Easton's theorem and large cardinals

Sy-David Friedman [a,1], Radek Honzik [b,2]

[a] *The University of Vienna, Kurt Gödel Research Center for Mathematical Logic, Währinger Strasse 25, A-1090 Wien, Austria*

[b] *The University of Vienna, Kurt Gödel Research Center for Mathematical Logic, Währinger Strasse 25, A-1090 Wien, Austria, Vienna; Charles University and the Mathematical Institute of the Academy of Sciences, Prague, Czech Republic*

Abstract

The continuum function $\alpha \mapsto 2^\alpha$ on regular cardinals is known to have great freedom. Say that F is an *Easton function* iff for regular cardinals α and β, $\mathrm{cf}(F(\alpha)) > \alpha$ and $\alpha < \beta \to F(\alpha) \leq F(\beta)$. The classic example of an Easton function is the *continuum function* $\alpha \mapsto 2^\alpha$ on regular cardinals. If GCH holds then any Easton function is the continuum function on regular cardinals of some cofinality-preserving extension $V[G]$; we say that F is *realised* in $V[G]$. However if we also wish to preserve measurable cardinals, new restrictions must be put on F. We say that κ is $F(\kappa)$-*hypermeasurable* iff there is an elementary embedding $j : V \to M$ with critical point κ such that $H(F(\kappa))^V \subseteq M$; j will be called a *witnessing embedding*. We will show that if GCH holds then for any Easton function F there is a cofinality-preserving generic extension $V[G]$ such that if κ, closed under F, is $F(\kappa)$-hypermeasurable in V and there is a witnessing embedding j such that $j(F)(\kappa) \geq F(\kappa)$, then κ will remain measurable in $V[G]$.

Key words: Easton's theorem, hypermeasurable and strong cardinals, lifting.
AMS subject code classification: 03E35,03E55.

1 Introduction

By Easton's results [3] the function $\alpha \mapsto 2^\alpha$ ("*continuum function*") on regular cardinals has great freedom. In fact, apart from the obvious restrictions of

Email addresses: sdf@logic.univie.ac.at (Sy-David Friedman), radek@logic.univie.ac.at (Radek Honzik).

[1] Supported by FWF Grant P 19375-N18
[2] Supported by grants FWF P 19375-N18 and GA ČR no. 401/03/H047

monotonicity, i.e. $\alpha < \beta \rightarrow 2^\beta \leq 2^\alpha$, and of Cantor's theorem $\alpha < 2^\alpha$, there is only one non-trivial condition, namely the König inequality $\mathrm{cf}(2^\alpha) > \alpha$ (which obviously implies Cantor's theorem). Such freedom is however not compatible with large cardinals possessing some kind of reflection. We for instance know, by Scott's theorem, that if κ is measurable, it cannot be the first cardinal where GCH fails. Acknowledging the importance of large cardinals in set theory, is seems reasonable to enquire what kind of restrictions the continuum function must satisfy if some large cardinal structure should be preserved.

In [9], Menas showed that if an Easton function F is "locally definable", then all supercompact cardinals are preserved in a generic extension realizing F. We show in this article how to extend this result to strong cardinals (see Section 3.2).

The focus of this article lies with the preservation of measurability, however. We know by results of Mitchell [10] and Gitik [5] that non-trivial, i.e. GCH-failing, values of the continuum function on measurable cardinals require more than just measurability. The exact strength is captured by sequences of measures or extenders which compose together to create elementary embeddings which are "stronger" than the usual measure ultrapower embeddings; for instance $2^\kappa = \kappa^{++}$ with κ being measurable can be forced from the assumption that κ has Mitchell order $o(\kappa) = \kappa^{++}$. Such sequences of measures or extenders are easily obtained from hypermeasurable embeddings, and we will use the slightly stronger assumption of hypermeasurability in our proofs.

Assuming GCH, we will show that for any Easton function F there is a cofinality-preserving generic extension realizing F which preserves the measurability of κ provided the following single non-trivial condition is satisfied:

There is an embedding j witnessing the $F(\kappa)$-hypermeasurability of κ such that $j(F)(\kappa) \geq F(\kappa)$ (1)

By way of illustration, if F is defined for some $n \in \omega$ as $F(\alpha) = \alpha^{+n}$ for every regular α, then if κ is κ^{+n}-hypermeasurable and j is any witnessing embedding, it follows by elementarity that $j(F)(\kappa) = F(\kappa) = \kappa^{+n}$ and consequently the conditions above are satisfied and the theorem implies that κ remains measurable.

2 Preliminaries

2.1 Product forcing

We first review some useful facts concerning product forcing (or "side-by-side" forcing).

To avoid confusion, we explicitly state that we use κ-distributive and κ-closed for $<\kappa$-distributive and $<\kappa$-closed (just as κ-cc is in fact used for antichains of size $<\kappa$).

The following lemma is often called "Easton's lemma" as it first appeared in the proof by Easton in [3].

Lemma 2.1 *Assume $\mathbb{P}, \mathbb{Q} \in V$ are forcing notions, $\mathbb{P}$ is κ-cc and $\mathbb{Q}$ is κ-closed. Then the following holds:*

(1) $1_{\mathbb{P}} \Vdash \check{\mathbb{Q}}$ *is κ-distributive;*
(2) $1_{\mathbb{Q}} \Vdash \check{\mathbb{P}}$ *is κ-cc;*
(3) As a corollary, if G is generic for $\mathbb{P}$ over V and H is generic for $\mathbb{Q}$ over V, then $G \times H$ is generic for $\mathbb{P} \times \mathbb{Q}$ over V, i.e. G and H are mutually generic.

For proof of (1), see [6] p. 234. (2) is easy.

We say that a forcing notion $\mathbb{P}$ is *κ-Knaster* if every subset $X \subseteq \mathbb{P}$ of size κ has a subfamily $Y \subseteq X$ of size κ such that the elements of Y are pairwise compatible. The property of being κ-Knaster is an obvious strengthening of the property of being κ-cc.

Lemma 2.2 *If $\mathbb{P}$ is κ-Knaster and $\mathbb{Q}$ is κ-cc, then $\mathbb{P} \times \mathbb{Q}$ is κ-cc.*

Proof. We will show that $1_{\mathbb{P}} \Vdash \mathbb{Q}$ is κ-cc. Assume $\tilde{p} \Vdash \dot{A}$ is an antichain in $\mathbb{Q}$ of size κ. Define a set $X = \{p_\alpha \,|\, \alpha < \kappa\}$ of conditions below $\tilde{p}$ such that $p_\alpha \Vdash \dot{A}(\alpha) = q_\alpha$ for some q_α. By κ-Knasterness of $\mathbb{P}$, there is a subfamily $X' = \{p_{\alpha_\xi} \,|\, \xi < \kappa\}$ of X such that the conditions in X' are pairwise compatible. It follows that the set $A' = \{q_{\alpha_\xi} \,|\, p_{\alpha_\xi} \in X'\}$ is an antichain in V. This is a contradiction. (Lemma 2.2) □

2.2 Hypermeasurable cardinals

We will review the definition of a hypermeasurable cardinal and give some related examples.

Definition 2.3 *A cardinal κ is λ-hypermeasurable (or λ-strong), where λ is a cardinal number, if there is an elementary embedding j with a critical point κ from V into a transitive class M such that $\lambda < j(\kappa)$ and $H(\lambda)^V \subseteq M$.*

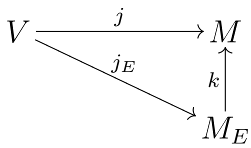
Definition 2.4 *An elementary embedding $j : V \to M$ is called an* extender embedding *if there are A and $B \subseteq j(A)$ such that $M = \{j(F)(a) \,|\, F : A \to V, a \in B\}$.*

In the context of this article, A will be the set of all finite subsets of the critical point of an embedding, i.e. $A = [\kappa]^{<\omega}$ and B will be $[\lambda]^{<\omega}$ for some $\lambda < j(\kappa)$. It is equally possible to consider all subsets of κ of size less than κ or all sets whose transitive closure has size less than κ, i.e. $A = [\kappa]^{<\kappa}$ or $A = H(\kappa)$. This choice of A is in fact convenient to show that an extender embedding $j : V \to M$ derived from a λ-hypermeasurable embedding satisfies that M is closed under $< \mu$-sequences in V where $\mu = \min(\mathrm{cf}(\lambda), \kappa^+)$. We call B as above the *support* of the extender embedding.

The following fact can be shown easily, for instance using the arguments in [8].

Fact 2.5 *(GCH) If κ is $F(\kappa)$-hypermeasurable, where F is an Easton function, and $j : V \to M$ is a witnessing embedding, then j can be factored through some $j_E : V \to M_E$ and $k : M_E \to M$ such that j_E is an extender embedding with $A = [\kappa]^{<\omega}$ and $B = [F(\kappa)]^{<\omega}$ witnessing the $F(\kappa)$-hypermeasurability of κ. Moreover, if $j(F)(\kappa) \geq F(\kappa)$, then also $j_E(F)(\kappa) \geq F(\kappa)$.*

Proof. Consider the following commutative triangle:



By the construction of the extender, it follows that k is the identity on $F(\kappa)$. The following holds: $k(j_E(F)(\kappa)) = k(j_E(F))(k(\kappa)) = k(j_E(F))(\kappa) = j(F)(\kappa)$. If $\mu = j_E(F)(\kappa) < F(\kappa)$ were true, then k would be the identity at μ, implying that $j(F)(\kappa) = \mu$, which is a contradiction. (Fact 2.5) □

The above fact allows us to use only extender embeddings in our arguments and these will be used tacitly throughout.

Remark 2.6 Note that Definition 2.3 is slightly different from the definition found in [6] or [8], where κ is called $\kappa + \alpha$ or just α-strong if $V_{\kappa+\alpha}$ is included in M. The main difference is that we use the $H_\alpha = H(\alpha)$ hierarchy instead of the V_α hierarchy to measure the strength of the embedding j.[3] The defini-

[3] Under GCH, there is a straightforward correspondence between the measurement

tion using the V_α-hierarchy has the drawback of depending on the continuum function in the given universe; thus it may happen that the degree of hypermeasurability of a given cardinal drops, although the witnessing embedding remains equally strong.

Example 2.7 In the article [4], one starts with GCH and κ being κ^{++}-hypermeasurable. Then one defines a forcing notion $\mathbb{P}$ of length $\kappa+1$ which iterates generalized Sacks forcing Sacks(α, α^{++}) for $\alpha \leq \kappa$ inaccessible, making $2^\kappa = \kappa^{++}$ in the generic extension $V[G]$, where G is $\mathbb{P}$-generic. It is shown that the original $j : V \to M$ witnessing the hypermeasurability of κ can be lifted to $j : V[G] \to M[j(G)]$. It is straightforward to verify that $H(\kappa^{++})^{V[G]}$ is included in $M[j(G)]$, so κ remains κ^{++}-hypermeasurable in $V[G]$ according to Definition 2.3. However, as 2^κ becomes κ^{++} in $V[G]$, κ may not stay $\kappa+2$-strong in $V[G]$ according to the definition as in [6] since this would require that $\mathscr{P}(\kappa^{++})$ of $V[G]$ is included in $M[j(G)]$.

We close this section with an observation concerning hypermeasurable cardinals which will be relevant later in the argument.

Observation 2.8 *(GCH) Let $j : V \to M$ for some transitive M be an embedding with critical point κ such that $H(\lambda)^V \subseteq M$, $\kappa < \lambda < j(\kappa)$ and λ is inaccessible in M (such an embedding exists for example if κ is λ-hypermeasurable for some V-inaccessible $\lambda > \kappa$). Then there exists $\bar\lambda \leq \lambda$ singular in V and an embedding $k : V \to N$ which witnesses that κ is $\bar\lambda$-hypermeasurable and $\bar\lambda$ is inaccessible in N.*

Proof. Without loss of generality assume that j is an extender embedding, that is $M = \{j(f)(a) \,|\, f : [\kappa]^{<\omega} \to V, a \in [\lambda]^{<\omega}\}$. If λ is singular in V, then we are done. So assume that λ is regular (and hence inaccessible) in V. For each $f : [\kappa]^{<\omega} \to \kappa$ define a function $\theta_f : [\lambda]^{<\omega} \to \lambda$ by setting $\theta_f(a) = j(f)(a)$ if $j(f)(a) < \lambda$ and $\theta_f(a) = 0$ otherwise. Working in V, let $C_f \subseteq \lambda$ by a closed unbounded set of limit cardinals closed under θ_f; as $\kappa^+ < \lambda$ and the number of all θ_f is κ^+, the intersection $C = \bigcap_f C_f$ is a closed unbounded set in λ. Let $\bar\lambda$ be some singular cardinal in C greater than κ and let $H = \{j(f)(a) \,|\, f : [\kappa]^{<\omega} \to V, a \in [\bar\lambda]^{<\omega}\}$. If $\pi : H \cong N$ is the transitive collapse map, we obtain that $\pi \circ j : V \to N$ witnesses that κ is $\bar\lambda$-hypermeasurable. We are done once we show that $\bar\lambda$ is regular in N. This will follow from the fact that $H \cap \lambda = \bar\lambda$. Let $\alpha \in H \cap \lambda$ be given; it is of the form $j(f)(a)$ for some $f : [\kappa]^{<\omega} \to \kappa$ and $a \in [\bar\lambda]^{<\omega}$. As $\alpha < \lambda$, $j(f)(a) = \theta_f(a) < \bar\lambda$ by the selection of $\bar\lambda$ in C_f. Conversely, if $\alpha < \bar\lambda$, then $\alpha = j(id)(\alpha)$. Finally, without loss of generality we may assume that there is some f_λ such that

---

of the strength of an embedding using the structures $H(\kappa^{+\alpha})$ and $V_{\kappa+\alpha}$ for an inaccessible κ and ordinal number α. It holds that if M is an inner model of ZFC then $V_{\kappa+\alpha} \subseteq M$ holds iff $H(\kappa^{+\alpha}) \subseteq M$ holds. This correspondence is however lost if GCH fails.

