows k lifts to

$$k : \mathscr{M}[G_\kappa][g(\beta^*)] \to \mathscr{N}[G_\kappa][g^*]. \tag{3.7}$$

Since Q is κ^+-distributive over P_κ it holds that both the models are still closed under κ-sequences in $V[g * G_\kappa]$.[17]

Thus for any $\dot{W}$ and β^* as above, we have in $V[g * G_\kappa]$ a fragment of a weakly compact embedding (3.7) such that all the relevant parameters are in $\mathscr{M}$, including the name $\dot{T}$, and the models are closed under κ-sequences in the universe.

[13] In more details, both Q and $\mathrm{Add}(\kappa,\lambda^+)$ (which is a part of $\mathbb{M}(\kappa,\lambda,\lambda^+)$) are products and therefore by the chain condition, we can ensure that only up to λ-many coordinates below λ^+ appear in a name for the tree.

[14] See [3] Theorem 16.1. To ensure $\beta^* < k(\lambda)$, define E in the proof of Theorem 16.1 so that it also codes a well-ordering of β^* of type λ: then $\mathscr{N} \models |\beta^*| = \lambda$ and therefore $k(\lambda) > \beta^*$ since by elementarity, $k(\lambda)$ is in $\mathscr{N}$ a limit cardinal greater than λ.

[15] Note that λ^+ is a fixed point of the mapping i so Q is $\mathrm{Add}(i(\kappa),\lambda^+)$ of the measure ultrapower N.

[16] The support of p is some set of size less than $i(\kappa)$ in the measure ultrapower N, but certainly less than λ in V: thus the k-image of the support is just its pointwise image.

[17] They are not closed under κ^+-sequences though.

3.2.2 Part 1

As we argued in Section 3.2.1, we can work in $V^1[G_\kappa]$ and assume for contradiction that $\dot{T}$ is a name for an $\aleph_{\omega+2}$-Aronszajn tree (we assume that $\dot{T}$ is a nice name for a subset of λ). There is $\bar{\beta}$, $\lambda \leq \bar{\beta} < \lambda^+$, such that all the coordinates in the forcing $\mathrm{Add}(\kappa, \lambda^+)$ which appear in $\dot{T}$ are below $\bar{\beta}$ (there are only λ-many of them by the chain condition of the forcing).

We wish to find β, $\bar{\beta} < \beta < \lambda^+$, which allows us to define a suitable truncation of $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ to β which we will analyse in Part 2.

Using standard arguments, construct an elementary submodel $\mathscr{A}$ of $H(\theta)$ for some large enough regular θ so that $\mathscr{A}$ satisfies the following conditions:

(i) $|\mathscr{A}| = \lambda$, and $\mathscr{A}$ is closed under $< \lambda$-sequences,
(ii) $\bar{\beta} + 1 \subseteq \mathscr{A}$,
(iii) $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, $\dot{U}$, $\dot{G}^g$, $\dot{T}$ are elements of $\mathscr{A}$,
(iv) $\mathscr{A} \cap \lambda^+ = \beta$ for some β of cofinality κ^+, $\bar{\beta} < \beta$,
(v) There is an $\mathbb{M}(\kappa, \lambda, \beta)$-name $\dot{U}_\beta$ which is forced by $\mathbb{M}(\kappa, \lambda, \lambda^+)$ to be a normal measure and a restriction of the measure $\dot{U}$ to $V^1[P_\kappa][\mathbb{M}(\kappa, \lambda, \beta)]$.

The last item (v) follows as in [4].

Let $c : \mathscr{A} \to \bar{\mathscr{A}}$ be the transitive collapse. The following hold (the proofs are routine):

(i) $c(\lambda^+) = \beta$,
(ii) $c(\mathbb{M}(\kappa, \lambda, \lambda^+)) = \mathbb{M}(\kappa, \lambda, \beta)$,
(iii) $c(\dot{U})$ is forced by $\mathbb{M}(\kappa, \lambda, \beta)$ to be equal to $\dot{U}_\beta$,
(iv) $c(\dot{G}^g)$, which we denote by $\dot{G}^g_\beta$, is forced by $\mathbb{M}(\kappa, \lambda, \beta)$ to be a guiding generic with respect to $\dot{U}_\beta$, and therefore $\mathbb{M}(\kappa, \lambda, \beta)$ forces that $\mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ is a Prikry forcing with collapses,
(v) $\mathbb{M}(\kappa, \lambda, \lambda^+)$ forces that $\mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ is a regular subforcing of $\mathrm{PrkCol}(\dot{U}, \dot{G}^g)$,
(vi) $c(\dot{T}) = \dot{T}$ is forced by $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ to be an $\aleph_{\omega+2}$-Aronszajn-tree.

