EASTON FUNCTIONS AND SUPERCOMPACTNESS

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Abstract. Suppose κ is λ-supercompact witnessed by an elementary embedding $j : V \to M$ with critical point κ, and further suppose that F is a function from the class of regular cardinals to the class of cardinals satisfying the requirements of Easton's theorem: (1) $\forall\alpha\ \alpha < \mathrm{cf}(F(\alpha))$ and (2) $\alpha < \beta \implies F(\alpha) \leq F(\beta)$. In this article we address the question: assuming GCH, what additional assumptions are necessary on j and F if one wants to be able to force the continuum function to agree with F globally, while preserving the λ-supercompactness of κ?

We show that, assuming GCH, if F is any function as above, and in addition for some regular cardinal $\lambda > \kappa$ there is an elementary embedding $j : V \to M$ with critical point κ such that κ is closed under F, the model M is closed under λ-sequences, $H(F(\lambda)) \subseteq M$, and for each regular cardinal $\gamma \leq \lambda$ one has $(|j(F)(\gamma)| = F(\gamma))^V$, *then* there is a cardinal-preserving forcing extension in which $2^\delta = F(\delta)$ for every regular cardinal δ and κ remains λ-supercompact. This answers a question of [CM13].

1. Introduction

In this article we address the following question, which is posed in [CM13].

Question 1.1. Given a λ-supercompact cardinal κ and assuming GCH, what behaviors of the continuum function on the regular cardinals can be forced while preserving the λ-supercompactness of κ, and from what hypotheses can such behaviors of the continuum function be obtained?

Let us first consider the special case where κ is κ-supercompact, in other words κ is measurable. Silver proved that if κ is κ^{++}-supercompact and GCH holds, then there is a cofinality-preserving forcing extension in which κ remains measurable and $2^\kappa = \kappa^{++}$, but one can also obtain such a model from a much weaker hypothesis. Woodin proved that the existence of a measurable cardinal κ such that $2^\kappa = \kappa^{++}$ is equiconsistent with the existence of an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \kappa^{++}$ and $M^\kappa \subseteq M$. The forward direction of Woodin's equiconsistency is trivial, and for the backward direction the embedding is lifted to a certain forcing extension $V[G][H][g_0]$ where g_0 is an "extra forcing"

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necessary for carrying out a surgical modification of a generic filter on the M-side (see [Cum10, Theorem 25.1] or [Jec03, Theorem 36.2]). A more uniform method for proving Woodin's equiconsistency, in which no "extra forcing" is required, is given in [FT08]. This method involves lifting an elementary embedding through Sacks forcing on uncountable cardinals, an idea which has found many additional applications (see [FM09], [FH08], [FZ10], [Hon10], [DF08], [FH12a] and [FH12b]). The uniformity of the method led to answers [FH08] to Question 1 in the case that κ is a measurable cardinal and in the case that κ is a strong cardinal.

In a result analagous to Woodin's equiconsistency mentioned above, the first author proved [Cod12] the equiconsistency of the following three hypotheses.

(i) There is a cardinal κ that is λ-supercompact and $2^\kappa > \lambda^{++}$.
(ii) There is a cardinal κ that is λ-supercompact and $2^\lambda > \lambda^{++}$.
(iii) There is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda^{++}$ and $M^\lambda \subseteq M$.

In the argument of [Cod12], a model of (ii) is obtained from a model of (iii) by lifting the embedding j to a forcing extension of the form $V[G][H][g_0]$ by using Woodin's technique of surgically modifying a generic filter. However, in the final model, κ is λ-supercompact and one has $2^\kappa = 2^\lambda = \lambda^{++}$, so the final model satisfies both (i) and (ii). Furthermore, it is remarked in [Cod12] that the surgery argument does not seem to yield a model with GCH on the interval $[\kappa, \lambda)$ and $2^\lambda = \lambda^{++}$, where κ is λ-supercompact.

The second and third authors showed that the more uniform method involving Sacks forcing on uncountable cardinals can be used to address this discordance. Indeed, it is shown in [FH12b] that from the hypothesis (iii) above, and assuming GCH, there is a cofinality-preserving forcing extension in which κ remains λ-supercompact, GCH holds on the interval $[\kappa, \lambda)$ and $2^\lambda = \lambda^{++}$. The following question is posed in [FH12b]. Starting with a model of (iii) and GCH, is there a cofinality-preserving forcing extension in which κ is λ-supercompact and for some regular cardinal γ with $\kappa < \gamma < \lambda$ one has GCH on $[\kappa, \gamma)$ and $2^\gamma = \lambda^{++}$? This question was recently answered in [CM13] where it is shown that Woodin's method of surgically modifying a generic filter to lift an embedding can be extended to include the case where modifications are made on "ghost-coordinates." Indeed [CM13] estabilshes that if GCH holds, $F : [\kappa, \lambda] \cap \mathrm{REG} \to \mathrm{CARD}$ is any function satisfying Easton's requirements

(E1) $\alpha < \mathrm{cf}(F(\alpha))$ and

(E2) $\alpha < \beta$ implies $F(\alpha) \leq F(\beta)$

where $\lambda > \kappa$ is a regular cardinal, and there is a $j : V \to M$ with critical point κ such that $j(\kappa) > F(\lambda)$ and $M^\lambda \subseteq M$, then there is a cofinality-preserving forcing extension in which κ remains λ-supercompact and $2^\gamma = F(\gamma)$ for every regular cardinal γ with $\kappa \leq \gamma \leq \lambda$. This provides an answer to the above Question 1.1 if we restrict our attention to controlling the continuum function only on the interval $[\kappa, \lambda]$ while preserving the λ-supercompactness of κ.

In this article we combine the methods of [FH08] and [CM13] to address Question 1.1 in the context of controlling the continuum function at all regular cardinals by proving the following theorem.

Theorem 1.2. *Assume* GCH. *Suppose* $F : \mathrm{REG} \to \mathrm{CARD}$ *is a function satisfying Easton's requirements* (E1) *and* (E2)*, for some regular cardinal* $\lambda > \kappa$ *there is an elementary embedding* $j : V \to M$ *with critical point* κ *such that* κ *is closed under* F*, the model* M *is closed under* λ*-sequences,* $H(F(\lambda)) \subseteq M$*, and for each regular cardinal* $\gamma \leq \lambda$ *one has* $(|j(F)(\gamma)| = F(\gamma))^V$*. Then there is a cardinal-preserving forcing extension in which* $2^\delta = F(\delta)$ *for every regular cardinal* δ *and* κ *remains* λ*-supercompact*

The forcing used to prove Theorem 1.2 will be an Easton-support iteration of Easton-support products of Cohen forcing. To lift the embedding through the first κ-stages of the forcing, we will use the technique of twisting a generic using an automorphism in order to obtain a generic for the M-side (see [FH08]). In order to lift the embedding through through a later portion of the iteration we will use the technique of surgically modifying a generic filter on ghost-coordinates (see [CM13]), which will require us to use an "extra forcing" over V. We will prove a lemma which establishes not only that the extra forcing preserves cardinals, but it also does not disturb the continuum function (see Lemma 3.10 below). Note that the later was not necessary in [CM13].