$j(f_\lambda)(\kappa) = \lambda$ and hence $\lambda \in H$. Since then $\pi(\lambda) = \bar{\lambda}$, the observation follows. (Observation 2.8) □

2.3 Preservation of measurability

As regards the preservation of measurability of a given cardinal κ in the generic extension, it turns out that the most suitable way to achieve this is to lift the original elementary embedding in the ground model.

Definition 2.9 *Let $\mathbb{P} \in M$ be a forcing notion and $j : M \to N$ an elementary embedding from M to N, both transitive models of ZFC. Let G be an M-generic filter for $\mathbb{P}$, and H an N-generic filter for $j(\mathbb{P})$. We say that j^* lifts the embedding j if j^* is an elementary embedding $j^* : M[G] \to N[H]$ extending j.*

There is a simple sufficient condition which guarantees the existence of a lifting j^*. The following lemma is due to Silver.

Lemma 2.10 (Lifting lemma) *Let $\mathbb{P} \in M$ be a forcing notion and $j : M \to N$ an elementary embedding from M to N, both transitive models of ZFC. Let G be an M-generic filter for $\mathbb{P}$, and H an N-generic filter for $j(\mathbb{P})$. If $j[G] \subseteq H$, i.e. if the pointwise image of G under j is included in H, then*

(1) j lifts to $j^* : M[G] \to N[H]$, and
(2) $j^*(G) = H$.

As we will be dealing with extender embeddings, it is useful to notice that by using names for the elements of $M[G]$, we can argue that if j is an extender embedding, then so will be the lift j^*:

Lemma 2.11 *Let the assumptions of Lemma 2.10 hold. Assume further that $j : M \to N$ is an extender embedding, i.e. $N = \{j(F)(a) \mid F \in M, F : A \to V, a \in B \subseteq j(A)\}$. Assume $j^* : M[G] \to N[j^*(G)]$ is a lift of j. Then j^* is also an extender embedding, and moreover the parameters A and B of the extender embedding j remain the same, i.e.*

$$N[j^*(G)] = \{j^*(F)(a) \mid F \in M[G], F : A \to M[G], a \in B\}.$$

Remark 2.12 If $j^* : M[G] \to N[j^*(G)]$ is a lift of an embedding $j : M \to N$ which witnessed the measurability of κ in M (i.e. j is definable in M), then κ is still measurable in $M[G]$, *providing* that j^* is definable in $M[G]$.

Remark 2.13 Assume $j^* : M[G] \to N[j^*(G)]$ is a lift of an embedding $j : M \to N$ which witnessed the measurability of κ in M. Assume further that j^* is definable in $M[G]$. Let $U_j = \{X \subseteq \kappa \mid X \in M, \kappa \in j(X)\}$ be the normal ultrafilter derived from j. Then U_{j^*} extends the ultrafilter U_j: $U_j \subseteq U_{j^*}$. Note however that the extension U_{j^*} is in general very difficult to find[4] unless some powerful structural information such as the embedding j is available.

In view of the Lifting lemma 2.10, the crucial part of the arguments dealing with the preservation of measurability consists in finding an N-generic H containing the pointwise image of G. This may be rather difficult in some cases, but if the forcing notion $\mathbb{P}$ is sufficiently distributive and the embedding to be lifted is an extender embedding, the existence of such an H is straightforward.

Lemma 2.14 *Assume $j : M \to N$ is an extender embedding as in Lemma 2.11 and $N = \{j(F)(a) \mid F \in M, F : A \to V, a \in B \subseteq j(A)\}$. Let G be M-generic for a forcing notion $\mathbb{P} \in M$. If M satisfies that $\mathbb{P}$ is $|A|^+$-distributive, then*

$$H = \{q \in j(\mathbb{P}) \mid \exists p \in G, j(p) \leq q\}$$

is N-generic for $j(\mathbb{P})$ and contains the pointwise image of G.

Proof. H is obviously a filter. We show it is a generic filter. Let $D = j(F)(d)$ be a dense open set. We may assume that the range of F consists of dense open sets in $\mathbb{P}$. Let $\{a_\xi \mid \xi < |A|\}$ be the enumeration of A. By distributivity, $X = \bigcap_{\xi<|A|} F(a_\xi)$ is dense. Let $p \in X$ be in G; then $M \models \forall a \in A, p \in F(a)$, and by elementarity it follows that $N \models \forall a \in j(A), j(p) \in j(F)(a)$. In particular, $j(p) \in j(F)(d) = D$. (Lemma 2.14) □

However, it is not true conversely that if $\mathbb{P}$ fails to be $|A|^+$-distributive, then H cannot be in some sense generated from the generic filter G. In fact, the construction in [4] shows that in the context of Sacks forcing, distributivity can be replaced by the weaker property of $|A|$-fusion (diverting from the notation in this paper in this case, $|A|$-fusion refers to sequences of length $|A|$ and not $< |A|$; see Fact 2.18 below).

We close this preliminary section by a technical lemma which is useful in constructing generic filters and is tacitly used throughout the arguments.

Lemma 2.15 *Assume $N \subseteq M$ are inner models of ZFC and $M \models {}^\lambda N \subseteq N$, i.e. N is closed under λ-sequences in M. If $\mathbb{P} \in N$ is λ^+-cc in M and G is $\mathbb{P}$-generic over M, then $M[G] \models {}^\lambda N[G] \subseteq N[G]$, i.e. $N[G]$ is closed under λ-sequences in $M[G]$.*

[4] With the notable exception when the forcing notion $\mathbb{P}$ is of size $<\kappa$; in this case any normal ultrafilter U in M generates a normal ultrafilter in the generic extension.

The proof of the theorem is centered around the technique developed in [4] which uses the Sacks forcing instead of the Cohen forcing in the lifting arguments. We will give a brief review here in a slightly generalized setting.

Though the concept of a perfect tree can be formulated for an arbitrary regular cardinal, see also [7], we will use the forcing at inaccessible cardinals only and this introduces further simplifications.

Definition 2.16 *If α is an inaccessible cardinal, then $p \subseteq 2^{<\alpha}$ is a perfect α-tree if the following conditions hold:*

(1) If $s \in p, t \subseteq s$, then $t \in p$;
(2) If $s_0 \subseteq s_1 \cdots$ is a sequence in p of length less than α, then the union of s_i's belongs to p;
(3) For every $s \in p$ there is some $s \subseteq t$ such that t is a splitting node, i.e. both $t * 0$ and $t * 1$ belong to p;
(4) Let $\mathrm{Split}(p)$ denote the set of s in p such that both $s * 0$ and $s * 1$ belong to p. Then for some (unique) closed unbounded set $C(p) \subseteq \alpha$, $\mathrm{Split}(p) = \{s \in p \,|\, \mathrm{length}(s) \in C(p)\}$.

A perfect α-tree is an obvious generalization of the perfect tree at ω ordered by inclusion; there is only one non-trivial condition, and this concerns the limit levels of the tree: if $s \in p$ is an element at a limit level and the splitting nodes $t \subseteq s$ are unbounded in s, then s must be a splitting node as well (continuous splitting). As α is inaccessible, and consequently every level of p is of size $<\alpha$, the trees obeying (4) above are dense in the trees having continuous splitting.

Generalized perfect trees can be used to define a natural forcing notion.

Definition 2.17 *The forcing notion $\mathrm{Sacks}(\alpha, 1)$ contains as conditions the perfect α-trees, the ordering is by inclusion (not the reverse inclusion), i.e. $p \leq q$ iff $p \subseteq q$. Or generally, the forcing notion $\mathrm{Sacks}(\alpha, \lambda)$, where $0 < \lambda$ is an ordinal number, is a product of length λ of the forcing $\mathrm{Sacks}(\alpha, 1)$ with support of size at most α, i.e. a condition p in $\mathrm{Sacks}(\alpha, \lambda)$ is a function from λ to $\mathrm{Sacks}(\alpha, 1)$ such that $\{\xi < \lambda \,|\, p(\xi) \neq 1_{\mathrm{Sacks}(\alpha,1)}\}$ has size at most α.*

For p a condition in $\mathrm{Sacks}(\alpha, 1)$, let $\langle \alpha_i \,|\, i < \alpha \rangle$ be the increasing enumeration of $C(p)$ and let $\mathrm{Split}_i(p)$ be the set of s in p of length α_i. For $p, q \in \mathrm{Sacks}(\alpha, 1)$ let us write $p \leq_\beta q$ iff $p \leq q$ and $\mathrm{Split}_i(p) = \mathrm{Split}_i(q)$ for $i < \beta$. In the generalization for the product $\mathrm{Sacks}(\alpha, \lambda)$ we write $p \leq_{\beta,X} q$ (where X is some subset of λ of size less than α) iff $p \leq q$ (i.e. for all $i < \lambda, p(i) \leq q(i)$) and moreover for each $i \in X$, $p(i) \leq_\beta q(i)$.

We will define several useful notions and state some facts.

Fact 2.18 *The forcing* Sacks(α, λ) *satisfies the following* α-fusion property*: Suppose* $p_0 \geq p_1 \geq \dots$ *is a descending sequence in* Sacks(α, λ) *of length* α *and suppose in addition that* $p_{i+1} \leq_{i,X_i} p_i$ *for each* i *less than* α*, where* X_i *form an increasing sequence of subsets of* λ *of size less than* α *whose union is the union of the supports of* p_i*'s; such a sequence will be called a* fusion sequence*. Then the* p_i*'s have a lower bound in* Sacks(α, λ) *(obtained by taking intersections at each component).*

Definition 2.19 *Assume* p *is a condition in* Sacks(α, λ)*,* X *is a subset of* λ *of size less than* α *and* β *is less than* α*. Then an* (X, β)-thinning of p *is an extension of* p *obtained by thinning each* $p(i)$ *for* $i \in X$ *to a subtree consisting of all nodes compatible with some particular node on the* β*-th splitting level of* $p(i)$*.*

Definition 2.20 *Assume* D *is a dense open set in* Sacks(α, λ)*. We say that* $p \in$ Sacks(α, λ) reduces D *iff for some subset* X *of* λ *of size less than* α *and some* $\beta < \alpha$ *any* (S, β)*-thinning of* p *meets* D*.*

The following important fact holds:

Fact 2.21 *Let* $\{D_i \,|\, i < \alpha\}$ *be a collection of* α*-many dense open sets in* Sacks(α, λ)*. Then for each* p *there is a condition* $q \leq p$*, obtained as a lower bound of a fusion sequence, such that* q *reduces each* D_i *in the above sense.*

Note that Sacks(α, λ) is obviously α^{++}-cc (by the GCH at α) and α-closed. Preservation of cardinals follows from Fact 2.21 which is used to show that α^+ is preserved as well. Also, Sacks(α, λ) adds λ-many new subsets of α (the intersection of all trees in a generic filter at a given coordinate determines a unique subset of α; this subset in turn determines the whole generic at the given coordinate).

As discussed after Lemma 2.14, it is the α-fusion property which is strong enough to replace the restrictive condition of distributivity in Lemma 2.14. We will briefly review here the argument of [4] in a slightly more general setting (for details consult [4]).