By elementarity, $c(\dot{T}) = \dot{T}$ is forced in $\bar{\mathscr{A}}$ by the forcing $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ to be a λ-Aronszajn tree. This by itself would not be enough to conclude that $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ adds such a tree over the universe $V^1[G_\kappa]$. However, since the collapse $c(\dot{T})$ is equal to $\dot{T}$, and (v) holds, any $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$-generic filter $h * x$ yields an $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$-generic filter $h' * x'$ over $V^1[G_\kappa]$ (and therefore over $\bar{\mathscr{A}}$) such that $(\dot{T})^{h*x} = (\dot{T})^{h'*x'}$. It follows that $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$ forces over $V^1[G_\kappa]$ that $\dot{T}$ is a λ-Aronszajn tree. In Part 2 we show this is not possible, and this will be the desired contradiction.

3.2.3 Part 2

Let $\mathbb{M}$ denote the forcing $\mathbb{M}(\kappa, \lambda, \beta)$, where β is as in Part 1. Let us work in $V^1[G_\kappa]$.

Using the arguments in Section 3.2.1, let us fix

$$k : \mathscr{M} \to \mathscr{N} \tag{3.8}$$

which is the fragment of a weakly compact embedding with critical point λ such that $\mathscr{M}$ and $\mathscr{N}$ are transitive models of size λ closed under κ-sequences, $\mathscr{M} \in \mathscr{N}$, $k \in \mathscr{N}$, $\beta < k(\lambda)$,[18] and $\mathscr{M}$ contains all relevant information (in particular, β, $\mathbb{M} * \mathrm{PrkCol}(\dot{U}_\beta, \dot{G}^g_\beta)$, $\dot{U}_\beta$ and $\dot{G}^g_\beta$ are elements of $\mathscr{M}$). Let $\mathbb{M}^*$ denote $k(\mathbb{M}(\kappa, \lambda, \beta))$, which is equal to $\mathbb{M}(\kappa, k(\lambda), k(\beta))$.

Let h^* be $\mathbb{M}^*$-generic over $V^1[G_\kappa]$; use h^* to define h which is $\mathbb{M}$-generic over $V^1[G_\kappa]$ and $k"h \subseteq h^*$. Now lift to

$$k : \mathscr{M}[h] \to \mathscr{N}[h^*]. \tag{3.9}$$

Let us abuse notation a little and write $U = (\dot{U}_\beta)^h$ and $G^g = (\dot{G}^g_\beta)^h$ instead of U_β and G^g_β (to simplify notation).

In $\mathscr{N}[h^*]$, consider $U^* = k(U)$, and $G^{g*} = k(G^g)$, and the forcing $\mathrm{PrkCol}(U^*, G^{g*})$. Note that by elementarity $U \subseteq U^*$ (since $k(X) = X$ for every $X \in U$), and all functions F whose equivalence class is in G^g appear in the forcing $\mathrm{PrkCol}(U^*, G^{g*})$ (since $k(F) = F$ for every $F : \kappa \to V_\kappa^{\mathscr{M}[h]}$, $F \in \mathscr{M}[h]$), and $k(\mathrm{PrkCol}(U, G^g)) = \mathrm{PrkCol}(U^*, G^{g*})$.[19]

It follows that k is a regular embedding:

$$k : \mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g) \to \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}), \tag{3.10}$$

as by the λ-cc of $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, if A is a maximal antichain in $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$, then $k(A) = k"A$ is a maximal antichain in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$.