Regarding the hypothesis of Theorem 1.2, notice that if $j : V \to M$ witnesses the λ-supercompactness of κ then it follows that for $\gamma \leq \lambda$ we have $2^\gamma \leq (2^\gamma)^M < j(\kappa)$ and futhermore, in V, the cardinality of $(2^\gamma)^M$ is equal to 2^γ. Thus, if one desires to lift an embedding $j : V \to M$ to a forcing extension in which the continuum function agrees with some F as in the statement of Theorem 1.2, then one must require that $(|j(F)(\gamma)| = F(\gamma))^V$.

2. Preliminaries

We assume familiarity with Easton's theorem [Eas70] as well as with lifting large cardinal embeddings through forcing, see [Cum10].

In the proof of Theorem 1.2 we will use the following forcing notion. Suppose F is a function from the regular cardinals to the cardinals satisfying the requirements (E1) and (E2) of Easton's theorem and that $\kappa < \lambda$ are regular cardinals. We will let $\mathbb{Q}_{[\kappa,\lambda]}$ denote the Easton-support product of Cohen forcing that will ensure that, assuming GCH in the ground model, the continuum function agrees with F on $[\kappa,\lambda] \cap \mathrm{REG}$ in the forcing extension. We can regard conditions $p \in \mathbb{Q}_{[\kappa,\lambda]}$ as functions satisfying the following.

- Every element in $\mathrm{dom}(p)$ is of the form (γ, α, β) where $\gamma \in [\kappa, \lambda]$ is a regular cardinal, $\alpha < \gamma$, and $\beta < F(\gamma)$.
- (Easton support) For each regular cardinal $\gamma \in [\kappa,\lambda]$ we have
$$|\{(\delta,\alpha,\beta) \in \mathrm{dom}(p) \mid \delta \leq \gamma\}| < \gamma.$$
- $\mathrm{ran}(p) \subseteq \{0,1\}$.

Lemma 2.1 ([Eas70])**.** *Assuming* GCH*, forcing with the poset* $\mathbb{Q}_{[\kappa,\lambda]}$ *preserves all cofinalities and achieves* $2^\gamma = F(\gamma)$ *for every regular cardinal* $\gamma \in [\kappa,\lambda]$ *while preserving* GCH *otherwise.*

Remark 2.2. Suppose F and j are as in the hypothesis of Theorem 1.2. Let us briefly show that one can assume, without loss of generality, that M is of the form

$$M = \{j(f)(j``\lambda, \alpha) \mid f : P_\kappa\lambda \times \kappa \to V \ \wedge \ \alpha < F(\lambda) \ \wedge \ f \in V\}.$$

Let $j : V \to M$ be as in the statement of Theorem 1.2. We will show that j can be factored through an embedding $j_0 : V \to M_0$ having all the desired properties. Let $X_0 = \{j(f)(j``\lambda, \alpha) \mid f : P_\kappa\lambda \times \kappa \to V \wedge \alpha < F(\lambda) \wedge f \in V\}$ and $X_1 = \{j(f)(j``\lambda, a) \mid f : P_\kappa\lambda \times H(\kappa) \to V \ \wedge \ a \in H(F(\lambda)) \ \wedge \ f \in V\}$. Now let $\pi_0 : X \to M_0$ and $\pi_1 : X_1 \to M_1$ be the Mostowski collapses of X_0 and X_1 respectively. Define $j_0 := \pi_0^{-1} \circ j : V \to M_0$ and $j_1 := \pi_1^{-1} \circ j : V \to M_1$. It follows that $j_0 : V \to M_0$ has critical point κ, $M_0^\lambda \subseteq M_0$, $j_0(\kappa) > F(\lambda)$ and M_0 has the desired form

$$M_0 = \{j_0(f)(j_0``\lambda, \alpha) \mid f : P_\kappa\lambda \times \kappa \to V \ \wedge \ \alpha < F(\lambda) \ \wedge \ f \in V\}.$$

It remains to show that $H(F(\lambda)) \subseteq M_0$. It is easy to see that $H(F(\lambda)) \subseteq M_1$ using the fact that

$$M_1 = \{j_1(f)(j_1``\lambda, a) \mid f : P_\kappa\lambda \times H(\kappa) \to V \ \wedge \ a \in H(F(\lambda)) \ \wedge \ f \in V\}.$$

Since the map $i : M_0 \to M_1$ defined by $i(j_0(f)(j_0"\lambda, \alpha)) := j_1(f)(j_1"\lambda, \alpha)$ is an elementary embedding with critical point greater than $F(\lambda)$, and since i is the identity on $F(\lambda)$, it follows that i is surjective, and thus $H(F(\lambda)) \subseteq M_0 = M_1$. To see that i is surjective onto $H(F(\lambda))$ (and thus onto M_1) one uses the fact that each $x \in H(F(\lambda))$ can be coded by a subset of some cardinal $\delta < F(\lambda)$.

3. Proof of Theorem 1.2

Proof of Theorem 1. Our final model will be a forcing extension of V by an ORD-length forcing iteration $\mathbb{P}$, which will be broken up as $\mathbb{P} \cong \mathbb{P}^1 * \dot{\mathbb{S}} * \dot{\mathbb{P}}^2$. The first factor $\mathbb{P}^1$, will be an iteration forcing the continuum function to agree with F at every regular cardinal less than or equal to $F(\lambda)$. The second factor $\mathbb{S}$ will be an "extra forcing" that will be necessary to cary out the surgery argument to lift the embedding j through $\mathbb{P}^1$. We will argue that the extra forcing $\mathbb{S}$ is mild in $V^{\mathbb{P}^1}$ in the sense that it preserves all cofinalities and preserves the continuum function. The last factor $\mathbb{P}^2 \in V^{\mathbb{P}^1 * \dot{\mathbb{S}}}$ will be a $\leq F(\lambda)$-closed, ORD length Easton-support product of Cohen forcing, which will force the continuum function to agree with F at all regular cardinals greater than or equal to $F(\lambda)^+$.

For an ordinal α let $\bar{\alpha}$ denote the least closure point of F greater than α. For a regular cardinal γ, the notation $\mathrm{Add}(\gamma, F(\gamma))$ denotes the forcing poset for adding $F(\gamma)$ Cohen subsets to γ.