Theorem 2.22 *(GCH) Let* κ *be a* λ*-hypermeasurable cardinal with* λ *greater than* κ *and of cofinality at least* κ^+*. Assume further that there is a witnessing embedding* j *and a function* $f_\lambda : \kappa \to \kappa$ *such that* $j(f_\lambda)(\kappa) = \lambda$*. Then there is a forcing iteration* $\mathbb{S} = \langle S_\alpha \,|\, \alpha \leq \kappa + 1\rangle$ *of generalized Sacks forcings which preserves measurability of* κ *and forces* $2^\kappa = \lambda$*. Moreover, the generic for the* $j(\kappa)$*-th stage of the iteration* $j(\mathbb{S})$ *is in some sense "generated" from the generic at stage* κ *of* $\mathbb{S}$*.*

We will give a sketch of the proof. Fix a λ-hypermeasurable extender embedding $j : V \to M$ with critical point κ; we may still assume that $j(f_\lambda)(\kappa) = \lambda$. As the cofinality of λ is at least κ^+, M can be taken to be closed under κ-sequences. Also, by GCH we have that $\lambda < j(\kappa) < \lambda^+$. We define the iteration $\mathbb{S} = \langle S_\alpha \mid \alpha \leq \kappa + 1\rangle$ as an Easton-supported forcing iteration of length $\kappa + 1$ which at every inaccessible $\alpha < \kappa$ adds $f_\lambda(\alpha)$-many new subsets of α using the forcing Sacks$(\alpha, f_\lambda(\alpha))$ and at stage κ adds λ-many new subsets of κ using Sacks(κ, λ). Let us write the generic $G_{\kappa+1}$ for $\langle S_\alpha \mid \alpha \leq \kappa + 1\rangle$ as $G * g$, where G is $\mathbb{S}_\kappa$-generic over V and g is Sacks(κ, λ)-generic over $V[G]$.

Our aim is to lift the embedding j to $V[G*g]$. Using the fact that $j(f_\lambda)(\kappa) = \lambda$, we can proceed as in [4] to lift partially to $j : V[G] \to M[G * g * H]$, where H is a generic for the iteration in the interval $(\kappa, j(\kappa))$.

Now it remains to find a generic h for Sacks$(j(\kappa), j(\lambda))$ over $M[G * g * H]$ containing the pointwise image of g. Denote $j[g]$ by h^*. As g is a set of conditions in Sacks(κ, λ) of $V[G]$, h^* is a set of conditions in Sacks$(j(\kappa), j(\lambda))$ of $M[G * g * H]$. The following lemma describes the "intersection" of the conditions in h^*.

Lemma 2.23 *For $\alpha < j(\lambda)$ let t_α be the intersection of the trees $j(p)(\alpha), p \in g$. If α belongs to the range of of j, then t_α is a $(\kappa, j(\kappa))$*-tuning fork*, i.e. a subtree of $2^{<j(\kappa)}$ which is the union of two cofinal branches which split at κ. If α does not belong to the range of j, then t_α consists of exactly one cofinal branch through $2^{<j(\kappa)}$.*

Proof. First notice that the intersection of $\bigcap_{C\in V[G]} j(C)$ where C is a closed unbounded set in κ is equal to $\{\kappa\}$. The intersection obviously contains κ. Now let $C_{f_\lambda} \subseteq \kappa$ be the closed unbounded set of closure points of f_λ, i.e. for all $\xi \in C_{f_\lambda}$, $f_\lambda(x) < \xi$ for each $x \in [\xi]^{<\omega}$ (without loss of generality, all elements of C_{f_λ} are limit cardinals). As $j(f_\lambda)(\kappa) = \lambda$ and $\bigcap_{C\in V[G]} j(C) \subseteq j(C_{f_\lambda})$, it is obvious that any element ξ of the intersection in the interval $(\kappa, j(\kappa))$ must be a limit cardinal greater than λ. But any such hypothetical ξ can be expressed as $j(f)(x)$ for some $f : [\kappa]^{<\omega} \to \kappa$ and $x \in [\lambda]^{<\omega}$. If C_f is the closed unbounded set of closure points of f, it is immediate that $j(C_f)$ cannot contain $\xi = j(f)(x)$. Notice that not only $j(f)(x) = \xi$ is not in $j(C_f)$, but the whole interval $[\lambda, \xi]$ is disjoint from $j(C_f)$ (as $x \in [\lambda]^{<\omega}$, the least closure point of $j(f)$ above λ must be greater than ξ). It follows that there is for each $\xi < j(\kappa)$ a closed unbounded set $C_\xi = C_{f_\lambda} \cap C_f$ such that

$$(\kappa, \xi] \cap j(C_\xi) = \emptyset. \tag{2}$$

The analysis of the intersection of closed unbounded sets is important as the following fact holds. If C is a closed unbounded subset of κ in $V[G]$ and X is a subset of λ of size at most κ, then any condition $p \in$ Sacks(κ, λ) in $V[G]$ has an extension q such that for all $i \in X$, $C(q(i))$ (= the set of splitting levels of the tree $q(i)$) is a subset of C. As every $\alpha < j(\lambda)$ can be expressed as some $j(f)(a)$ where f is a function from $[\kappa]^{<\omega}$ to λ, by applying the above fact with $X = \mathrm{rng}(f)$ we obtain that $C(j(q)(\alpha))$ is a subset of $j(C)$. It follows by (2) that there is for each $\xi < j(\kappa)$ a condition r_ξ in g such that the tree $j(r_\xi)(\alpha)$ does not split between κ and ξ (though it may split at κ).

As $M[G * g * H]$ contains all subsets of κ existing in $V[G * g]$, it follows that the intersection t_α of the $j(p)(\alpha)$, $p \in g$, is a subtree of $2^{<j(\kappa)}$ which is the union of at most two cofinal branches which can only differ at κ.

If α is in the range of j than it is obvious that all trees $j(p)(\alpha)$, $p \in g$, do branch at κ (by elementarity and by the "continuous splitting" of a perfect tree). If α is not in the range of j, then the intersection t_α does not split at κ (the proof can be found in [4]). (Lemma 2.23) □

Definition 2.24 *For $\alpha < j(\lambda)$ in the range of j, let $(x(\alpha)_0, x(\alpha)_1)$ be the branches that make up the $(\kappa, j(\kappa))$-tuning fork at α, where $x(\alpha)_0(\kappa) = 0$ and $X(\alpha)_1(\kappa) = 1$. For $\alpha < j(\lambda)$ not in the range of j let $x(\alpha)_0$ denote the unique branch constituting the intersection of the $j(p)(\alpha)$, $p \in g$.*

Lemma 2.25 *Let h consist of all conditions p in* Sacks$(j(\kappa), j(\lambda))$ *of $M[G * g * H]$ such that for each $\alpha < j(\lambda)$, $x(\alpha)_0$ is contained in $p(\alpha)$. Then h is generic for* Sacks$(j(\kappa), j(\lambda))$ *of $M[G * g * H]$ and contains $j[g]$.*

Proof. Let D be a dense open set in Sacks$(j(\kappa), j(\lambda))$ in $M[G * g * H]$. As an element of $M[G*g*H]$, it can be written as $j(f)(d)$ for some f and $d \in [\lambda]^{<\omega}$. Without loss of generality we may assume that $f(a)$ is a dense open set in Sacks(κ, λ) for every $a \in [\kappa]^{<\omega}$.

Using Fact 2.21, there is a condition $q \in g$ such that q reduces all dense open sets $f(a)$. By elementarity, $j(q)$ reduces all dense open sets $j(f)(a)$ for $a \in [\lambda]^{<\omega}$ and in particular reduces $j(f)(d) = D$.

In $M[G * g * H]$ choose a subset X of $j(\lambda)$ of size less than $j(\kappa)$ and $\alpha < j(\kappa)$ such that any (X, α)-thinning of $j(q)$ meets D. Now for each $i \in X$ thin $j(q)$ by choosing an initial segment of $x(i)_0$ on the α-th splitting level of $j(q)(i)$. As this sequence of choices is in $M[G * g * H]$ (for proof see [4]), it follows that this thinned out condition belongs to h and meets D. So h is generic for Sacks$(j(\kappa), j(\lambda))$ of $M[G*g*H]$ over $M[G*g*H]$ as desired. [5] (Lemma 2.25) □

Amongst the main advantages of [4], apart from the fact that we avoid the "modification" argument as in the Woodin-style approach (see for instance [1]

[5] One must also verify that any two conditions in h are compatible with each other; for argument, see [4].

or a slightly different argument in [2]) is that we don't have to enlarge the universe $V[G * g]$ to complete the lifting. This adds a degree of uniformity which will be used later in this article.

3 Easton's theorem and large cardinals

3.1 Preservation of measurable cardinals

Definition 3.1 *A class function F defined on regular cardinals is called an* Easton function *if it satisfies the following two conditions which were shown by Easton to be the only conditions provable about the continuum function on regular cardinals in ZFC. Let κ, μ be arbitrary regular cardinals:*

(1) If $\kappa < \mu$, then $F(\kappa) \leq F(\mu)$;
(2) $\kappa < \mathrm{cf}(F(\kappa))$.

Note that Cantor's theorem $\kappa < 2^\kappa = F(\kappa)$ is implied by (2) above.

It is obvious, however, that if a given large cardinal κ should remain measurable in a generic extension realizing a given Easton function F, the properties of the cardinal κ and the properties of the function F need to combine in a suitable way which requires more than the conditions given in Definition 3.1.

Example 3.2 (1) If for some $\lambda < \kappa$, $\kappa \leq F(\lambda)$, then κ will not be even strongly inaccessible if F is realised.
(2) By a theorem of Scott, κ cannot be the least cardinal where GCH fails if it should remain measurable.
(3) Or more generally, F should not "jump" at κ. For instance if $F(\lambda) \leq \lambda^{++}$ for $\lambda < \kappa$ and $F(\kappa) = \kappa^{+3}$, then κ cannot remain measurable if F is realised.

We capture a sufficient condition for preservation of measurability in the following definition.

Definition 3.3 *We say that a cardinal κ is* good *for F, or shortly F-*good*, if the following properties hold:*

(1) $F[\kappa] \subseteq \kappa$, i.e. κ is closed under F;
(2) κ is $F(\kappa)$-hypermeasurable and this is witnessed by an embedding $j : V \to M$ such that $j(F)(\kappa) \geq F(\kappa)$.

Our forcing to realise a given Easton function F will be a combination of the Sacks forcing $\mathrm{Sacks}(\bar{\alpha}, \bar{\beta})$ (see Definition 2.17) and of the Cohen forcing

$\mathrm{Add}(\alpha,\beta)$, where $\bar\alpha,\alpha$ are regular cardinals and $\bar\beta,\beta$ are ordinal numbers. For notational convenience we will construe $\mathrm{Add}(\alpha,\beta)$ as the $<\alpha$-supported product of $\mathrm{Add}(\alpha,1)$ of length β, where conditions in $\mathrm{Add}(\alpha,1)$ are functions from α to 2 with domain of size less than α.

As our aim is the preservation of large cardinals, we cannot use the standard Easton product-style forcing, but we need to use some kind of (reverse Easton) iteration. The iteration however needs some "space" as otherwise it would collapse cardinals, as the following observation shows.

Observation 3.4 *Assume $\kappa<\lambda$ are regular cardinals. If κ^* is a cardinal greater than λ, then forcing with $\mathrm{Add}(\kappa,\kappa^*)*\mathrm{Add}(\lambda,1)$ collapses κ^* to λ.*

Proof. Let $\langle x_\xi \,|\, \xi<\kappa^*\rangle$ be the enumeration of subsets of κ in $V^{\mathrm{Add}(\kappa,\kappa^*)}$; for each $\xi<\kappa^*$, the set $D_\xi=\{p\in \mathrm{Add}(\lambda,1)\,|\,\exists\alpha<\lambda, p\restriction[\alpha,\alpha+\kappa) \text{ determines } x_\xi\}$ is dense. Consequently, there is a surjection from λ onto κ^* in the generic extension of V by $\mathrm{Add}(\kappa,\kappa^*)*\mathrm{Add}(\lambda,1)$. (Observation 3.4) $\Box$

The concept of the "space" mentioned above is technically captured by the closure points of the function F. We say that a cardinal κ is a closure point of F if $\mu<\kappa$ implies $F(\mu)<\kappa$. We will enumerate in the increasing order the closed unbounded class of closure points of F as $\langle i_\alpha\,|\,\alpha<\mathrm{On}\rangle$. Note that every i_α must be a limit cardinal and $i_{\beta+1}$ has cofinality ω for every β. If κ is a regular closure point, then κ equals i_κ.

We will now give a full definition of the forcing notion to realise an Easton function F.

Definition 3.5 *Let an Easton function F satisfying the conditions (1), (2) of 3.1 be given. Let $\langle i_\alpha\,|\,\alpha<\mathrm{On}\rangle$ be an increasing enumeration of the closure points of F.*

We will define an iteration $\mathbb{P}^F=\Big\langle\langle\mathbb{P}_{i_\alpha}\,|\,\alpha<\mathrm{On}\rangle,\langle\dot{\mathbb{Q}}_{i_\alpha}\,|\,\alpha<\mathrm{On}\rangle\Big\rangle$ indexed by $\langle i_\alpha\,|\,\alpha<\mathrm{On}\rangle$ such that:

- *If i_α is not an inaccessible cardinal, then*

$$\mathbb{P}_{i_{\alpha+1}}=\mathbb{P}_{i_\alpha}*\dot{\mathbb{Q}}_{i_\alpha}, \tag{3}$$

  where $\dot{\mathbb{Q}}_{i_\alpha}$ is a name for $\prod_{i_\alpha<\lambda<i_{\alpha+1}}\mathrm{Add}(\lambda,F(\lambda))$ (λ ranges over regular cardinals and the product has the Easton support).
- *If i_α is an inaccessible cardinal, then*

$$\mathbb{P}_{i_{\alpha+1}}=\mathbb{P}_{i_\alpha}*\dot{\mathbb{Q}}_{i_\alpha}, \tag{4}$$

  where $\dot{\mathbb{Q}}_{i_\alpha}$ is a name for $\mathrm{Sacks}(i_\alpha,F(i_\alpha))\times\prod_{i_\alpha<\lambda<i_{\alpha+1}}\mathrm{Add}(\lambda,F(\lambda))$ (λ ranges over regular cardinals and the product has the Easton support).