Let x^* be $\mathrm{PrkCol}(U^*, G^{g*})$-generic over $V^1[G_\kappa][h^*]$; the pull-back of x^* via k^{-1} is a generic filter x for $\mathrm{PrkCol}(U, G^g)$ such that $k"x \subseteq x^*$. Let us lift k further to

$$k : \mathscr{M}[h][x] \to \mathscr{N}[h^*][x^*]. \tag{3.11}$$

By (3.10) and (3.11), we can define in $\mathscr{N}[h^*][x^*]$ a generic filter $h*x$ for $\mathbb{M}*\mathrm{PrkCol}(\dot{U}, \dot{G}^g) \in \mathscr{N}$ using the inverse of k (by our assumptions in (3.8), k is an element of $\mathscr{N}$). By standard arguments for complete Boolean algebras it follows that there is a projection π,

$$\pi : \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \to \mathrm{RO}^+(\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)). \tag{3.12}$$

Notice that if $((p, q), r)$ is a condition in $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$, then we can identify $\pi(p) = \pi((p, \emptyset), 1_{\mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})})$ with $(k^{-1})"p$, i.e. with

$$p \restriction (\kappa \times \lambda) \cup \{((\gamma, \alpha), i) \mid \gamma < \kappa, \alpha \in [\lambda, \beta), i \in \{0, 1\}, ((\gamma, k(\alpha)), i) \in p\}. \tag{3.13}$$

In the analysis of the quotient determined by π, it will be important to control the names for the conditions in $\mathrm{PrkCol}(U^*, G^{g*})$. Recall that by Lemma 1.4, we can adopt the following convention:

[18] Choose β^* in Section 3.2.1 high enough to ensure this inequality.

[19] However, note that the equivalence classes of a fixed F with respect to U and U^* may be different objects (after the transitive collapse).

Convention. We now view $\mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g})$ as consisting of conditions $((p,q),r)$, where r depends only on Cohen information, and its stem is a checked name (such conditions are dense by Lemma 1.4). With this convention in mind, let us denote the quotient determined by π in (3.12) as $\mathbb{Q}_\pi$:

$$\mathbb{Q}_\pi = \{((p,q),r) \in \mathbb{M}^* * \mathrm{PrkCol}(\dot{U}^*, \dot{G}^{*g}) \mid \pi(((p,q),r)) \in h * x\}, \tag{3.14}$$

where we identify $h * x$ with the generic filter for the associated complete Boolean algebra.

The following product analysis reformulates the analysis in Section 1.1.3 in a quotient setting.

Define:

$$\mathbb{C} = \{((p,\emptyset),r) \mid ((p,\emptyset),r) \in \mathbb{Q}_\pi\}, \tag{3.15}$$

The ordering is the one inherited from $\mathbb{Q}_\pi$.

Define:

$$\mathbb{T} = \{(\emptyset,q) \in \mathbb{M}^* \mid (\emptyset,q) \restriction \lambda \in h\}. \tag{3.16}$$

The ordering is the one inherited from $\mathbb{M}^*$.

Define a function τ from $\mathbb{C} \times \mathbb{T}$ to $\mathbb{Q}_\pi$ as follows: τ applied to the pair composed of $((p,\emptyset),r)$ and $(\emptyset,q)$ is equal to the condition $((p,q),r)$. Note that if $((p,\emptyset),r)$ is in $\mathbb{C}$ and $(\emptyset,q)$ is in $\mathbb{T}$, then $((p,q),r)$ is a condition in the quotient $\mathbb{Q}_\pi$ since $\pi((p,q),r)$ is the infimum of $\pi((p,\emptyset),r)$ and $((\emptyset,q) \restriction \lambda, 1_{\mathrm{PrkCol}(\dot{U},\dot{G}^g)})$ in $\mathrm{RO}^+(\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g))$.

Lemma 3.4 is proved exactly as Lemma 1.5(i):

Lemma 3.4 *τ is a projection from $\mathbb{C} \times \mathbb{T}$ onto $\mathbb{Q}_\pi$.*

The following lemma is obvious:

Lemma 3.5 *$\mathbb{T}$ is κ^+-closed in $\mathscr{N}[h]$.*

Finally, we analyse the chain condition of $(\mathbb{C})^2$ in $\mathscr{N}[h][x]$.

Lemma 3.6 *$(\mathbb{C})^2$ is κ^+-cc in $\mathscr{N}[h][x]$.*

Proof. Assume for contradiction that A is an antichain in $(\mathbb{C})^2$ in $\mathscr{N}[h][x]$ of size κ^+. Denote the elements of A by $(1a_i, 2a_i)$ for $i < \kappa^+$. By thinning out the antichain if necessary, we can choose a condition $((p,q),r)$ in $h * x$ which forces that A is an antichain and also forces that stems of all conditions $1a_i$, $i < \kappa^+$, are the same and the stems of all conditions $2a_i$, $i < \kappa^+$, are the same (they may not equal each other, but they are compatible; denote them $1t$, and $2t$). Now choose $((p_i,q_i),r_i)$ in $h * x$ which decide the $1a_i$'s and $2a_i$'s; let us write $1a_i = ((1p_i^*, \emptyset), 1r_i^*)$ and $2a_i = ((2p_i^*, \emptyset), 2r_i^*)$, $i < \kappa^+$.