Let λ_0 be the greatest closure point of F which is less or equal to λ. We now recursively define an Easton-support forcing iteration $\mathbb{P}_{\lambda_0+1} = \langle (\mathbb{P}_\eta, \dot{\mathbb{Q}}_\eta) : \eta \leq \lambda_0 \rangle$ as follows.

(1) If $\eta < \lambda_0$ is a closure point of F, then $\dot{\mathbb{Q}}_\eta$ is a $\mathbb{P}_\eta$-name for the Easton support product
$$\mathbb{Q}_{[\eta,\bar{\eta})} = \prod_{\gamma \in [\eta,\bar{\eta}) \cap \mathrm{REG}} \mathrm{Add}(\gamma, F(\gamma))$$
as defined in $V^{\mathbb{P}_\eta}$ and $\mathbb{P}_{\eta+1} = \mathbb{P}_\eta * \dot{\mathbb{Q}}_\eta$.

(2) If $\eta = \lambda_0$, then $\dot{\mathbb{Q}}_\eta$ is a $\mathbb{P}_{\lambda_0}$-name for
$$\mathbb{Q}_{[\lambda_0,F(\lambda)]} = \prod_{\gamma \in [\lambda_0,F(\lambda)] \cap \mathrm{REG}} \mathrm{Add}(\gamma, F(\gamma))$$
as defined in $V^{\mathbb{P}_{\lambda_0}}$ and $\mathbb{P}_{\lambda_0+1} = \mathbb{P}_{\lambda_0} * \dot{\mathbb{Q}}_{\lambda_0}$.

(3) Otherwise, if $\eta < \lambda_0$ is not a closure point of F, then $\dot{\mathbb{Q}}_\eta$ is a $\mathbb{P}_\eta$-name for trivial forcing and $\mathbb{P}_{\eta+1} = \mathbb{P}_\eta * \dot{\mathbb{Q}}_\eta$.

Let G_{λ_0+1} be generic for $\mathbb{P}_{\lambda_0+1}$ over V.

Remark 3.1 (Notation)**.** We will adopt the notation and conventions used in [FH08]. We will use $\prod_{[\eta,\bar{\eta})} Q_\gamma$ to denote $\mathbb{Q}_{[\eta,\bar{\eta})}$ where $Q_\gamma := \mathrm{Add}(\gamma, F(\gamma))$ denotes an individual factor of the product, and similarly $g_{[\eta,\bar{\eta})}$ denotes the corresponding generic filter. It will be understood that, for example, $g_{[\eta,\bar{\eta})}$ is a product over just the regular cardinals in the interval $[\eta,\bar{\eta})$ of the relevant generic filters. In particular, if η is a singular cardinal then there is no forcing over η in the product $g_{[\eta,\bar{\eta})} \subseteq \prod_{[\eta,\bar{\eta})} Q_\gamma$.

3.1. **Lifting the embedding through $\mathbb{P}_\kappa$ by twisting a generic using an automorphism.** By Remark 2.2 we can assume that $j : V \to M$ is an embedding as in the statement of Theorem 1.2 such that

$$M = \{j(f)(j``\lambda, \alpha) \mid f : P_\kappa\lambda \times \kappa \to V \ \wedge \ \alpha < F(\lambda) \ \wedge \ f \in V\}.$$

First we will lift j through $G_\kappa \subseteq \mathbb{P}_\kappa$ by finding a filter for $j(\mathbb{P}_\kappa)$ that is generic over M. We will need the following definitions of various cardinals relating to F and λ.

Definition 3.2. The first three definitions will be needed because the forcings $\mathbb{P}_{\lambda_0+1}$ and $j(\mathbb{P}_{\lambda_0+1})$ are iterations of products over intervals determined by closure points of F and $j(F)$ respectively, and these three cardinals are important such closure points.

- $\lambda_0 :=$ "the greatest closure point of F that is at most λ"
- $\lambda_1 :=$ "the least closure point of F greater than λ_0"
- $\lambda_1^M :=$ "the least closure point of $j(F)$ greater than λ_0"

The way one builds a generic for the forcing $\mathrm{Add}(\gamma, F(\gamma))$ depends, of course, on the size of $F(\gamma)$ and the regular cardinals γ_0 and γ_1 defined below are important transition points in the size of $F(\gamma)$.

- $\gamma_0 :=$ "the least regular cardinal less than or equal to λ such that $F(\gamma_0) = F(\lambda)$"
- $\gamma_1 :=$ "the least regular cardinal such that $F(\gamma_1) > F(\lambda)$"

We have $\kappa \leq \lambda_0 \leq \gamma_0 \leq \lambda < \gamma_1 \leq F(\lambda) = F(\gamma_0) \leq j(F)(\gamma_0) < \lambda_1^M < j(\kappa) < F(\lambda)^+ < \lambda_1$. Furthermore, if $\gamma \in [\kappa, \gamma_0)$ is a regular cardinal we have $|j(F)(\gamma)|^V = F(\gamma)$ and since M and V have the same cardinals $\leq F(\lambda)$, it follows that $j(F)(\gamma) = F(\gamma)$. In other words, F and $j(F)$ agree on $[\kappa, \gamma_0) \cap \mathrm{REG}$. This implies that we may let $G^M_{[\kappa,\lambda_0)} = G_{[\kappa,\lambda_0)}$ and $g^M_{[\lambda_0,\gamma_0)} = g_{[\lambda_0,\gamma_0)}$. Note that F and $j(F)$ may disagree at γ_0 because M has cardinals strictly between $F(\gamma_0) = F(\lambda)$ and $(F(\lambda)^+)^V$.

Suppose $\gamma \in [\gamma_0, F(\lambda)]$ is a regular cardinal. Since $j(\kappa)$ is a closure point of $j(F)$ we have $F(\lambda) \leq j(F)(\gamma) < j(\kappa)$, and since $|j(\kappa)|^V \leq F(\lambda)$ it follows that $|j(F)(\gamma)|^V = F(\lambda)$. Let us define a forcing $\prod_{[\gamma_0,F(\lambda)]} Q^+_\gamma$ in $V[G_{\lambda_0}]$ that will be used to obtain a generic for $\mathbb{Q}^M_{[\gamma_0,F(\lambda)]}$ over $M[G_{\lambda_0}]$. Working in $V[G_{\lambda_0}]$, for regular $\gamma \in [\gamma_0,\gamma_1)$ let $Q^*_\gamma = \mathrm{Add}(\gamma, j(F)(\gamma))$ and notice that Q^*_γ is isomorphic to $\mathrm{Add}(\gamma, F(\gamma))$ since $|j(F)(\gamma)|^V = |j(\kappa)|^V = F(\lambda) = F(\gamma)$. For regular $\gamma \in [\gamma_1, F(\lambda)]$, let $Q^{**}_\gamma = \mathrm{Add}(\gamma, j(F)(\gamma))$ and notice that Q^{**}_γ is a truncation of $\mathrm{Add}(\gamma, F(\gamma))$ because for such γ one has $j(F)(\gamma) < j(\kappa) < F(\lambda)^+ \leq F(\gamma)$. Now define

$$\prod_{[\gamma_0,F(\lambda)]} Q^+_\gamma := \prod_{[\gamma_0,\gamma_1)} Q^*_\gamma \times \prod_{[\gamma_1,F(\lambda)]} Q^{**}_\gamma. \tag{3.1}$$