- *If γ is a limit ordinal, then $\mathbb{P}_{i_\gamma}$ is an inverse limit unless i_γ is a regular cardinal, in which case $\mathbb{P}_{i_\gamma}$ is a direct limit (the usual Easton support).*

Lemma 3.6 *Under GCH, $\mathbb{P}^F$ preserves all cofinalities.*

Proof. The only non-standard feature of $\mathbb{P}^F$ is the inclusion of the Sacks forcing. It is enough to show that $\mathrm{Sacks}(\kappa, F(\kappa)) \times \mathrm{Add}(\kappa^+, F(\kappa^+))$ preserves cofinalities. The product is κ-closed and κ^{++}-cc, hence just the cardinal κ^+ needs a special argument. For this, it suffices to use the usual fusion-style argument for the Sacks forcing in $V^{\mathrm{Add}(\kappa^+, F(\kappa^+))}$. (Lemma 3.6) □

In fact, a more detailed analysis of the product $\mathrm{Sacks}(\kappa, \alpha)$ and $\mathrm{Add}(\kappa^+, \beta)$ for arbitrary ordinals α, β shows that the Cohen forcing $\mathrm{Add}(\kappa^+, \beta)$ remains κ^+-distributive after forcing with the Sacks forcing $\mathrm{Sacks}(\kappa, \alpha)$. This will be useful in further arguments.

Lemma 3.7 *Let κ be an inaccessible cardinal and α an ordinal number. Let $\mathbb{P}$ be any κ^+-closed forcing notion.*

(1) $\mathrm{Sacks}(\kappa, 1)$ *forces that $\check{\mathbb{P}}$ is κ^+-distributive.*
(2) Or more generally, $\mathrm{Sacks}(\kappa, \alpha)$ *forces that $\check{\mathbb{P}}$ is κ^+-distributive.*

Proof. Ad (1). Denote $\mathbb{S} = \mathrm{Sacks}(\kappa, 1)$. The proof is a generalization of the usual argument which shows that a κ^+-closed forcing notion does not add new κ-sequences. The difference is in the treatment of the Sacks coordinates in $\mathbb{S} \times \mathbb{P}$ which are obviously not κ^+-closed; however, they are closed under fusion limits of length κ, and this will suffice to argue that new κ sequences cannot appear between $V^{\mathbb{S}}$ and $V^{\mathbb{S}\times\mathbb{P}}$.

Let $\langle s, p\rangle$ force that $\dot{f} : \kappa \to \mathrm{On}$. It is enough to find a condition $\langle \tilde{s}, \tilde{p}\rangle \leq \langle s, p\rangle$ such that if $\langle \tilde{s}, \tilde{p}\rangle \in G \times H$, where $G \times H$ is a generic for $\mathbb{S} \times \mathbb{P}$, then $\dot{f}^{G\times H} = f$ can be defined in $V[G]$.

We will define a decreasing sequence of conditions $\langle \langle s_\alpha, p_\alpha\rangle \,|\, \alpha < \kappa\rangle$ deciding the values of $\dot{f}(\alpha)$ for $\alpha < \kappa$ where $\tilde{s}$ will be the fusion limit of $\langle s_\alpha \,|\, \alpha < \kappa\rangle$ and $\tilde{p}$ will be the lower bound of $\langle p_\alpha \,|\, \alpha < \kappa\rangle$.

Set $\langle s_0, p_0\rangle = \langle s, p\rangle$. Assume $\langle s_{\alpha'}, p_{\alpha'}\rangle$ are constructed for $\alpha' < \alpha$ and let first $\langle \bar{s}_\alpha, \bar{p}_\alpha\rangle$ be a lower bound of $\langle s_{\alpha'}, p_{\alpha'}\rangle$'s; we show how to construct $\langle s_\alpha, p_\alpha\rangle$. Let S_α denote the set of splitting nodes of rank α in $\bar{s}_\alpha$ (the first splitting node has rank 0). Pick some $t \in S_\alpha$, and considering its immediate continuations $t*0$ and $t*1$, find conditions $\langle r_{t*0}, p_{t*0}\rangle$, $\langle r_{t*1}, p_{t*1}\rangle$ and ordinals α_{t*0}, α_{t*1} such that the following conditions hold:

(1) $\bar{p}_\alpha \geq p_{t*0} \geq p_{t*1}$;
(2) $r_{t*0} \leq \bar{s}_\alpha \restriction t*0$, $r_{t*1} \leq \bar{s}_\alpha \restriction t*1$;

(3) $\langle r_{t*0}, p_{t*0}\rangle \Vdash \dot{f}(\alpha) = \alpha_{t*0}$ and $\langle r_{t*1}, p_{t*1}\rangle \Vdash \dot{f}(\alpha) = \alpha_{t*1}$.

Continue in this fashion considering successively all $t \in S_\alpha$, taking care to form a decreasing chain $\bar{p}_\alpha \geq p_{t*0} \geq p_{t*1} \ldots \geq p_{t'*0} \geq p_{t'*1} \geq \ldots$, for $t, t' \in S_\alpha$ (where t' is considered after t) in the Cohen forcing. We define:

(1) p_α = the lower bound of $\bar{p}_\alpha \geq p_{t*0} \geq p_{t*1} \ldots \geq p_{t'*0} \geq p_{t'*1} \geq \ldots$;
(2) s_α = the amalgamation of the subtrees r_{t*0}, r_{t*1} for all $t \in S_\alpha$.

Finally, define $\langle \tilde{s}, \tilde{p}\rangle$ as the fusion limit of s_α's at the first coordinate and as the lower bound at the second coordinate.

Let $G \times H$ be a generic for $\mathbb{S} \times \mathbb{P}$ containing $\langle \tilde{s}, \tilde{p}\rangle$. In $V[G]$ define a function $f' : \kappa \to \mathrm{On}$ as follows: $f'(\alpha) = \beta$ iff $\beta = \alpha_{t*i}$, for $i \in \{0, 1\}$, where t is a splitting node of rank α in $\tilde{s}$ and $t * i \subseteq \bigcup_{s\in G} \mathrm{stem}(s)$.

It is straightforward to verify that $f' = f = \dot{f}^{G\times H}$.

Ad (2). The proof proceeds in exactly the same way as (1) except that a generalized fusion is used for the Sacks(κ, α) forcing (it is essential here that the conditions in the Sacks forcing can have support of size κ). (Lemma 3.7) $\Box$

We can now state the main theorem.

Theorem 3.8 *Assume GCH and let F be an Easton function according to Definition 3.1. Then the generic extension by $\mathbb{P}^F$ preserves all cofinalities and realises F, i.e. $2^\kappa = F(\kappa)$ for every regular cardinal κ. Moreover, if a cardinal κ is good for F, then it will remain measurable.*

The proof will be given in a sequence of lemmas.

It is obvious that the Easton function F is realised in $V^{\mathbb{P}^F}$. It remains to prove that each F-good cardinal κ remains measurable in the generic extension. Let an F-good cardinal κ be fixed. Fix also a $j : V \to M$ an $F(\kappa)$-hypermeasurable extender embedding witnessing the F-goodness of κ.

$$V \xrightarrow{\quad j \quad} M$$

The properties of the Easton function F imply that $\mathrm{cf}(F(\kappa)) > \kappa$, so in particular M is closed under κ-sequences in V. It also holds that $F(\kappa) < j(\kappa) < F(\kappa)^+$, $j(F)(\kappa) \geq F(\kappa)$ (by goodness), $M = \{j(f)(a) \,|\, f : [\kappa]^{<\omega} \to V, a \in [F(\kappa)]^{<\omega}\}$, and $H(F(\kappa))^V = H(F(\kappa))^M$. Note that M is not closed even under κ^+-sequences in V, but the correct capturing of $H(F(\kappa))$ implies that ${}^{<\mathrm{cf}(F(\kappa))}H(F(\kappa)) \subseteq M$, so M is closed under $< \mathrm{cf}(F(\kappa))$-sequences providing that they refer to objects in $H(F(\kappa))$.

We fix some notation first. Let G be a generic for $\mathbb{P}^F$. As usual, we will write

G_α for the generic G restricted to $\mathbb{P}_\alpha$. The generic for $\dot{\mathbb{Q}}_\alpha$ taken in $V[G_\alpha]$ will be denoted as g_α; it follows that $G_{\alpha+1} = G_\alpha * g_\alpha$.

For reasons of notational simplicity, we write $\mathbb{P}^M$ for $j(\mathbb{P}^F)$. Recall that $\mathbb{P}^F$ is defined as an iteration along the closure points $\langle i_\alpha \,|\, \alpha < \mathrm{On}\rangle$ of F; by elementarity, $\mathbb{P}^M$ is defined using the closure points of $j(F)$, which we will denote as $\langle i^M_\alpha \,|\, \alpha < \mathrm{On}\rangle$. Since j is the identity on $H(\kappa)$, the closure points of F and $j(F)$ coincide up to and including κ, i.e. $\langle i_\alpha \,|\, \alpha \leq \kappa\rangle = \langle i^M_\alpha \,|\, \alpha \leq \kappa\rangle$. Because κ is regular, we also have that $i_\kappa = \kappa$. By elementarity, $j(\kappa)$ is closed under $j(F)$, and as $j(\kappa)$ is regular in M, it follows that $j(\kappa) = i^M_{j(\kappa)}$ and so in particular $F(\kappa) \leq j(F)(\kappa) < i^M_{\kappa+1} < j(\kappa) < F(\kappa)^+ < i_{\kappa+1}$.

The general strategy of the proof is to lift the embedding j to $V[G]$. This amounts to finding a suitable generic for $\mathbb{P}^M$. As the cardinal structure between V and M is the same up to and including $F(\kappa)$, it follows that the generics for the V-regular cardinals $\leq F(\kappa)$ need to be "copied" from the $V[G]$-side. The forcing $\mathbb{P}^M$ at the M-cardinals in the interval $(F(\kappa), j(\kappa))$ (and at $F(\kappa)$ if $F(\kappa)$ is singular in V but regular in M) will be shown to be sufficiently well-behaved so that the corresponding generics can be constructed in $V[G]$. The next step is the forcing $\mathbb{P}^M$ at $j(\kappa)$ where the task is twofold: not only do we need to find a generic, but we need to find one which contains the pointwise image under j of g_κ. Precisely to resolve this difficult point, we have included the Sacks forcing $\mathrm{Sacks}(\kappa, F(\kappa))$ at stage κ because by [4] the point-wise image of the generic g_κ (or rather of its Sacks part) will (almost) generate the correct generic for $j(\kappa)$. Finally, we lift to all of $V[G]$ using Lemma 2.14.

We will first lift the embedding j to $V[G_\kappa]$. As $H(\kappa)^V = H(\kappa)^M$, $\mathbb{P}_\kappa = \mathbb{P}^M_\kappa$ and it follows we can copy the generic G_κ.

Note: In order to keep track of where we are, we will use the following dotted arrow convention to indicate that we are in the process of lifting the embedding j to $V[G_\kappa]$, but we have not yet completed the lifting. Once we lift the embedding, the arrow will be printed in solid line.

$$V[G_\kappa] \overset{j}{\dashrightarrow} M[G_\kappa]$$

Recall by the definition of $\mathbb{P}^F$ that the next step of iteration $\mathbb{Q}_\kappa$ in $V[G_\kappa]$ is the product $\mathrm{Sacks}(\kappa, F(\kappa)) \times \prod_{\kappa<\lambda<i_{\kappa+1}} \mathrm{Add}(\lambda, F(\lambda))$, where λ ranges over regular cardinals in V and the product has the Easton support; the corresponding forcing in $M[G_\kappa]$, to be denoted $\mathbb{Q}^M_\kappa$, is $\mathrm{Sacks}(\kappa, j(F)(\kappa)) \times \prod_{\kappa<\lambda<i^M_{\kappa+1}} \mathrm{Add}(\lambda, j(F)(\lambda))$, where λ ranges over regular cardinals in M.

Note. For typographical reasons, we employ the following notation for $\mathbb{Q}_\kappa$ and $\mathbb{Q}^M_\kappa$.