By further thinning and extending the stems if necessary, we may assume that the stems of $((p,q),r)$ and $((p_i,q_i),r_i)$, $i<\kappa^+$, are all the same; denote this stem s. Note that s extends both $1t$ and $2t$.

Now, we need to handle together $1a_i$ and $2a_i$, for all $i<\kappa^+$, to get mutual compatibility in Claim 3.9 below: Let $((1p_i^{**},1q_i^{**}),1r_i^{**})$ be a lower bound of $((1p_i^*,\emptyset),1r_i^*)$, $((p_i,q_i),r_i)$, and $((p,q),r)$ with stem s such that $\pi(1p_i^{**})$[20] is in the Cohen part of the generic $h*x$ (this can be done since such conditions are dense). Analogously, let $((2p_i^{**},2q_i^{**}),2r_i^{**})$ be a lower bound of $((2p_i^*,\emptyset),2r_i^*)$, $((p_i,q_i),r_i)$, and $((p,q),r)$ with stem s such that $\pi(2p_i^{**})$ is in the Cohen part of the generic $h*x$. Note that in particular $\pi(1p_i^{**})$ is compatible with $\pi(2p_i^{**})$.

Using a Δ-system argument, find $i<j$ such that $1p_i^{**}$ is compatible with $1p_j^{**}$ and $2p_i^{**}$ is compatible with $2p_j^{**}$. Let us define:

(3.17) $\quad 1(*)$ is the greatest lower bound (glb) of$((1p_i^{**},\emptyset),1r_i^{**})$ and $((1p_j^{**},\emptyset),1r_j^{**})$

and

(3.18) $\quad 2(*)$ is the greatest lower bound (glb) of$((2p_i^{**},\emptyset),2r_i^{**})$ and $((2p_j^{**},\emptyset),2r_j^{**})$.

Note that both $1(*)$ and $2(*)$ have the same stem s.

Denote

$$p'=\pi(1p_i^{**})\cup\pi(1p_j^{**})\cup\pi(2p_i^{**})\cup\pi(2p_j^{**}).$$

Note that p' is correctly defined by the construction of the $1p_i^{**}$'s and $2p_i^{**}$'s. Let $((\bar p,\bar q),\bar r)$ denote the glb of the conditions $((p',\emptyset),\emptyset)$, $((p,q),r)$, $((p_i,q_i),r_i)$, $((p_j,q_j),r_j)$. Note that $((\bar p,\bar q),\bar r)$ has stem s.

We need the following claims to finish the proof.

Claim 3.7 *Assume $((p,q),r)$ is a condition in $\mathbb{M}^*\mathrm{PrkCol}(\dot U,\dot G^g)$ and $((p^*,\emptyset),r^*)$ is a condition in $\mathbb{M}^**\mathrm{PrkCol}(\dot U^*,\dot G^{*g})$ and the following conditions are satisfied:*

(i) Stems of r and r^* are checked names,
(ii) $p\le\pi(p^*)$,
(iii) The length of the stems of r and r^* is the same and the ordinals on the stems coincide,
(iv) The collapsing information in the stem of r extends the collapsing information in the stem of r^*.

Then $((p,q),r)$ does not force $((p^*,\emptyset),r^*)$ out of the quotient $\mathbb{C}$.

Proof. It suffices to find a generic filter $h'*x'$ for $\mathbb{M}^**\mathrm{PrkCol}(\dot U^*,\dot G^{*g})$ such that $\pi((p,q),r)$ is in the $\mathbb{M}*\mathrm{PrkCol}(\dot U,\dot G^g)$-generic filter obtained by π from $h'*x'$. Any filter $h'*x'$ containing a lower bound of $((p,q),r)$ and $((p^*,\emptyset),r^*)$ (such a lower bound exists by conditions (i)-(iv)) satisfies this. □

Recall that the Prikry forcing with collapses satisfies the Prikry condition: any statement in the forcing language is decidable by a direct extension (note that a direct extension does not lengthen the stem, but is allowed to extend the collapsing information).