It is easy to see that $\prod_{[\gamma_0,F(\lambda)]} Q^+_\gamma$ completely embeds into $\prod_{[\gamma_0,F(\lambda)]} Q_\gamma$, and hence there is a filter $g^+_{[\gamma_0,F(\lambda)]} \in V[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times g_{[\gamma_0,F(\lambda)]})]$ generic over $V[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$ for $\prod_{[\gamma_0,F(\lambda)]} Q^+_\gamma$.

The lifting of j through G_κ will be broken up into two cases, depending on the regularity or singularity of $F(\lambda)$. If $F(\lambda)$ is regular, the proof is substantially simpler because it almost directly follows from the assumption $H(F(\lambda)) \subseteq M$ (see Lemma 3.3 below). If $F(\lambda)$ is singular, there are two cases to distinguish depending on whether the V-cofinality of $F(\lambda)$ is λ^+ or not; in both cases the assumption of $H(F(\lambda)) \subseteq M$ is again essential, but an additional argument is required. Assuming $F(\lambda)$ is singular, the case in which $\mathrm{cf}(F(\lambda))^V = \lambda^+$ is easier to handle than the case where $\mathrm{cf}(F(\lambda)) > \lambda^+$. The later case requires an induction along a matrix of coordinates (see Lemma 3.4). To avoid long repetitions of the relevant proofs in [FH08], we include only outlines of the proofs of Lemma 3.3 and Lemma 3.4, with detailed references to [FH08] where appropriate (the proofs in [FH08] apply almost verbatim here when one identifies κ with λ).

Lemma 3.3. *Assume $F(\lambda)$ is regular. There is in $V[G_{\lambda_0}*(g_{[\lambda_0,\gamma_0)}\times g_{[\gamma_0,F(\lambda)]})]$ an $M[G_{\lambda_0}*g_{[\lambda_0,\gamma_0)}]$-generic for $\prod^M_{[\gamma_0,\lambda_1^M)} Q^M_\gamma$, which we will denote as $g^M_{[\gamma_0,\lambda_1^M)}$. Furthermore, we can take $g^M_{[\gamma_0,\lambda]}$ to agree with $g^+_{[\gamma_0,\lambda]}$, that is, $g^M_{[\gamma_0,\lambda_1^M)} = g^+_{[\gamma_0,\lambda]} \times g^M_{(\lambda,\lambda_1^M)}$.*

Proof. Since $F(\lambda)$ is regular in V and hence also in M, it follows that $\prod^M_{[\gamma_0,F(\lambda)]} Q^M_\gamma$ is $(F(\lambda)^+)^M$-c.c. in $M[G_{\lambda_0}*g_{[\lambda_0,\gamma_0)}]$. Furthermore, $\prod^M_{(F(\lambda),\lambda_1^M)} Q^M_\gamma$ is $(F(\lambda)^+)^M$-closed in $M[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$. It follows by Easton's Lemma that generic filters for these forcings are mutually generic and therefore it suffices to obtain generic filters for them separately.

As in [FH08, Lemma 3.9], one may check that $g^M_{[\gamma_0,F(\lambda)]} := g^+_{[\gamma_0,F(\lambda)]} \cap \prod^M_{[\gamma_0,F(\lambda)]} Q^M_\gamma$ is $M[G_{\lambda_0}*g_{[\lambda_0,\gamma_0)}]$-generic and one can build an $M[G_{\lambda_0}*g_{[\lambda_0,\gamma_0)}]$-generic filter $g^M_{(F(\lambda),\lambda_1^M)}$ for $\prod^M_{(F(\lambda),\lambda_1^M)} Q^M_\gamma$ in $V[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$.

Now we define $g^M_{[\gamma_0,\lambda_1^M)} := g^M_{[\gamma_0,F(\lambda)]} \times g^M_{(F(\lambda),\lambda_1^M)}$ and it remains to show that $g^M_{[\gamma_0,\lambda_1^M)} = g^+_{[\gamma_0,\lambda]} \times g^M_{(\lambda,\lambda_1^M)}$. Since $M[G_{\lambda_0}]$ is closed under λ-sequences in $V[G_{\lambda_0}]$ we have $\prod^M_{[\gamma_0,\lambda]} Q^M_\gamma = \prod_{[\gamma_0,\lambda]} Q^*_\gamma$. Now use (3.1) to obtain the desired conclusion. □

Lemma 3.4. *Assume $F(\lambda)$ is singular. There is in $V[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times g_{[\gamma_0,F(\lambda))})]$ an $M[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$-generic for $\prod^M_{[\gamma_0,\lambda_1^M)} Q^M_\gamma$, which we will denote as $g^M_{[\gamma_0,\lambda_1^M)}$. Furthermore, we can take $g^M_{[\gamma_0,\lambda_1^M)}$ to be of the form $\sigma[g^+_{[\gamma_0,\lambda]}] \times g^M_{(\lambda,\lambda_1^M)}$ where σ is an automorphism of $\prod_{[\gamma_0,\lambda]} Q^+_\gamma$ in $V[G_{\lambda_0}]$ and $g^M_{(\lambda,\lambda_1^M)}$ is $M[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$-generic for $\prod^M_{(\lambda,\lambda_1^M)} Q^M_\gamma$.*

Proof. **Case I:** Suppose $F(\lambda)$ is singular in V with $\mathrm{cf}(F(\lambda))^V = \lambda^+$ ($F(\lambda)$ can be singular or regular in M). As in [FH08, Sublemma 3.12], we can find a condition $p_\infty \in \prod_{[\gamma_0,\lambda_1^M)} Q^+_\gamma$ (which may only exist in $V[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$) such that if h is generic for $\prod_{[\gamma_0,F(\lambda))} Q^+_\gamma$ with $p_\infty \upharpoonright [\gamma_0, F(\lambda)) \in h$ and $h' = \{p_\infty \upharpoonright [F(\lambda),\lambda_1^M)\} \cup \{q \in \prod^M_{[F(\lambda),\lambda_1^M)} Q^M_\gamma \mid p_\infty \upharpoonright [F(\lambda),\lambda_1^M) \leq q\}$, then $(h \times h') \cap M[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$ is $M[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$-generic for $\prod^M_{[\gamma_0,\lambda_1^M)} Q^M_\gamma$.