– We write $i(\kappa+1)$ for $i_{\kappa+1}$ and $i^M(\kappa+1)$ for $i^M_{\kappa+1}$;

– If λ is a regular cardinal in V in the interval $[\kappa, i(\kappa+1))$, then $\mathcal{Q}_\lambda$ stands for the forcing Sacks$(\kappa, F(\kappa))$ if $\lambda = \kappa$, and for the forcing Add$(\lambda, F(\lambda))$ if $\lambda \neq \kappa$;
– If $\mu < \mu'$ are cardinals in V (μ, μ' may be singular) in the interval $[\kappa, i(\kappa + 1))$ then we write $\prod_{[\mu,\mu')} \mathcal{Q}_\lambda$ for the product $\mathbb{Q}_\kappa$ restricted to the interval $[\mu, \mu')$ (and similarly for other intervals (μ, μ') etc.). Thus for instance $\prod_{[\kappa,i(\kappa+1))} \mathcal{Q}_\lambda = \mathbb{Q}_\kappa$.
– Analogously, if $\bar{\lambda}$ is a regular cardinal in M in the interval $[\kappa, i^M(\kappa + 1))$, then $\mathcal{Q}^M_{\bar{\lambda}}$ stands for the forcing Sacks$(\kappa, j(F)(\kappa))$ in $M[G_\kappa]$ if $\bar{\lambda} = \kappa$, and for the forcing Add$(\bar{\lambda}, j(F)(\bar{\lambda}))$ in $M[G_\kappa]$ if $\bar{\lambda} \neq \kappa$;
– If $\mu < \mu'$ are cardinals in M (μ, μ' may be singular in M) in the interval $[\kappa, i^M(\kappa+1))$ then we write $\prod^M_{[\mu,\mu')} \mathcal{Q}^M_{\bar{\lambda}}$ for the $M[G_\kappa]$-product $\mathbb{Q}^M_\kappa$ restricted to the interval $[\mu, \mu')$ (and similarly for other intervals (μ, μ') etc.);
– Generic filters for these forcings (once they are found in the case of the forcing in $M[G_\kappa]$) shall be denoted in the same fashion using the notation $g_{[\mu,\mu')}$ and $g^M_{[\mu,\mu')}$, respectively.

Now we return to the proof. We will proceed to show that $g_{[\kappa,F(\kappa)]}$ can be used to find in $V[G_\kappa]$ an $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$.

We will first correct the possible discrepancy between the values of $F(\lambda)$ and $j(F)(\lambda)$ for V-regular λ in the interval $[\kappa, F(\kappa)]$ (recall that $\lambda \le F(\kappa)$ is a cardinal in V if and only if it is a cardinal in M, but $F(\kappa)$ may be regular M but singular in V, so we need to remember in which universe we are: V or M). By elementarity of j, $j(\kappa)$ is closed under $j(F)$ and by the F-goodness of κ, $F(\kappa) \le j(F)(\kappa) < j(\kappa)$. Let λ_0 be the least regular cardinal greater than κ such that $F(\kappa) < F(\lambda_0)$ ($\lambda_0 \le \mathrm{cf}((F(\kappa))$ as $F(\mathrm{cf}(F(\kappa)))$ has cofinality greater than $\mathrm{cf}(F(\kappa))$ and therefore cannot equal $F(\kappa)$). For a regular $\lambda \in [\kappa, \lambda_0)$, $F(\lambda) = F(\kappa) \le j(F)(\kappa) \le j(F)(\lambda)$. Also $\lambda < \lambda_0 \le \mathrm{cf}(F(\kappa)) \le F(\kappa) < j(\kappa)$ and $j(\kappa)$ is closed under $j(F)$; it follows that $j(F)(\lambda) < j(\kappa)$ and hence $F(\lambda)$, $j(F)(\lambda)$ both have V-cardinality $F(\kappa)$. Any bijection between $F(\lambda)$ and $j(F)(\lambda)$ for a given λ generates an isomorphism between the forcings Sacks$(\kappa, F(\kappa))$ and Sacks$(\kappa, j(F)(\kappa))$ if $\lambda = \kappa$ and between Add$(\lambda, F(\lambda))$ and Add$(\lambda, j(F)(\lambda))$ otherwise. Denote these isomorphic forcings as $\mathcal{Q}^*_\lambda$, i.e. $\mathcal{Q}_\lambda \cong \mathcal{Q}^*_\lambda$. If a V-regular λ lies in the interval $[\lambda_0, F(\kappa)]$, then $j(F)(\lambda) < j(\kappa) < F(\kappa)^+ \le F(\lambda)$ and so $j(F)(\lambda) < F(\lambda)$. It follows we can truncate the product $\mathcal{Q}_\lambda$ at the ordinal $j(F)(\lambda)$; let $\mathcal{Q}^{**}_\lambda$ denote this truncation. It is immediate that

$$\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda =_{df} \prod_{[\kappa,\lambda_0)} \mathcal{Q}^*_\lambda \times \prod_{[\lambda_0,F(\kappa)]} \mathcal{Q}^{**}_\lambda \tag{5}$$

is completely embeddable into $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}_\lambda$ and so there is a generic filter, to be denoted $g^+_{[\kappa,F(\kappa)]}$, existing in $V[G]$, which is $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$-generic over $V[G_\kappa]$. The generic $g^+_{[\kappa,F(\kappa)]}$ will be used to find a generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_{\bar{\lambda}}$.

The manipulation to obtain $\prod_{[\kappa,F(\kappa)]} \mathcal{Q}^+_\lambda$ ensures agreement for $\lambda \le F(\kappa)$ between the lengths of the products $\mathcal{Q}_\lambda$ and $\mathcal{Q}_\lambda^M$ in $V[G_\kappa]$ and $M[G_\kappa]$, respectively, but a word of caution is in order. For instance if $F(\kappa) > \kappa^+$ is regular, $\prod_{[\kappa,F(\kappa)]}\mathcal{Q}_\lambda^+$ is never identical with $\prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M$: Already for $F(\kappa) = \kappa^{++}$ the forcing $\mathrm{Add}(\kappa^{++}, j(F)(\kappa^{++}))$ in $M[G_\kappa]$ fails to capture all conditions in $\mathrm{Add}(\kappa^{++}, j(F)(\kappa^{++}))$ in $V[G_\kappa]$ as the supports in this forcing are κ^+-sequences extending above κ^{++}, and some such sequences are missing in $M[G_\kappa]$ (for instance if $F(\kappa) = \kappa^{++}$, $(\kappa^{+3})^M$ has cofinality κ^+ in V). Accordingly, we only have (when $F(\kappa)$ is regular in V greater than κ^+) $\prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M \subseteq \prod_{[\kappa,F(\kappa)]}\mathcal{Q}_\lambda^+$.

We will deal separately with the two cases: $F(\kappa)$ regular in V, and $F(\kappa)$ singular in V.

Lemma 3.9 *Assume $F(\kappa)$ is* regular *in V. There is in $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$ an $M[G_\kappa]$-generic for $\prod_{[\kappa,i^M(\kappa+1))}^M\mathcal{Q}_\lambda^M$, which we will denote as $g_{[\kappa,i^M(\kappa+1))}^M$.*

Proof. As $F(\kappa)$ is regular in V, it is also regular in M. Consequently, the forcing $\prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M$ is $F(\kappa)^+$-cc in $M[G_\kappa]$ and as $\prod_{(F(\kappa),i^M(\kappa+1))}^M\mathcal{Q}_\lambda^M$ is $F(\kappa)^+$-closed, the forcings are mutually generic in the sense of Lemma 2.1. It follows that we can deal with $\prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M$ and $\prod_{(F(\kappa),i^M(\kappa+1))}^M\mathcal{Q}_\lambda^M$ separately.

A) The product $\prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M$.

We will use $g_{[\kappa,F(\kappa)]}^+$ to obtain the required generic; in fact we will show that the intersection $g_{[\kappa,F(\kappa)]}^M = g_{[\kappa,F(\kappa)]}^+ \cap \prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M$ is $M[G_\kappa]$-generic for $\prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M$.

We will argue that a maximal antichain $A \in M[G_\kappa]$ in $\prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M$ will stay maximal in $\prod_{[\kappa,F(\kappa)]}\mathcal{Q}_\lambda^+$, and so will be hit by $g_{[\kappa,F(\kappa)]}^+$.

For $p \in \prod_{[\kappa,F(\kappa)]}\mathcal{Q}_\lambda^+$ write

$$\mathrm{supp}(p) = \{\langle\lambda,\alpha\rangle \mid p(\lambda)(\alpha) \neq 1\}, \tag{6}$$

where 1 stands for the empty condition in the relevant forcing, and analogously for $A \subseteq \prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M$,

$$\mathrm{supp}(A) = \{\langle\lambda,\alpha\rangle \mid \exists p \in A, \langle\lambda,\alpha\rangle \in \mathrm{supp}(p)\}. \tag{7}$$

We will show that if $A \in M[G_\kappa]$ is a maximal antichain in $\prod_{[\kappa,F(\kappa)]}^M\mathcal{Q}_\lambda^M$ and $p \in \prod_{[\kappa,F(\kappa)]}\mathcal{Q}_\lambda^+$ is arbitrary, then

$$X = \mathrm{supp}(p) \cap \mathrm{supp}(A) \in M[G_\kappa] \text{ and } p \restriction X \in M[G_\kappa]. \tag{8}$$

Providing we know (8), $p \restriction X$ must be compatible with some $a \in A$, and because p and a are compatible on the supports, they must be compatible everywhere. It follows that A stays maximal in $V[G_\kappa]$. To argue for (8), the

$F(\kappa)^+$-cc of $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ in $M[G_\kappa]$ implies that the size of supp(A) in $M[G_\kappa]$ is at most $F(\kappa)$. Since the size of supp(p) is strictly less than $F(\kappa)$, (8) will follow from the following property (9).

$$\text{If a set } x \in M[G_\kappa] \text{ has size at most } F(\kappa) \text{ in } M[G_\kappa], \\ \text{then } {}^{<F(\kappa)}x \cap V[G_\kappa] \subseteq M[G_\kappa]. \tag{9}$$

Let $f : x \to F(\kappa)$ be a 1-1 function, $f \in M[G_\kappa]$, and let $\vec{s} \in {}^{<F(\kappa)}x \cap V[G_\kappa]$ be given. Working in $V[G_\kappa]$, it is obvious that $f[\vec{s}] \in H(F(\kappa))$. Since $H(F(\kappa))$ is the same in $V[G_\kappa]$ and $M[G_\kappa]$, $f[\vec{s}] \in M[G_\kappa]$. But as f is in $M[G_\kappa]$, so is $f^{-1}[f[\vec{s}]] = \vec{s}$.

B) The product $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_\lambda$.

Notice that every dense open set of $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_\lambda$ in $M[G_\kappa]$ is of the form $(j(f)(a))^{G_\kappa}$, $a \in [F(\kappa)]^{<\omega}$, where $j(f)(a)$ is a $\mathbb{P}^M_\kappa$-name, for some $f : [\kappa]^{<\omega} \to H(\kappa^+)$. Without loss of generality, we may assume that the range of all such f contains just names for dense open sets.[6] For each such f, the set $\{\langle j(f)(a), 1\rangle \,|\, a \in [F(\kappa)]^{<\omega}\}$ is a $\mathbb{P}^M_\kappa$-name in M, which interprets as a family $\{(j(f)(a))^{G_\kappa} \,|\, a \in [F(\kappa)]^{<\omega}\}$ of at most $F(\kappa)$ many dense open sets in $M[G_\kappa]$ – it follows the intersection $\mathscr{D}_f = \bigcap_{a\in[F(\kappa)]^{<\omega}} (j(f)(a))^{G_\kappa}$ is dense in $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_\lambda$ since the forcing notion $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_\lambda$ is $F(\kappa)^+$-distributive in $M[G_\kappa]$. As there are only $(\kappa^+)^\kappa = \kappa^+$ such f's, and $M[G_\kappa]$ is closed under κ-sequences in $V[G_\kappa]$, we can construct a generic in $V[G_\kappa]$ meeting all the dense sets $\mathscr{D}_f$ for all suitable f. Let us denote this generic as $g^M_{(F(\kappa),i^M(\kappa+1)}$.

We finish the proof by setting $g^M_{[\kappa,i^M(\kappa+1))} = g^M_{[\kappa,F(\kappa)]} \times g^M_{(F(\kappa),i^M(\kappa+1))}$. (Lemma 3.9) □

Lemma 3.10 *Assume $F(\kappa)$ is* singular *in V with cofinality $\delta < F(\kappa)$ (recall that $\kappa^+ \le \delta$ by the definition of Easton function). There is in $V[G_\kappa * g_{[\kappa,F(\kappa))}]$ an $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_\lambda$, which we will denote as $g^M_{[\kappa,i^M(\kappa+1))}$.*

The singularity of $F(\kappa)$ implies that $M[G_\kappa]$ may not be closed in $V[G_\kappa]$ under $<F(\kappa)$-sequences of elements of $F(\kappa)$, but just under $<\delta$-sequences. It follows that the argument given in Lemma 3.9, in particular (9), cannot be used as it stands. However, we will argue that the desired generic can be constructed via "approximations" by induction along some sequence of regular cardinals cofinal in $F(\kappa)$.