[20] See (3.13) for definition.

Claim 3.8 *Let $((p, q), r)$ and $((p^*, \emptyset), r^*)$ are as in Claim 3.7. Then there is a direct extension $((p', q'), r')$ of $((p, q), r)$ which forces $((p^*, \emptyset), r^*)$ into $\mathbb{C}$.*

Proof. By the Prikry property, there is a direct extension of $((p, q), r)$ which decides the statement "$((p^*, \emptyset), r^*)$ is in $\mathbb{C}$". The negative decision contradicts Claim 3.7 (when applied to the direct extension); it follows that the decision must be positive. □

Returning to our proof, we get:

Claim 3.9 *There is a direct extension of $((\bar{p}, \bar{q}), \bar{r})$ which forces $1(*)$ and $2(*)$ into $\mathbb{C}$.*

Proof. $((\bar{p}, \bar{q}), \bar{r})$ and $1(*)$ satisfy the conditions in Claim 3.7, and therefore by Claim 3.8, there is a direct extension $a_1 \leq ((\bar{p}, \bar{q}), \bar{r})$ which forces $1(*)$ into $\mathbb{C}$. a_1 and $2(*)$ satisfy the conditions in Claim 3.7, and therefore by Claim 3.8 there is a direct extension $a_2 \leq a_1$ as desired. □

This finishes the proof since a_2 forces that $(1(*), 2(*))$ is in $\mathbb{C}$ a witness for compatibility of $(1a_i, 2a_i)$ and $(1a_j, 2a_j)$ in the antichain A. As a_2 is below $((p, q), r)$, it also forces that A is an antichain. Contradiction. □

The good chain condition of $(\mathbb{C})^2$ and the closure of $\mathbb{T}$ are enough to argue that $\mathbb{C} \times \mathbb{T}$, and therefore $\mathbb{Q}_\pi$, do not add branches to λ-trees, finishing the argument in the standard way.

For completeness we state the relevant facts below.

The following Fact is implicit in Mitchell's [13], and stated in Unger's [16].

Fact 3.10 *Suppose γ is a regular infinite cardinal and P adds a subset x of γ such that x is not in V but $x \cap \alpha$ is in V for all $\alpha < \gamma$. Then $P \times P$ is not γ-cc.*

Proof. It suffices to show that if G is P-generic then P is not γ-cc in $V[G]$. In $V[G]$ let $x = (\dot{x})^G$ be a subset of γ as in the hypothesis and choose a sequence $\langle p_i \mid i < \gamma \rangle$ of conditions in G and an increasing sequence of ordinals $\langle \alpha_i \mid i < \gamma \rangle$ less than γ such that p_i fixes $\dot{x} \cap \alpha_i$ (i.e. forces it to equal a specific element of V) but does not fix $\dot{x} \cap \alpha_{i+1}$. This is possible as $x \cap \alpha$ is fixed by some condition in G for each $\alpha < \gamma$ but x itself is fixed by no condition in G. Now choose q_{i+1} extending p_i to disagree with p_{i+1} about $\dot{x} \cap \alpha_{i+1}$. This is possible as p_i does not fix $\dot{x} \cap \alpha_{i+1}$. But then the q_{i+1}'s form an antichain as any condition extending q_{i+1} disagrees with p_{i+1} (and therefore with p_j for all $j > i$) about $\dot{x}$ and therefore cannot extend q_{j+1} for $j > i$, as q_{j+1} extends p_j. □

The following Fact is due to Silver:

Fact 3.11 *Suppose γ is an infinite cardinal and P is γ^+-closed. Suppose $\mu > \gamma$ is a regular cardinal and T is a tree of height μ which has all levels of size less than 2^γ. Then P does not add cofinal branches to T.*

Finally, the following Fact is due to Unger [15], generalising Fact 3.11:

Fact 3.12 *Suppose γ is an infinite cardinal, P is γ^+-cc, Q is γ^+-closed, and $2^\gamma > \gamma^+$. If T is a γ^{++}-tree in $V[P]$, then in $V[P][Q]$, T has no new cofinal branches.*

Suppose T is a λ-tree in the model $\mathscr{N}[h][x]$, where $\lambda = \kappa^{++}$. By Lemma 3.6 and Fact 3.10, $\mathbb{C}$ does not add new branches to T. As $\mathrm{PrkCol}(U, G^g) * \mathbb{C}$ has the κ^+-cc in $\mathscr{N}[h]$, we can apply Fact 3.12 over $\mathscr{N}[h]$ (with Q being $\mathbb{T}$), and conclude that $\mathbb{T}$ does not add branches to trees in $\mathscr{N}[h][\mathrm{PrkCol}(U, G^g) * \mathbb{C}]$, and therefore $\mathbb{C} \times \mathbb{T}$ does not add new branches to trees in $\mathscr{N}[h][x]$.