We define $g^M_{[\gamma_0,\lambda_1^M)}$ as follows. A homogeneity argument can be used to find an automorphism σ of $\prod_{[\gamma_0,F(\lambda))} Q^+_\gamma$ such that $p_\infty \upharpoonright [\gamma_0, F(\lambda)) \in \sigma[g^+_{[\gamma_0,F(\lambda))}]$. We obtain the desired generic by letting

$$g^M_{[\gamma_0,\lambda_1^M)} := \left(\sigma[g^+_{[\gamma_0,F(\lambda))}] \times h'\right) \cap M[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]. \tag{3.2}$$

Since $M[G_{\lambda_0}]$ is closed under λ-sequences in $V[G_{\lambda_0}]$ we have $\prod^M_{[\gamma_0,\lambda]} Q^M_\gamma = \prod_{[\gamma_0,\lambda]} Q^*_\gamma$. Now using (3.1) and the definition (3.2), we obtain $g^M_{[\gamma_0,\lambda]} = \sigma[g^+_{[\gamma_0,\lambda]}]$.

Case II: Suppose $F(\lambda)$ is singular in V and $\mathrm{cf}(F(\lambda))^V \neq \lambda^+$ ($F(\lambda)$ can be singular or regular in M). If $F(\lambda)$ is regular in M then, as in [FH08, Sublemma 3.13] we can use a "matrix of confitions" argument to find a p_∞ as above. As in Case I, we have $g^M_{[\gamma_0,\lambda_1^M)} := \sigma[g^+_{[\gamma_0,F(\lambda))}] \times h'$ is $M[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$-generic for $\prod^M_{[\gamma_0,\lambda_1^M)} Q^M_\gamma$ where h' is some $M[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$-generic filter for $\prod^M_{[F(\kappa),\lambda_1^M)} Q^M_\gamma$. As in Case I we get $g^M_{[\gamma_0,\lambda]} = \sigma[g^+_{[\gamma_0,\lambda]}]$.

If $F(\lambda)$ is singular in M then an easier argument will suffice (see [FH08, Case (2), page 205]). □

By Lemmas 3.3 and 3.4 above, if $F(\lambda)$ is regular or singular in V, there is an $M[G_{\lambda_0} * g_{[\lambda_0,\gamma_0)}]$-generic filter $g^M_{[\gamma_0,\lambda_1^M)}$ for $\prod^M_{[\gamma_0,\lambda_1^M)} Q^M_\gamma$ in

$$V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}] = V[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times g_{[\gamma_0,F(\lambda)]})].$$

Define $g^M_{[\lambda_0,\lambda_1^M)} := g_{[\lambda_0,\gamma_0)} \times g^M_{[\gamma_0,\lambda_1^M)}$. We will now use the fact that, depending on whether $F(\lambda)$ is regular or singular, $g^M_{[\gamma_0,\lambda_1^M)}$ agrees with either $g^+_{[\gamma_0,\lambda]}$ or an automorphic image of $g^+_{[\gamma_0,\lambda]}$ to establish the following.

Lemma 3.5. *$M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda_1^M)}]$ is closed under λ-sequences in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$.*

Proof. It will suffice to argue that if X is a λ-sequence of ordinals in $V[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times g_{[\gamma_0,F(\lambda)]})]$ then X is in $M[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times g^M_{[\gamma_0,\lambda_1^M)})]$. Since $\prod_{(\lambda,F(\lambda)]} Q_\gamma$ is $\leq \lambda$-distributive in $V[G_{\lambda_0} * g_{[\lambda_0,\lambda]}]$ we have $X \in V[G_{\lambda_0} * g_{[\lambda_0,\lambda]}]$. Furthermore, since $\lambda < \gamma_1$ it follows from (3.1) that

$$X \in V[G_{\lambda_0} * g_{[\lambda_0,\lambda]}] = V[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times g^+_{[\gamma_0,\lambda]})].$$

Since $M[G_{\lambda_0}]$ is closed under λ-sequences in $V[G_{\lambda_0}]$, it follows that

$$\prod\nolimits^M_{[\lambda_0,\lambda]} Q^M_\gamma = \prod\nolimits_{[\lambda_0,\gamma_0)} Q_\gamma \times \prod\nolimits_{[\gamma_0,\lambda]} Q^*_\gamma. \tag{3.3}$$

First let us assume that $F(\lambda)$ is regular so that, by Lemma 3.3, we have $g^M_{[\lambda_0,\lambda]} = g_{[\lambda_0,\gamma_0)} \times g^+_{[\gamma_0,\lambda]}$. As the forcing in (3.3) is isomorphic to $\prod_{[\lambda_0,\lambda]} Q_\gamma$ in $V[G_{\lambda_0}]$, we see that it is λ^+-c.c. in $V[G_{\lambda_0}]$, and therefore the model

$$M[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times g^+_{[\gamma_0,\lambda]})] = M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda]}]$$

is closed under λ-sequences in $V[G_{\lambda_0} * g_{[\lambda_0,\lambda]}]$. Thus $X \in M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda]}] \subseteq M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda_1^M)}]$.

Now let us assume $F(\lambda)$ is singular. By Lemma 3.4 we have $g^M_{[\lambda_0,\lambda]} = g_{[\lambda_0,\gamma_0)} \times \sigma[g^+_{[\gamma_0,\lambda]}]$ for some automorphism σ of $\prod_{[\gamma_0,\lambda]} Q^+_\gamma$ in $V[G_{\lambda_0}]$. Since

$$V[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times g^+_{[\gamma_0,\lambda]})] = V[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times \sigma[g^+_{[\gamma_0,\lambda]}])]$$

and since $g_{[\lambda_0,\gamma_0)} \times \sigma[g^+_{[\gamma_0,\lambda]}]$ is $V[G_{\lambda_0}]$-generic for the λ^+-c.c. forcing in (3.3), it follows as before that the model $M[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times \sigma[g^+_{[\gamma_0,\lambda]}])] = M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda]}]$ is closed under λ-sequences in $V[G_{\lambda_0} * g_{[\lambda_0,\lambda]}]$. Thus $X \in M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda]}] \subseteq M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda_1^M)}]$. □

Lemma 3.6. *We can build an $M[G_{\lambda_0} * (g_{[\lambda_0,\gamma_0)} \times g^M_{[\gamma_0,\lambda_1^M)})]$-generic filter $G^M_{[\lambda_1^M,j(\kappa))}$ for $\mathbb{P}^M_{[\lambda_1^M,j(\kappa))}$ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$.*