---

[6] Formally, $f(s)$ will be a name for a dense open set in the forcing $\mathbb{P}_\kappa$, and so $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ will be a name for a dense open set in $\mathbb{P}^M_{j(\kappa)}$. We will abuse notation and identify every $j(f)(a)$ with a $\mathbb{P}_\kappa$-name for a dense open set in $\prod^M_{(F(\kappa),i^M(\kappa+1))} \mathcal{Q}^M_\lambda$.

In preparation for the argument, we will define a certain procedure which will be used in the argument. Let $\langle \gamma_i \,|\, i < \delta \rangle$ be a sequence of regular cardinals cofinal in $F(\kappa)$, with $\delta < \gamma_0$ (we may assume that this sequence belongs to M if $F(\kappa)$ is singular in M, as in that case, $F(\kappa)$ has the same cofinality in M as it has in V). Generalizing our notation, if $\gamma_{i+1} < \mu$, where μ is an M-cardinal (μ will in fact be always either $F(\kappa)$ or $i^M(\kappa+1)$), and $p \in \prod^M_{[\kappa,\mu)} \mathcal{Q}^M_\lambda$ is a condition, let p_{γ_i} denote p restricted to $\prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_\lambda$ (the "lower part of p") and p^{γ_i} denote p restricted to $\prod^M_{(\gamma_i,\mu)} \mathcal{Q}^M_\lambda$ (the "upper part of p") (the parameter μ will be understood from the context). Note that for each γ_i, $\prod^M_{(\gamma_i,\mu)} \mathcal{Q}^M_\lambda$ is γ_i^+-closed and $\prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_\lambda$ is γ_i^+-cc in $M[G_\kappa]$.

Let γ_i, $f : [\kappa]^{<\omega} \to H(\kappa^+)$, and $a \in [\gamma_i]^{<\omega}$ be arbitrary and assume that $j(f)(a)$ is a $\mathbb{P}^M_\kappa$-name for a dense open set in $\prod^M_{[\kappa,\mu)} \mathcal{Q}^M_\lambda$, where μ is either $F(\kappa)+1$ or $i^M_{\kappa+1}$. Let us denote $(j(f)(a))^{G_\kappa}$ as D. Assume further that p is a condition in $\prod^M_{[\kappa,\mu)} \mathcal{Q}^M_\lambda$.

Definition 3.11 $\overline{q} \in \prod^M_{(\gamma_i,\mu)} \mathcal{Q}^M_\lambda$ *is said to* γ_i-reduce D below p *if the following holds:*

(1) $\overline{q}$ *extends the upper part of* p*, i.e.* $\overline{q} \leq p^{\gamma_i}$ *in* $\prod^M_{(\gamma_i,\mu)} \mathcal{Q}^M_\lambda$*;*
(2) The set $\overline{D} = \{q \leq p_{\gamma_i} \in \prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_\lambda \,|\, q \cup \overline{q} \in D\}$ *is dense open in* $\prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_\lambda$ *below the lower part of* p*.*

We will show how to construct such a reduction $\overline{q}$ (the argument is essentially the one used to prove the Easton lemma 2.1 (1)). Choose some (r_0, s_0) such that $r_0 \cup s_0 \in D$ and $r_0 \leq p_{\gamma_i}$ and $s_0 \leq p^{\gamma_i}$. At stage ξ of the construction, let r'_ξ be any condition which is incompatible with the set of all previous conditions $\{r_\zeta \,|\, \zeta < \xi\}$ (if there is such) and let s'_ξ be a lower bound of $\langle s_\zeta \,|\, \zeta < \xi \rangle$. Choose $r_\xi \leq r'_\xi$ and $s_\xi \leq s'_\xi$ such that $r_\xi \cup s_\xi \in D$. The construction is well-defined since $\prod^M_{[\kappa,\gamma_i]} \mathcal{Q}^M_\lambda$ is γ_i^+-cc and consequently the process will stop at some $\rho < \gamma_i^+$. Set $\overline{q}$ to be the lower bound of all s_ζ for $\zeta < \rho$. We will show that $\overline{q}$ indeed γ_i-reduces D below p according to Definition 3.11. We only need to check the condition (2) as (1) is obvious. Let $q \leq p_{\gamma_i}$ be given. It follows from the construction that there is some r_ζ such that q and r_ζ are compatible with some lower bound $\tilde{r}$. Also, $r_\zeta \cup s_\zeta \in D$ and consequently $\tilde{r} \cup \overline{q} \in D$ by openness. Note that as $p \in M[G_\kappa]$ by assumption, the construction can be carried out in $M[G_\kappa]$ and consequently $\overline{q}$ will also be in $M[G_\kappa]$.

We will need to distinguish several cases which will be handled in a sequence of Sublemmas. Notice that by Observation 2.8, we cannot disregard the possibility that $F(\kappa)$ is singular in V while it is regular in M.

Sublemma 3.12 *If the cofinality of* $F(\kappa)$ *in* V *is* κ^+ *and* $F(\kappa)$ *is singular in* V *(*$F(\kappa)$ *can be either regular or singular in* M*) then there is in* $V[G_\kappa * g_{[\kappa,F(\kappa))}]$

an $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))}\mathcal{Q}^M_\lambda$.

Proof. Fix the two following sequences:

(1) Sequence $\langle\gamma_i\,|\,i<\kappa^+\rangle$ of regular cardinals cofinal in $F(\kappa)$, where $\kappa^+<\gamma_0$;
(2) Sequence $\langle j(f_\alpha)\,|\,\alpha<\kappa^+\rangle$, where $\langle f_\alpha\,|\,\alpha<\kappa^+\rangle$ enumerates all $f:[\kappa]^{<\omega}\to H(\kappa^+)$ such that $f(s)$ is a name for a dense open set in $\mathbb{P}_\kappa$ for every $s\in[\kappa]^{<\omega}$; $j(f)(a)$ for $a\in[F(\kappa)]^{<\omega}$ will thus range over names for dense open sets in $\mathbb{P}^M_{j(\kappa)}$ but we will abuse notation and identify every $j(f)(a)$ for $a\in[F(\kappa)]^{<\omega}$ with a name restricted to $\prod^M_{[\kappa,i^M(\kappa+1))}\mathcal{Q}^M_\lambda$ in $M[G_\kappa]$.

By induction on $i<\kappa^+$, we will construct conditions $p_i\in M[G_\kappa]$ the tails of which will reduce all dense open sets in $\prod^M_{[\kappa,i^M(\kappa+1))}\mathcal{Q}^M_\lambda$ according to Definition 3.11. We will also consider their limit – a "master condition" – p_∞ (possibly outside $M[G_\kappa]$) .

Fix in advance some wellordering $<_0\in M[G_\kappa]$ of the pairs in $\kappa^+\times[F(\kappa)]^{<\omega}$ such that the restriction of $<_0$ to $k\times[\gamma_k]^{<\omega}$ for each $k<\kappa^+$ has order type γ_k. Assume that p_i have been constructed for all $i<k$ and we need to construct p_k. First let r_k be a lower bound of p_i for $i<k$ and work below this condition. Carry out the following construction in $M[G_\kappa]$. By induction on $<_0$ restricted to $k\times[\gamma_k]^{<\omega}$ construct a decreasing chain of conditions $\overline{q}_{(\xi,a)}$ in $\prod^M_{(\gamma_k,i^M(\kappa+1))}\mathcal{Q}^M_\lambda$ as follows. At stage (ξ,a), let first $r_{(\xi,a)}$ be the lower bound of $\overline{q}_{(\xi',a')}$ for $(\xi',a')<_0(\xi,a)$. Using the argument below Definition 3.11, set $\overline{q}_{(\xi,a)}$ to be a condition which γ_k-reduces the dense open set with the name $j(f_\xi)(a)$ below $(r_k)_{\gamma_k}\cup r_{(\xi,a)}$. Since the induction has length γ_k and we consider only the initial segment of order type k of the functions in the sequence $\langle j(f_\xi)\,|\,\xi<\kappa^+\rangle$ (which exists in $M[G_\kappa]$), the lower bound of all $\overline{q}_{(\xi,a)}$ exists in $M[G_\kappa]$. Denoting this lower bound $\overline{q}$, we set p_k to be equal to the union of the lower part of r_k and $\overline{q}$, i.e. $p_k=(r_k)_{\gamma_k}\cup\overline{q}$.

Set p_∞ to be a lower bound of $\langle p_i\,|\,i<\kappa^+\rangle$ (p_∞ may exist only in $V[G_\kappa]$). Let us write $p^{\leftarrow}_\infty$ for p_∞ restricted to the interval $[\kappa,F(\kappa))$ and $p^{\rightarrow}_\infty$ for the rest of p_∞ defined at the interval $[F(\kappa),i^M_{\kappa+1})$. Note that $p^{\leftarrow}_\infty$ is an element of the forcing $\prod_{[\kappa,F(\kappa))}\mathcal{Q}^+_\lambda$, while $p^{\rightarrow}_\infty$ is not an element of any of the forcings introduced so far (it is just a union of certain conditions which exists in $V[G_\kappa]$).

Define the desired generic $g^M_{[\kappa,i^M(\kappa+1))}$ as follows. Assume now that h is a $\prod_{[\kappa,F(\kappa))}\mathcal{Q}^+_\lambda$-generic filter over $V[G_\kappa]$ containing the condition $p^{\leftarrow}_\infty$, and set $h'=\{p^{\rightarrow}_\infty\}\cup\{q\in\prod^M_{[F(\kappa),i^M(\kappa+1))}\mathcal{Q}^M_\lambda\,|\,p^{\rightarrow}_\infty\le q\}$. We claim that $g^M_{[\kappa,i^M(\kappa+1))}=(h\times h')\cap M[G_\kappa]$ is $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))}\mathcal{Q}^M_\lambda$.

Let $D=(j(f)(a))^{G_\kappa}$ dense open be given, where $a\in[\gamma_{k'}]^{<\omega}$ for some $k'<\kappa^+$. We will show that $g^M_{[\kappa,i^M(\kappa+1))}$ meets D. Assume that the set D was dealt with at substage (ξ,a) of the inductive construction of p_∞ at stage $k\ge k'$, where $j(f)$ is considered. Under this notation, recall that the set $\overline{D} = \{q \leq (r_k)_{\gamma_k} \mid q \cup \overline{q}_{(\xi,a)} \in D\}$ is dense in $M[G_\kappa]$ below $(r_k)_{\gamma_k}$ in $\prod^M_{[\kappa,\gamma_k]} \mathcal{Q}^M_\lambda$. If A is a maximal antichain contained in $\overline{D}$, then

$$A \text{ remains maximal in } \prod_{[\kappa,\gamma_k]} \mathcal{Q}^+_\lambda \text{ in } V[G_\kappa] \tag{10}$$

To see that (10) is true, we argue as in Lemma 3.9 (8): Since A is a maximal antichain contained in a dense set, it is a maximal antichain in the whole forcing $\prod^M_{[\kappa,\gamma_k]} \mathcal{Q}^M_\lambda$. As $\prod^M_{[\kappa,\gamma_k]} \mathcal{Q}^M_\lambda$ is γ_k^+-cc in $M[G_\kappa]$ and a support of a condition p in $\prod_{[\kappa,\gamma_k]} \mathcal{Q}^+_\lambda$ has size $< \gamma_k$, the closure property of $M[G_\kappa]$

$$\begin{aligned}&\text{If a set } x \in M[G_\kappa] \text{ has size at most } \gamma_k \text{ in } M[G_\kappa],\\ &\quad \text{then } {}^{<\gamma_k}x \cap V[G_\kappa] \subseteq M[G_\kappa]\end{aligned} \tag{11}$$

ensures that (10) is true. It follows that h restricted to $\prod_{[\kappa,\gamma_k]} \mathcal{Q}^+_\lambda$ must hit A; let a be an element of h such that $a_{\gamma_k} \in \overline{D}$. It follows that $a_{\gamma_k} \cup \overline{q}_{(\xi,a)}$ meets D. As both a and $p^{\leftarrow}_\infty$ are in h, there is some $a' \in h$ below both of them. But then $a' \cup p^{\rightarrow}_\infty \in h \times h'$ and $a' \cup p^{\rightarrow}_\infty \leq a_{\gamma_k} \cup \overline{q}_{(\xi,a)}$, and so $a_{\gamma_k} \cup \overline{q}_{(\xi,a)}$ is in $g^M_{[\kappa,i^M(\kappa+1))}$ and meets D.

We finish the proof by arguing that $g^+_{[\kappa,F(\kappa))}$ can be used to find in $V[G]$ some such generic h containing $p^{\leftarrow}_\infty$. By the homogeneity [7] of the forcing $\prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$ there is $r \in g^+_{[\kappa,F(\kappa))}$ and an automorphism $\pi : \prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda \cong \prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$ such that $\pi(r) = p^{\leftarrow}_\infty$; it follows that $h = \pi[g^+_{[\kappa,F(\kappa))}]$ is $\prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$-generic containing $p^{\leftarrow}_\infty$ as desired. (Sublemma 3.12) □

Sublemma 3.13 *If the cofinality of $F(\kappa)$, which we denote δ, is greater than κ^+ in V, and $F(\kappa)$ is singular in V, then there is in $V[G_\kappa * g_{[\kappa,F(\kappa))}]$ a $M[G_\kappa]$-generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_\lambda$.*

Proof. We will need to distinguish two cases.