This finishes the proof of Theorem 3.3.

4 The tree property with a finite gap

We would like to generalise the result of the previous section to a finite gap m, i.e. obtain the tree property at $\aleph_{\omega+2}$ and have $2^{\aleph_\omega} = \aleph_{\omega+m}$ for any $2 < m < \omega$. To this end, we need to do some straightforward modifications to definitions and lemmas we used to obtain gap 3. To simplify indexing of the forcing notions, we will use the index n, where $m = n + 2$ (thus gap 3 is obtained with $n = 1$).

As in the previous section, let κ be the large cardinal which gets collapsed to $\aleph_\omega$, and λ the least weakly compact cardinal above κ.

We now list the modifications we need to do:

- In Section 2.1, we choose $\mu = \lambda^{+n}$ so that the preparation, which we now call Q^n, ensures that κ stays measurable after adding μ-many Cohen subsets of κ. Let us denote the resulting model as V^1.
- The definition of P_κ in (2.3) is to be modified as follows:

  $$P^n_\kappa = \langle (P^n_\alpha, \dot{Q}^n_\alpha) \mid \alpha < \kappa \text{ is measurable} \rangle, \tag{4.19}$$

  where $\dot{Q}^n_\alpha$ denotes the forcing $\mathbb{M}(\alpha, \lambda_\alpha, \lambda_\alpha^{+n})$.
- Let $G_\kappa * H$ be a generic filter for $P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n})$, and let $j : V^1[G_\kappa * H] \to M^1(j(G_\kappa * H))$ be the lifting of j as in Theorem 2.3.
- Let Coll^n denote the forcing $\mathrm{Coll}((\kappa^{+3+n}, < j(\kappa)))^{M^1[j(G_\kappa * H)]}$. As in Lemma 3.1, we can fix a guiding generic G^g for Coll^n over $M^1[j(G_\kappa * H)]$.
- The definition of the forcing $\mathbb{P}$ in (3.5) is modified as follows:

  $$\mathbb{P}^n = Q^n * P^n_\kappa * \mathbb{M}(\kappa, \lambda, \lambda^{+n}) * \mathrm{PrkCol}(\dot{U}, \dot{G}^g), \tag{4.20}$$

  where $\dot{U}$ is a name for a normal measure and $\dot{G}^g$ is a name for a guiding generic (defined with respect to $\dot{U}$).

Now we get the following generalisation of Theorem 3.3:

Theorem 4.1 *Let $1 \leq n < \omega$ be fixed. The forcing $\mathbb{P}^n$ in (4.20) forces $\kappa = \aleph_\omega$, $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2+n}$, and the tree property holds at $\lambda = \aleph_{\omega+2}$.*

Proof. The basic strategy of the proof is to reduce the general case to gap 3.

Let G_κ be a P^n_κ-generic filter over V^1. Let us denote $\mathbb{M} = \mathbb{M}(\kappa, \lambda, \lambda^{+n})$. Let us assume for simplicity that the weakest condition of the forcing $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$ forces in $V^1[G_\kappa]$ that $\dot{T}$ is a λ-Aronszajn tree (otherwise work below a condition which forces it).