Proof. There are at most λ^+ functions in V that represent names for dense subsets of a tail of $j(\mathbb{P}_\kappa)$. Thus every dense subset of $\mathbb{P}^M_{[\lambda_1^M,j(\kappa))}$ in $M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda_1^M)}]$ has a name represented by one of these functions. We may use the fact that $\mathbb{P}^M_{[\lambda_1^M,j(\kappa))}$ is $\leq F(\lambda)$-closed in $M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda_1^M)}]$ and that $M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda_1^M)}]$ is closed under λ-sequences in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ to build a decreasing λ^+-sequence of conditions from $\mathbb{P}^M_{[\lambda_1^M,j(\kappa))}$ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ meeting every dense subset of $\mathbb{P}^M_{[\lambda_1^M, j(\kappa))}$ in $M[G_{\lambda_0} * g^M_{[\lambda_0,\lambda_1^M)}]$. It follows that this λ^+-sequence of conditions generates the desired generic filter. $\square$

Thus we may lift j to

$$j : V[G_\kappa] \to M[j(G_\kappa)],$$

where $j(G_\kappa) = G_{\lambda_0} * g^M_{[\lambda_0,\lambda_1^M)} * G^M_{[\lambda_1^M, j(\kappa))}$ and j is a class of $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$. Furthermore, we have that $M[j(G_\kappa)]$ is closed under λ-sequences in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda))}]$ and

$$M[j(G_\kappa)] = \{j(f)(j\text{"}\lambda, \alpha) \mid f : P_\kappa\lambda \times \kappa \to V \ \wedge \ \alpha < F(\lambda) \ \wedge \ f \in V[G_\kappa]\}.$$

3.2. **Outline.** Our goal is to lift j through the forcing $\mathbb{P}_{[\kappa,\lambda_0)} * \dot{\mathbb{Q}}_{[\lambda_0,\lambda]} = \mathbb{P}_{[\kappa,\lambda_0)} * \prod_{[\lambda_0,\lambda]} Q_\gamma$. Our strategy will be to first use a master condition for lifting j through $\mathbb{P}_{[\kappa,\lambda_0)}$ of this forcing and then to use the surgery argument of [CM13] to lift j through $\mathbb{Q}_{[\lambda_0,\lambda]}$.

3.3. **Lifting the embedding through $\mathbb{P}_{[\kappa,\lambda_0)}$ via a master condition argument.** In $V[G_\kappa]$, the poset $\mathbb{P}_{[\kappa,\lambda_0)}$ has size no larger than λ and thus, $j\text{"}G_{[\kappa,\lambda_0)}$ has size at most λ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda))}]$. Hence $j\text{"}G_{[\kappa,\lambda_0)} \in M[j(G_\kappa)]$ and since $j(\mathbb{P}_{[\kappa,\lambda_0)})$ is $<j(\kappa)$-directed closed in $M[j(G_\kappa)]$, there is a master condition $p_{[\kappa,\lambda_0)} \in j(\mathbb{P}_{[\kappa,\lambda_0)})$ extending every element of $j\text{"}G_{[\kappa,\lambda_0)}$. We now build an $M[j(G_\kappa)]$-generic filter below $p_{[\kappa,\lambda_0)}$. First notice that every dense subset of $j(\mathbb{P}_{[\kappa,\lambda_0)})$ in $M[j(G_\kappa)]$ can be written as $j(h)(j\text{"}\lambda, \alpha)$ where $h \in V[G_\kappa]$ is a function from $P_\kappa\lambda \times \kappa$ into the collection of dense subsets of $\mathbb{P}_{[\kappa,\lambda_0)}$ and $\alpha < F(\lambda)$. Since in $V[G_\kappa]$ there are no more than λ^+ such functions, it follows that we can enumerate them as $\langle h_\xi \mid \xi < \lambda^+\rangle \in V[G_\kappa]$ so that every dense subset of $j(\mathbb{P}_{[\kappa,\lambda_0)})$ in $M[j(G_\kappa)]$ is of the form $j(h_\xi)(j\text{"}\lambda, \alpha)$ for some $\xi < \lambda^+$ and some $\alpha < F(\lambda)$. One can build a decreasing λ^+-sequence of conditions $\langle p_\xi \mid \xi < \lambda^+\rangle \in V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda))}]$ below $p_{[\kappa,\lambda_0)}$, such that for every $\xi < \lambda^+$ the condition $p_\xi \in j(\mathbb{P}_{[\kappa,\lambda_0)})$ meets every dense subset of $j(\mathbb{P}_{[\kappa,\lambda_0)})$ in $M[j(G_\kappa)]$ appearing in the sequence $\langle j(h_\xi)(j\text{"}\lambda, \alpha) \mid \alpha < F(\lambda)\rangle$. Let $G^M_{[j(\kappa), j(\lambda_0))} \in V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda))}]$ be the filter generated by $\langle p_\xi \mid \xi < \lambda^+\rangle$. It follows by construction that $G^M_{[j(\kappa), j(\lambda_0))}$ is $M[j(G_\kappa)]$-generic and $j\text{"}G_{[\kappa,\lambda_0)} \subseteq G^M_{[j(\kappa), j(\lambda_0))}$. Thus we may lift j to

$$j : V[G_\kappa * G_{[\kappa,\lambda_0)}] \to M[j(G_\kappa) * j(G_{[\kappa,\lambda_0)})] \tag{3.4}$$

where $j(G_{[\kappa,\lambda_0)}) = G^M_{[j(\kappa), j(\lambda_0))}$ and where j is a class of $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$. Furthermore, $M[j(G_\kappa) * j(G_{[\kappa,\lambda_0)})]$ is closed under λ-sequences in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$.

3.4. **Obtaining a generic for $j(\mathbb{Q}_{[\lambda_0,\lambda]})$ for use in surgery.** Now we will lift j through the forcing $\mathbb{Q}_{[\lambda_0,\lambda]}$ by applying the surgery technique of [CM13]. We will factor the embedding in (3.4) through an ultrapower embedding j_0, force with $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ over $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ and then modify the generic to lift the embedding.

Let $X = \{j(h)(j"\lambda) \mid h : P_\kappa\lambda \to V[G_{\lambda_0}], h \in V[G_{\lambda_0}]\}$. Then it follows that $X \prec M[j(G_{\lambda_0})]$. Let $k : M_0' \to M[j(G_{\lambda_0})]$ be the inverse of the Mostowski collapse $\pi : X \to M_0'$ and let $j_0 : V[G_{\lambda_0}] \to M_0'$ be defined by $j_0 := k^{-1} \circ j$. It follows that j_0 is the ultrapower embedding by the measure $U_0 := \{X \subseteq P_\kappa\lambda \mid j"\lambda \in j(X)\}$ and we will see that $U_0 \in V[G_{\lambda_0} * \pi(g^M_{[\lambda_0,F(\lambda)]})]$. Using a theorem of Laver [Lav07], which says that the ground model is always definable from a parameter in any set forcing extension, it follows by elementarity that M_0' is of the form $M_0[j_0(G_{\lambda_0})]$, where $M_0 \subseteq M_0'$ and $j_0(G_{\lambda_0}) \subseteq j_0(\mathbb{P}_{\lambda_0}) \in M_0'$ is M_0-generic.