Case (1)*: $F(\kappa)$ is regular in M.*

Recall the sequences $\langle \gamma_i \mid i < \delta\rangle$, where $\kappa^+ < \gamma_0$, and $\langle j(f_\alpha) \mid \alpha < \kappa^+\rangle$ which we used in the inductive construction in Sublemma 3.12. Unlike in Sublemma 3.12, we do not make the assumption that $\delta = \kappa^+$. Thus the two inductions cannot be merged together as in Sublemma 3.12 and a more complicated argument is called for. We will construct the desired generic for $\prod^M_{[\kappa,i^M(\kappa+1))} \mathcal{Q}^M_\lambda$ in two steps.

[7] In fact, we need homogeneity only for the Cohen forcing part of the forcing $\prod_{[\kappa,F(\kappa))} \mathcal{Q}^+_\lambda$ above γ_0 as the master condition $p^{\leftarrow}_\infty$ is trivial below γ_0. This implies that we can disregard the Sacks forcing here.

A) The forcing $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$.

Intuitively, we need to define a generic for $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ by building a decreasing list of conditions using induction along $\langle \gamma_i \,|\, i < \delta \rangle$ and simultaneously along $\langle j(f_\alpha) \,|\, \alpha < \kappa^+ \rangle$.[8] As both inductions can lead the construction outside the model $M[G_\kappa]$, we need to find a way to compatibly extend conditions "locally" without leaving the class $M[G_\kappa]$. We shall do this by dividing the supports of the conditions into segments corresponding to some elementary substructures existing in $M[G_\kappa]$.

Let m_α for $\alpha < \kappa^+$ denote the following elementary substructure of some large enough $H(\theta)^{M[G_\kappa]}$ which is closed under $<F(\kappa)$-sequences existing in $M[G_\kappa]$:

$$m_\alpha = \mathrm{SkolemHull}^{H(\theta)^{M[G_\kappa]}} \Big(\{ \prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda \} \cup F(\kappa) + 1 \cup \{ j(f_\xi) \,|\, \xi \le \alpha \} \Big). \quad (12)$$

Notice that each m_α has size $F(\kappa)$ in $M[G_\kappa]$ and contains as elements all dense open sets of the form $(j(f_\xi)(a))^{G_\kappa}$ for $a \in [F(\kappa)]^{<\omega}$ and $\xi \le \alpha$.

We will build a matrix of conditions $\{p_{i,\alpha} \,|\, i < \delta, \alpha < \kappa^+\}$ in $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ with δ-many rows each of length κ^+ such that the conditions will be decreasing both in the rows and the columns. Moreover, for every $i < \delta$ and every $\alpha < \kappa^+$, the sequence of conditions in the α-th column up to i, i.e. $\langle p_{k,\alpha} \,|\, k < i \rangle$, will exist in m_α. We will construct the matrix in δ-many steps, each of length κ^+ (i.e. we will be completing rows first).

The first "square" of the matrix $p_{0,0}$ will be filled in as follows. By definition of m_0, all dense open sets in $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$ of the form $(j(f_0)(a))^{G_\kappa}$ for $a \in [\gamma_0]^{<\omega}$ are in m_0; by elementarity, they are dense open in m_0. Working inside m_0, carry out the reduction argument described in Sublemma 3.12. In particular, $p_{0,0}$ will γ_0-reduce all dense open sets $(j(f_0)(a))^{G_\kappa}$ for $a \in [\gamma_0]^{<\omega}$ (below the trivial condition 1 as we are filling in the first square). The square $p_{0,1}$ will be filled in in exactly the same way (considering f_0 and f_1), but working below the condition $p_{0,0}$ which is present in m_1. In particular $p_{0,1}$ will γ_0-reduce below $p_{0,0}$ all dense open sets of the form $(j(f_1)(a))^{G_\kappa}$ for $a \in [\gamma_0]^{<\omega}$. Proceed this way at every successor ordinal, obtaining $p_{0,\alpha+1}$. At a limit ordinal $\lambda < \kappa^+$, first take a lower bound q of $\langle p_{0,\alpha} \,|\, \alpha < \lambda \rangle$ which by the closure properties of m_λ exists in m_λ, and then work below this lower bound; the resulting $p_{0,\lambda}$ will γ_0-reduce below q all dense open sets of the form $(j(f_\xi)(a))^{G_\kappa}$ for $a \in [\gamma_0]^{<\omega}$ and $\xi \le \lambda$. After κ^+ steps we have completed the 0-th row of the matrix. Note that the limit of $\langle p_{0,\alpha} \,|\, \alpha < \kappa^+ \rangle$ may not exist in $M[G_\kappa]$.

We now need to complete row 1. In order to complete the first square in row

[8] This time, $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ will be identified with $\mathbb{P}_\kappa$-names for dense open sets in $\prod^M_{[\kappa,F(\kappa)]} \mathcal{Q}^M_\lambda$.

1, we need to find $p_{1,0}$ compatible with all conditions in the 0-th row of the matrix. Though the lower bound of these conditions may not exist in $M[G_\kappa]$, we will argue that an intersection of the union of the conditions (or more precisely of their supports) in the 0-th row with m_0 is in $M[G_\kappa]$, and even in m_0, i.e.

$$m_0 \cap \bigcup_{\alpha<\kappa^+} \mathrm{supp}(p_{0,\alpha}) \in m_0 \tag{13}$$

To see that (13) is true, we argue similarly as in Lemma 3.9. Each $p_{0,\alpha}$ is obviously in $M[G_\kappa]$, and consequently $\mathrm{supp}(p_{0,\alpha}) \cap m_0$ is in $M[G_\kappa]$ and in particular in m_0. The intersection (13) can thus be viewed as the union of a κ^+-sequence of elements in m_0. But as m_0 has size $F(\kappa)$ in $M[G_\kappa]$, such a sequence exists in $M[G_\kappa]$ due to the following closure property

$$^{\kappa^+}F(\kappa) \in M[G_\kappa], \tag{14}$$

which is implied by the $F(\kappa)$-hypermeasurability of κ and the fact that κ^+ is smaller than the cofinality of $F(\kappa)$.

It follows there is $p_{1,0}$ which γ_1-reduces all dense open sets $(j(f_0)(a))^{G_\kappa}$ for $a \in [\gamma_1]^{<\omega}$ below the condition $\bigcup_{\alpha<\kappa^+} p_{0,\alpha}$ restricted to m_0. In general for $\alpha < \kappa^+$, the condition $p_{1,\alpha}$ will reduce the relevant dense open sets below the common lower bound of $\bigcup_{\alpha<\kappa^+} p_{0,\alpha}$ restricted to m_α and the union of previous $p_{1,\beta}$ for $\beta < \alpha$.

It is immediate that the above construction can be repeated for any successor ordinal $i+1$ below δ, i.e. if the matrix has been completed up to the stage i, we can fill in the $i+1$-th row by the above argument.

Assume now that $i < \delta$ is a limit ordinal. First consider the sequence $\langle p_{k,\alpha} \,|\, k < i\rangle$ for a single $\alpha < \kappa^+$. As the sequence is of length less then cofinality $F(\kappa)$ in $M[G_\kappa]$ and contains elements from m_α, which has size $F(\kappa)$ in $M[G_\kappa]$, we can infer from

$$^{i}F(\kappa) \in M[G_\kappa] \tag{15}$$

that the sequence exists in $M[G_\kappa]$, and in particular in m_α. Let $q_{i,\alpha} \in m_\alpha$ denote the lower bound of the sequence $\langle p_{k,\alpha} \,|\, k < i\rangle$ for each $\alpha < \kappa^+$. Now repeat the above argument for the successor step considering the restrictions of $\bigcup_{\alpha<\kappa^+} q_{i,\alpha}$ to m_β's for $\beta < \kappa^+$.

We finish the construction by taking the limit of the whole matrix $\{p_{i,\alpha} \,|\, i < \delta, \alpha < \kappa^+\}$, obtaining some set p_∞ existing in $V[G_\kappa]$ (for instance first taking limits of the rows and then the single limit of this sequence). Let $p_\infty^{\leftarrow}$ denote the restriction of p_∞ to the interval $[\kappa, F(\kappa))$ (note that $p_\infty^{\leftarrow}$ is a condition in $\prod_{[\kappa,F(\kappa))} \mathcal{Q}_\lambda^+$) and $p_\infty^{\rightarrow}$ the restriction of p_∞ to $\{F(\kappa)\}$ (note that $p_\infty^{\rightarrow}$ is a union of conditions in $(\mathrm{Add}(F(\kappa), j(F)(F(\kappa))))^{M[G_\kappa]}$ which exists in $V[G_\kappa]$). Arguing as at the end of Sublemma 3.12, we find a $\prod_{[\kappa,F(\kappa))} \mathcal{Q}_\lambda^+$-generic h, where $p_\infty^{\leftarrow}$ is in h, and define h' to be generated by $p_\infty^{\rightarrow}$, such that $h \times h' \cap M[G_\kappa]$ is

$\prod^{M}_{[\kappa,F(\kappa)]}\mathcal{Q}^{M}_{\lambda}$-generic over $M[G_\kappa]$. Let us denote this generic as $g^{M}_{[\kappa,F(\kappa)]}$.

B) The forcing $\prod^{M}_{(F(\kappa),i^M(\kappa+1))}\mathcal{Q}^{M}_{\lambda}$ *in* $M[G_\kappa]$.

The regularity of $F(\kappa)$ in M implies that $\prod^{M}_{[\kappa,F(\kappa)]}\mathcal{Q}^{M}_{\lambda}$ and $\prod^{M}_{(F(\kappa),i^M(\kappa+1))}\mathcal{Q}^{M}_{\lambda}$ are mutually generic. Working in $V[G_\kappa]$, we construct the generic $g^{M}_{(F(\kappa),i^M(\kappa+1))}$ for $\prod^{M}_{(F(\kappa),i^M(\kappa+1))}\mathcal{Q}^{M}_{\lambda}$ exactly as in case B) of Lemma 3.9.

It follows that $g^{M}_{[\kappa,i^M(\kappa+1))} = g^{M}_{[\kappa,F(\kappa)]} \times g^{M}_{(F(\kappa),i^M(\kappa+1))}$ is the desired $M[G_\kappa]$-generic for $\prod^{M}_{[\kappa,i^M(\kappa+1))}\mathcal{Q}^{M}_{\lambda}$.

Case (2): *$F(\kappa)$ is singular in M.*

Recall once again the sequences $\langle \gamma_i \,|\, i < \delta\rangle$, where $\kappa^+ < \gamma_0$, and $\langle j(f_\alpha) \,|\, \alpha < \kappa^+\rangle$ which we used in the inductive construction in Sublemma 3.12 and Case (1) of the present Sublemma.

The singularity of $F(\kappa)$ in $M[G_\kappa]$ introduces an important simplification into the construction: $\langle \gamma_i \,|\, i < \delta\rangle$ can be picked in $M[G_\kappa]$ this time. Just run the argument in Sublemma 3.12 with the following modification: Start with $j(f_0)$ and run the argument using just this one function $j(f_0)$, obtaining some master condition $p^{f_0}_\infty$. Since the sequence $\langle \gamma_i \,|\, i < \delta\rangle$ is in $M[G_\kappa]$, so is $p^{f_0}_\infty$. Now deal with $j(f_1)$ and so on by induction on $\alpha < \kappa^+$. At each $\alpha < \kappa^+$ we can take the lower bound of the conditions $p^{f_\beta}_\infty$ for $\beta < \alpha$ as we have closure under κ-sequences. Denote the constructed generic as $g^{M}_{[\kappa,i^M(\kappa+1))}$.

(Sublemma 3.13) □

(Lemma 3.10) □

It follows we have completed one more step in finding suitable generics for $\mathbb{P}^M$.

$$V[G_\kappa] \xrightarrow{\ j\ } M[G_\kappa * g^{M}_{[\kappa,i^M(\kappa+1))}]$$

In order to construct another generic, we need to verify that we have preserved closure under κ-sequences of $M[G_\kappa * g^{M}_{[\kappa,i^M(\kappa+1))}]$ in $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$.