Let $\mathscr{A}$ be an elementary substructure of large enough $H(\theta)^{V^1[G_\kappa]}$ which has size λ^+, is closed under λ-sequences, and contains the name $\dot{T}$ and the forcing $\mathbb{M} * \mathrm{PrkCol}(\dot{U}, \dot{G}^g)$. Let $c : \mathscr{A} \to \bar{\mathscr{A}}$ be the transitive collapse. Then the following hold:

(i) $c(\lambda^{+n})$ is an ordinal between λ^+ and λ^{++}, let us denote this ordinal as β.
(ii) $c(\mathbb{M}(\kappa, \lambda, \lambda^{+n})) = \mathbb{M}(\kappa, \lambda, \beta)$.
(iii) $c(\dot{U})$ is forced by $\mathbb{M}(\kappa, \lambda, \lambda^{+n})$ to be a normal ultrafilter on κ in the generic extension of $V^1[G_\kappa]$ by $\mathbb{M}(\kappa, \lambda, \beta)$.
(iv) $c(\dot{G}^g)$ is forced by $\mathbb{M}(\kappa, \lambda, \beta)$ to be a guiding generic with respect to $c(\dot{U})$, and therefore $\mathbb{M}(\kappa, \lambda, \beta)$ forces that $\mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ is a Prikry forcing with collapses.
(v) $c(\dot{T})$ is forced by $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ to be a λ-Aronszajn tree.

In contrast to the analogous construction in Section 3.2.2, we cannot claim now that $c(\dot{T})$ is equal to $\dot{T}$. However, since this time the model $\bar{\mathscr{A}}$ has size λ^+ and is closed under λ sequences, the forcing $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ adds a λ-Aronszajn tree not only over $\bar{\mathscr{A}}$ (which follows by elementarity), but also over $V^1[G_\kappa]$. The reason is that by λ-closure of $\bar{\mathscr{A}}$, a name for a cofinal branch in $c(\dot{T})$ would appear already in $\bar{\mathscr{A}}$.

Let f be any bijection between β and λ^+ which is the identity on λ. This bijection extends in $V^1[G_\kappa]$ into an isomorphism between $\mathbb{M}(\kappa, \lambda, \beta) * \mathrm{PrkCol}(c(\dot{U}), c(\dot{G}^g))$ and $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}_{\lambda^+}, \dot{G}^g_{\lambda^+})$, where $\dot{U}_{\lambda^+}$ and $\dot{G}^g_{\lambda^+}$ are names obtained naturally from f.

This is a contradiction since we can argue as in Theorem 3.3 that the forcing $\mathbb{M}(\kappa, \lambda, \lambda^+) * \mathrm{PrkCol}(\dot{U}_{\lambda^+}, \dot{G}^g_{\lambda^+})$ does not add a λ-Aronszajn tree over $V^1[G_\kappa]$. □

5 Open questions

The following questions are not solved by the methods of this paper:

Q1. Can we obtain an infinite gap at $2^{\aleph_\omega}$? More precisely, given an $\omega \leq \alpha < \omega_1$, is there a model where $\aleph_\omega$ is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+\alpha+1}$, and the tree property holds at $\aleph_{\omega+2}$?

It seems that an entirely different method is required for this configuration (perhaps based on the methods of Magidor [12] and Shelah [14]).

Q2. Can we obtain a similar result for $\aleph_{\omega_1}$? Or in general, for any uncountable cofinality?

The last question is more general. Let $P * \dot{Q}$ be a forcing notion and assume P is κ^+-cc for a cardinal κ. Then the following are equivalent:

(i) $P * \dot{Q}$ is κ^+-cc.

(ii) $P \Vdash$ "$\dot{Q}$ is κ^+-cc".

Consider Lemma 3.6 where we argued that $(\mathbb{C})^2$ is κ^+-cc in $\mathscr{N}[h][x]$, which is useful by Fact 3.10 for showing that certain trees do not get new branches in a generic extension by $\mathbb{C}$. It would be helpful in various contexts to have a generalisation of the equivalence (i) and (ii) for the "square-cc condition" for some – rich enough – class of forcing notions.

Q3. Let $P * \dot{Q}$ be a forcing notion. Assume that

$(*)$ $P * \dot{Q}$ is κ^+-Knaster (and therefore P is κ^+-Knaster as well).

Is there a useful characterisation of the forcings $P * \dot{Q}$ for which $(*)$ already implies $P \Vdash$ "$(\dot{Q})^2$ is κ^+-cc"?

Note that this cannot hold for all $P * \dot{Q}$ by the following example: Work in a model where MA (Martin's Axiom) holds and assume P is $\mathrm{Add}(\omega, 1)$ and $\dot{Q}$ is a name for the Souslin tree constructed from the generic filter for P (see Jech [10] for details). Then $P * \dot{Q}$ is $\aleph_1$-Knaster by MA, and yet P forces that $(\dot{Q})^2$ is not $\aleph_1$-cc.

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