Remark 3.7. Since $j"\lambda \in X$ it follows that X is closed under λ-sequences in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$. Thus $\lambda^+ \subseteq X$ and hence the transitive collapse π is the identity on $[0, \lambda^+)$. In fact λ^+ also belongs to X so the critical point of k is greater than λ^+.

In Lemma 3.8 and Lemma 3.9 below, we show that the forcing $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ behaves well in the model $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$, in the sense that it is highly distributive and has a good chain condition. Then it easily follows that forcing with $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ over $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ preserves cardinals, and since SCH holds in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$, this forcing does not disturb the continuum function (see Lemma 3.10).

Lemma 3.8. $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ *is* $\leq\lambda$*-distributive in* $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$.

Proof. Define $S := j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ and $g^{M_0}_{[\lambda_0,\lambda]} := \pi(g^M_{[\lambda_0,\lambda]})$. It follows that $g^{M_0}_{[\lambda_0,\lambda]}$ is generic over $M_0[G_{\lambda_0}]$ for $T := \pi(\prod^M_{[\lambda_0,\lambda]} Q^+_\gamma)$. Notice that T is a "truncated" version of $\prod^M_{[\lambda_0,\lambda]} Q^+_\gamma$ because π is the identity on $[0, \lambda]$; moreover, $g^{M_0}_{[\lambda_0,\lambda]}$ is generic for T over $V[G_{\lambda_0}]$ and T is λ^+-c.c. over $V[G_{\lambda_0}]$.

We prove the lemma in two steps: (i) Firstly, we show that $M_0[j_0(G_{\lambda_0})]$ is closed under λ-sequences in $V^* := V[G_{\lambda_0}][g^{M_0}_{[\lambda_0,\lambda]} \times g_{(\lambda,F(\lambda)]}]$; this will imply that S is $\leq \lambda$-closed in V^*. (ii) Secondly, we show that S remains $\leq \lambda$-distributive in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$, which can be written – as we will argue – as $V^*[g_{[\lambda_0,\lambda]}]$.

As for (i), notice that $j_0|V : V \to M$ is elementary, and M is closed under λ-sequences in V. The generic $G_{\lambda_0} * g^{M_0}_{[\lambda_0,\lambda]}$ is added by a λ^+-c.c. forcing over V, and hence $M_0[G_{\lambda_0}][g^{M_0}_{[\lambda_0,\lambda]}]$ is still closed under λ-sequences in $V[G_{\lambda_0}][g^{M_0}_{[\lambda_0,\lambda]}]$. Finally, the forcing adding $g_{(\lambda,F(\lambda)]}$ is, by the Easton's lemma, $\leq \lambda$-distributive over $V[G_{\lambda_0}][g^{M_0}_{[\lambda_0,\lambda]}]$ (and therefore does not add new λ-sequences); now (i) follows because $M_0[j_0(G_{\lambda_0})]$ is included in V^*.

As for (ii), notice that $\prod_{[\lambda_0,\lambda]} Q_\gamma$ (with the associated generic $g_{[\lambda_0,\lambda]}$), is isomorphic in $V[G_{\lambda_0}]$ to $T \times \prod_{[\lambda_0,\lambda]} Q_\gamma$. Now (ii), and hence the lemma, follows by another application of the Easton's lemma, using the λ^+-cc of $\prod_{[\lambda_0,\lambda]} Q_\gamma$. □

Lemma 3.9. *$j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ is λ^{++}-c.c. in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$.*

Proof. Notice that each condition $p \in j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ can be written as $j_0(h_p)(j\text{"}\lambda)$ for some function $h_p : P_\kappa\lambda \to \mathbb{Q}_{[\lambda_0,\lambda]}$ in $V[G_{\lambda_0}]$. Thus, each condition $p \in j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ leads to a function $\bar{h}_p : \lambda \to \mathbb{Q}_{[\lambda_0,\lambda]}$ in $V[G_{\lambda_0}]$, which is a condition in the full-support product of λ copies of $\mathbb{Q}_{[\lambda_0,\lambda]}$ taken in $V[G_{\lambda_0}]$, denoted by $\bar{\mathbb{Q}} = (\mathbb{Q}_{[\lambda_0,\lambda]})^\lambda$.

Let us argue that $\bar{\mathbb{Q}}$ is λ^{++}-c.c. in $V[G_{\lambda_0} * g_{[\lambda^+,F(\lambda)]}]$. We define the domain of a condition $p = \langle p_\xi \mid \xi < \lambda\rangle \in \bar{\mathbb{Q}}$ to be the disjoint union of the domains of its coordinates: $\text{domain}(p) := \bigsqcup_{\xi<\lambda} \text{dom}(p_\xi)$. It follows that each $p \in \bar{\mathbb{Q}}$, being the union of λ sets, each of size less than λ, has domain of size at most λ. Suppose A is an antichain of $\bar{\mathbb{Q}}$ in $V[G_{\lambda_0} * g_{[\lambda^+,F(\lambda)]}]$ of size λ^{++}. If there are λ^{++} conditions in A that have a common domain, say d, then we immediately get a contradiction because, in $V[G_{\lambda_0} * g_{[\lambda^+,F(\lambda)]}]$, there are at most $2^\lambda = \lambda^+$ functions in 2^d. Otherwise, the set $\text{domain}(A) = \{\text{domain}(p) \mid p \in A\}$ has size λ^{++}. Since $2^\lambda = \lambda^+$ in $V[G_{\lambda_0} * g_{[\lambda^+,F(\lambda)]}]$, it follows that $(\lambda^+)^{<\lambda^+} = \lambda^+$, and hence, by the Δ-system lemma, $\text{domain}(A)$ contains a Δ-system of size λ^{++} with root r. This produces a contradiction because, in $V[G_{\lambda_0} * g_{[\lambda^+,F(\lambda)]}]$ we have $|2^r| = 2^\lambda = \lambda^+$.