Lemma 3.14 *$M[G_\kappa * g^{M}_{[\kappa,i^M(\kappa+1))}]$ is closed under κ-sequences in $V[G_\kappa * g_{[\kappa,F(\kappa)]}]$.*

Proof. As mentioned above, $M[G_\kappa]$ remains closed under κ-sequences in $V[G_\kappa]$ as the forcing $\mathbb{P}_\kappa$ is κ-cc. Let us denote as g_S the projection of $g_{[\kappa,i(\kappa+1))}$ to the Sacks forcing. By Lemma 3.7, the forcing $\mathrm{Add}(\kappa^+, F(\kappa^+))$ is κ^+-distributive after the Sacks forcing $\mathrm{Sacks}(\kappa, F(\kappa))$, and consequently it is enough to show closure just in $V[G_\kappa * g_S]$. Recall the "manipulation" argument just before Lemma 3.9 which removes the discrepancy between the values $F(\kappa)$ and $j(F)(\kappa)$; the modification of $\mathrm{Sacks}(\kappa, F(\kappa))$ changes this forcing to $S^* = \mathrm{Sacks}(\kappa, j(F)(\kappa))$. Due to closure of $M[G_\kappa]$ under κ-sequences, S^* is the same in $V[G_\kappa]$ and in $M[G_\kappa]$ and is the first step of the product $\prod^{M}_{[\kappa,i^M(\kappa+1))}\mathcal{Q}^{M}_{\lambda}$ in the iteration $\mathbb{P}^M$

at stage κ. We are going to work in $V[G_\kappa * g_S^*] = V[G_\kappa * g_S]$, where g_S^* is the generic for S^*, and as such is present in $M[G_\kappa * g^M_{[\kappa, i^M(\kappa+1))}]$.

Let X be a κ-sequence of ordinal numbers in $V[G_\kappa * g_S^*]$, and let this be forced by some $p_0 \in g_S^*$. By the fusion argument (carried out in $V[G_\kappa]$), there is for every $r \leq p_0$ some $p_X \leq r$ such that if p_X is in g_S^*, then X can be uniquely determined from p_X and g_S^* restricted to the support of p_X. Since such p_X are dense below p_0, some such p_X is in g_S^*, and as p_X and g_S^* are present in $M[G_\kappa * g^M_{[\kappa, i^M(\kappa+1))}]$, so is X. (Lemma 3.14) □

The preservation of closure allows us to prove:

Lemma 3.15 *We can construct in $V[G_\kappa * g_{[\kappa, F(\kappa)]}]$ an $M[G_\kappa * g^M_{[\kappa, i^M(\kappa+1))}]$-generic for the stage $\mathbb{P}^M_{[i^M_{\kappa+1}, j(\kappa))}$.*

Proof. As in Lemma 3.9, case B), work in $V[G_\kappa * g_{[\kappa, F(\kappa)]}]$ and construct a generic H hitting all dense sets. (Lemma 3.15) □

It follows we can lift partially to $V[G_\kappa]$:

$$V[G_\kappa] \xrightarrow{\quad j \quad} M[G_\kappa * g^M_{[\kappa, i^M(\kappa+1))} * H]$$

The next step is to lift to the $\prod_{[\kappa, i(\kappa+1))} \mathcal{Q}_\lambda$-generic $g_{[\kappa, i(\kappa+1))}$ over $V[G_\kappa]$.

Again due to Lemma 3.7, coupled with Lemma 2.14, the only non-trivial part of this step is to lift to the generic filter g_S for Sacks$(\kappa, F(\kappa))$. This follows directly from the technique in [4], which is reviewed (and sufficiently generalized) in Section 2.4 (note that the condition $j(f_\lambda)(\kappa) = \lambda$ in the proof of Theorem 2.22 can be easily replaced by our assumption that $j(F)(\kappa) \geq F(\kappa)$).

Let us denote the generic generated by $j[g_S]$ as h_0. It follows we can lift to $V[G_\kappa * g_S]$:

$$V[G_\kappa * g_S] \xrightarrow{\quad j \quad} M[G_\kappa * g^M_{[\kappa, i^M(\kappa+1))} * H * h_0]$$

We finish the lifting by an application of Lemma 2.14 in two stages, first to the rest of the product $\prod_{[\kappa^+, i(\kappa+1))} \mathcal{Q}_\lambda$ and then to the rest of the iteration $\mathbb{P}^F$:

$$V[G] \xrightarrow{\quad j \quad} M[j(G)]$$

This proves Theorem 3.8 and shows that κ remains measurable in the generic extension by $\mathbb{P}^F$.

In [9], Menas showed using a "master condition" argument that *locally definable* (see Definition 3.16 below) Easton functions F can be realised while preserving supercompact cardinals. We will show how to adapt his result to strong cardinals using the above arguments.

Definition 3.16 *An Easton function F, see Definition 3.1, is said to be* locally definable *if the following condition holds:*

There is a sentence ψ and a formula $\varphi(x, y)$ with two free variables such that ψ is true in V and for all cardinals γ, if $H(\gamma) \models \psi$, then $F[\gamma] \subseteq \gamma$ and

$$\forall \alpha, \beta \in \gamma(F(\alpha) = \beta \Leftrightarrow H(\gamma) \models \varphi(\alpha, \beta)). \tag{16}$$

Theorem 3.17 *(GCH) Assume F is locally definable in the sense of Definition 3.16. If $\mathbb{P}^F$ is the forcing notion as in Definition 3.5, then $V^{\mathbb{P}^F}$ realises F and preserves all strong cardinals.*

Proof. First note that since ψ is true in V, there exists a closed unbounded class of cardinals C_ψ such that if $\beta \in C_\psi$, then $H(\beta) \models \psi$. It also holds that the closed unbounded class C_ψ is included in the closed unbounded class C_F of closure points of F.

Assume κ is a strong cardinal. We first show that κ is closed under F. Choose some β greater than κ such that $H(\beta)$ satisfies ψ and let $j : V \to M$ be an embedding witnessing β-hypermeasurability of κ; in particular $H(\beta)^V \subseteq M$ and $\beta < j(\kappa)$. Notice that for every $\alpha < \kappa$, the following equivalence is true by elementarity of j:

$$\exists \xi \in (\alpha, \kappa), \xi \text{ closed under } F \text{ iff } \exists \xi \in (\alpha, j(\kappa)), \xi \text{ closed under } j(F) \tag{17}$$

Since β in the interval $(\alpha, j(\kappa))$ was chosen to satisfy ψ and thus it is closed under F (and $j(F)$), we conclude that the closure points of F are unbounded in κ, and consequently κ is closed under F.

Let G be a generic filter for $\mathbb{P}^F$. Assume that $\beta > \kappa$ is a singular cardinal such that $H(\beta)$ satisfies ψ (it follows that β is a closure point of F). We claim that every extender embedding $j : V \to M$ witnessing the β^{++}-hypermeasurability of κ can be lifted to a $j^* : V[G] \to M[j(G)]$ with $H(\beta^+)$ of $V[G]$ included in $M[j(G)]$, thereby witnessing that κ is still β^+-hypermeasurable in $V[G]$. As β can be arbitrarily large, this implies that κ is still strong in $V[G]$.

Let $\beta > \kappa$ singular such that $H(\beta) \models \psi$ be given. Let $j : V \to M$ be a β^{++}-hypermeasurable witnessing embedding; that is $\beta^{++} < j(\kappa) < \beta^{+++}$ and $M = \{j(f)(a) \,|\, f : [\kappa]^{<\omega} \to V, a \in [\beta^{++}]^{<\omega}\}$. Since κ is closed under F, $j(\kappa)$

is closed under $j(F)$. Moreover, since $j(F)$ is locally definable in M via the formulas ψ and $\varphi(x,y)$ and $H(\beta)^M = H(\beta)^V$, it follows that $H(\beta)^M \models \psi$ and consequently F and $j(F)$ are identical on the interval $[\omega,\beta)$; in particular β is closed under $j(F)$. The fact that $H(\beta^{++})$ is correctly captured by M implies that $\mathbb{P}^F$ and $j(\mathbb{P}^F)$ coincide up to stage β, i.e. $\mathbb{P}^F_\beta = j(\mathbb{P}^F)_\beta$, and thus we may "copy" the generic G_β, i.e. G restricted to β, and use it as a generic for $j(\mathbb{P}^F)_\beta$. Moreover, as $\mathbb{P}^F_\beta$ is β^{++}-cc, all nice $\mathbb{P}^F_\beta$-names for subsets of β^+ in V are included in M, and consequently all subsets of β^+ existing in $V[G_\beta]$ are also present in $M[G_\beta]$. It follows that $H(\beta^{++})$ of $V[G_\beta]$ equals $H(\beta^{++})$ of $M[G_\beta]$.

Applying the notation of the previous section, we denote $\beta = i(\bar\beta) = i^M(\bar\beta)$, where i and i^M enumerate the closure points of F and $j(F)$, respectively, and $\bar\beta \leq \beta$ is some ordinal. The singularity of β in M implies that the next step of the iteration, the product $\mathbb{Q}^M_\beta$ in $M[G_\beta]$, is trivial at β, and so

$$\mathbb{Q}^M_\beta = \textstyle\prod^M_{[\beta^+, i^M(\bar\beta+1))} \mathcal{Q}^M_\lambda \tag{18}$$

is the Easton-supported product of Cohen forcings in the interval $[\beta^+, i^M(\bar\beta + 1))$, where $i^M(\bar\beta+1) < j(\kappa)$ is the next closure point of $j(F)$ after β.

Due to Lemma 2.1, we know that $\prod^M_{[\beta^+,\beta^{++}]} \mathcal{Q}^M_\lambda = (\mathrm{Add}(\beta^+, j(F)(\beta^+)))^{M[G_\beta]} \times (\mathrm{Add}(\beta^{++}, j(F)(\beta^{++})))^{M[G_\beta]}$ and $\prod^M_{[\beta^{+3M}, i^M(\bar\beta+1))} \mathcal{Q}^M_\lambda$ are mutually generic, and hence we can deal with them separately.

As $\beta^{++} \leq F(\beta^+) \leq F(\beta^{++})$ and the size of $j(F)(\beta^+)$ and $j(F)(\beta^{++})$ is β^{++} in V (due to closure of $j(\kappa)$ under $j(F)$), we can "manipulate" the forcing $\mathrm{Add}(\beta^+, F(\beta^+)) \times \mathrm{Add}(\beta^{++}, F(\beta^{++}))$ of $V[G_\beta]$ just like in (5) before Lemma 3.9. We obtain a forcing notion $\prod_{[\beta^+,\beta^{++}]} \mathcal{Q}^+_\lambda$ and a $V[G_\beta]$-generic $g^+_{[\beta^+,\beta^{++}]}$ for $\prod_{[\beta^+,\beta^{++}]} \mathcal{Q}^+_\lambda$. Since $H(\beta^{++})$ of $V[G_\beta]$ is correctly captured in $M[G_\beta]$, we can argue as in Lemma 3.9, case A) that maximal antichains in $\prod^M_{[\beta^+,\beta^{++}]} \mathcal{Q}^M_\lambda$ existing in $M[G_\beta]$ remain maximal in $\prod_{[\beta^+,\beta^{++}]} \mathcal{Q}^+_\lambda$. It follows that

$$g^+_{[\beta^+,\beta^{++}]} \cap M[G_\beta] \text{ is } M[G_\beta]\text{-generic for } \textstyle\prod^M_{[\beta^+,\beta^{++}]} \mathcal{Q}^M_\lambda. \tag{19}$$

Arguing as in Lemma 3.14, $M[G_\beta]$ is easily seen to be still closed under κ-sequences in $V[G_\beta]$. Consequently, we may construct a $M[G_\beta]$-generic for $\prod^M_{[\beta^{+3M}, i^M(\bar\beta+1))} \mathcal{Q}^M_\lambda$ just like in Lemma 3.9, Case B). Similarly, we construct a generic for the iteration $j(\mathbb{P}^F)$ up to the closure point $j(\kappa)$ (see Lemma 3.15). We finish the proof by first lifting to the Sacks forcing at κ, using [4] and the generalization in Section 2.4 of this paper, and then to the rest of the forcing above κ (see the end of the proof for Theorem 3.8, just before this Section 3.2), finally obtaining

$$j^* : V[G] \to M[j^*(G)]. \tag{20}$$

Notice that $M[j^*(G)]$ captures all subsets of β in $V[G]$, and hence κ is still β^+-hypermeasurable in $V[G]$. (Theorem 3.17) □

References

[1] James Cummings. Iterated forcing and elementary embeddings. To be included in the Handbook of Set Theory.

[2] James Cummings. A model where GCH holds at successors but fails at limits. *Transactions of the American Mathematical Society*, 329(1), 1992.

[3] William B. Easton. Powers of regular cardinals. *Annals of Mathematical Logic*, 1, 1970.

[4] Sy-David Friedman and Katherine Thompson. Perfect trees and elementary embeddings. To appear in the *Journal of Symbolic Logic*.

[5] Moti Gitik. The negation of singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$. *Annals of Pure and Applied Logic*, 43(3), 1989.

[6] Tomáš Jech. *Set Theory*. Springer, 2003.

[7] Akihiro Kanamori. Perfect-set forcing for uncountable cardinals. *Annals of Mathematical Logic*, 19, 1980.

[8] Akihiro Kanamori. *The Higher Infinite*. Springer, 2003.

[9] Telis K. Menas. Consistency results concerning supercompactness. *Transactions of the American Mathematical Society*, 223, 1976.

[10] William J. Mitchell. The core model for sequences of measures. I. *Math. Proc. Cambridge Philos. Soc.*, 95(2), 1984.