To see that $\bar{\mathbb{Q}}$ is λ^{++}-c.c. in

$$V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}] = V[G_{\lambda_0} * g_{[\lambda^+,F(\lambda)]}][g_{[\lambda_0,\lambda]}] \tag{3.5}$$

we will use the fact that the product of θ^+-Knaster forcing with θ^+-c.c. forcing is θ^+-c.c., where $\theta > \omega$ is a cardinal. Since the forcing $g_{[\lambda_0,\lambda]} \subseteq \mathbb{Q}_{[\lambda_0,\lambda]}$ is λ^{++}-Knaster and $\bar{\mathbb{Q}}$ is λ^{++}-c.c. in $V[G_{\lambda_0} * g_{[\lambda^+,F(\lambda)]}]$, it follows that $\bar{\mathbb{Q}}$ is λ^{++}-c.c. in the model $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}] = V[G_{\lambda_0} * g_{[\lambda^+,F(\lambda)]}][g_{[\lambda_0,\lambda]}]$.

It remains to show that an antichain of $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ with size λ^{++} would lead to an antichain of $\bar{\mathbb{Q}}$ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ of size λ^{++}, but this is quite easy. Suppose A is an antichain of $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ with size δ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$. Each $p \in A$ is of the form $j_0(h_p)(j\text{"}\lambda)$ where $h_p : P_\kappa\lambda \to \mathbb{Q}_{[\lambda_0,\lambda]}$. As mentioned above, each h_p leads to a condition $\bar{h}_p \in$

$\bar{\mathbb{Q}}$. It is easy to check that $\bar{A} := \{\bar{h}_p \mid p \in A\}$ is an antichain of $\bar{\mathbb{Q}}$ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ of size λ^{++}. □

Lemma 3.10. *Forcing with $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ over $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ preserves cardinals and does not disturb the continuum function.*

Proof. By Lemma 3.8, $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ is $\leq\lambda$-distributive in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ and thus preserves cardinals in $[\omega, \lambda^+]$ and does not disturb the continuum function on the interval $[\omega, \lambda]$.

Lemma 3.9 implies that cardinals in $[\lambda^{++}, \infty)$ are preserved. Furthermore, by counting nice names we will now show that the continuum function is not disturbed on $[\lambda^+, \infty)$. Working in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$, since $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ has size at most $|{}^{\lambda}F(\lambda) \cap V[G_{\lambda_0}]| = F(\lambda)$ and is λ^{++}-c.c., it follows that if $\delta \in [\lambda^+, \infty)$ is a cardinal then there are at most $F(\lambda)^{\lambda^+ \cdot \delta} = F(\lambda)^{\delta}$ nice $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$-names for subsets of δ. Since SCH holds in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$, it follows that for all infinite cardinals μ and ν, if $\mu \leq 2^{\nu}$ then $\mu^{\nu} = 2^{\nu}$ (see [Jec03, Theorem 5.22(ii)(a)]). In particular, we have $F(\lambda) \leq F(\lambda^+) \leq 2^{\delta}$ and thus $F(\lambda)^{\delta} = 2^{\delta}$ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$. Thus there are at most 2^{δ} nice $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$-names for subsets of δ, and the result follows. □

Let J be a $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$-generic filter for $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$.

Lemma 3.11. *k"J generates an $M[j(G_{\lambda_0})]$-generic filter for $j(\mathbb{Q}_{[\lambda_0,\lambda]})$, which we will call K.*

Proof. Suppose $D \in M[j(G_{\lambda_0})]$ is an open dense subset of $j(\mathbb{Q}_{[\lambda_0,\lambda]})$ and let $D = j(h)(j"\lambda, \alpha)$ for some $h \in V[G_{\lambda_0}]$ with $\mathrm{dom}(h) = P_\kappa\lambda \times \kappa$ and $\alpha < F(\lambda)$. Without loss of generality, let us assume that every element of the range of h is a dense subset of $\mathbb{Q}_{[\lambda_0,\lambda]}$ in $V[G_{\lambda_0}]$. We have $D = j(h)(j"\lambda, \alpha) = k(j_0(h))(j"\lambda, \alpha)$. Define a function $\widetilde{h} \in M_0[j_0(G)]$ with $\mathrm{dom}(\widetilde{h}) = \pi(F(\lambda))$ by $\widetilde{h}(\xi) = j_0(h)(j_0"\lambda, \xi)$. Then $\mathrm{dom}(k(\widetilde{h})) = k(\pi(F(\lambda))) = F(\lambda)$ and we have $D = k(\widetilde{h})(\alpha)$. Now the range of $\widetilde{h}$ is a collection of $\pi(F(\lambda))$ open dense subsets of $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$. Since $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ is $\leq\pi(F(\lambda))$-distributive in $M_0[j_0(G)]$, one sees that $\widetilde{D} = \bigcap \mathrm{ran}(\widetilde{h})$ is an open dense subset of $j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$. Hence there is a condition $p \in J \cap \widetilde{D}$ and by elementarity, $k(p) \in k"J \cap k(\widetilde{D}) \subseteq D$. □

3.5. **Performing surgery.** We will modify the $M[j(G_{\lambda_0})]$-generic filter $K \subseteq j(\mathbb{Q}_{[\lambda_0,\lambda]})$ to get K^* with $j"g_{[\lambda_0,\lambda]} \subseteq K^*$. Then we will argue that K^* remains an $M[j(G_{\lambda_0})]$-generic filter for $j(\mathbb{Q}_{[\lambda_0,\lambda]})$ using the main lemma from [CM13].

Let us define K^*. Working in $V[G_{\lambda_0}][g_{[\lambda_0,F(\lambda)]}]$, define

$$\mathrm{dom}(j(\mathbb{Q}_{[\lambda_0,\lambda]})) := \bigcup\{\mathrm{dom}(p) \mid p \in j(\mathbb{Q}_{[\lambda_0,\lambda]})\}$$

and let Q be the partial function with $\mathrm{dom}(Q) \subseteq \mathrm{dom}(j(\mathbb{Q}_{[\lambda_0,\lambda]}))$, defined by $Q = \bigcup j\text{"} g_{[\lambda_0,\lambda]}$. Given $p \in K$, let p^* be the partial function with $\mathrm{dom}(p^*) = \mathrm{dom}(p)$, obtained from p by altering p on $\mathrm{dom}(p) \cap \mathrm{dom}(Q)$ so that p^* agrees with Q. Let

$$K^* = \{p^* \mid p \in K\}.$$

Clearly, $j\text{"} g_{[\lambda_0,\lambda]} \subseteq K^*$ and it remains to argue that p^* is a condition in $j(\mathbb{Q}_{[\lambda_0,\lambda]})$ for each $p \in K$ and that K^* is an $M[j(G_{\lambda_0})]$-generic filter. This follows from the next lemma, which appears in [CM13].

Lemma 3.12 ([CM13])**.** *Suppose $B \in M[j(G_{\lambda_0})]$ with $B \subseteq j(\mathrm{dom}(\mathbb{Q}_{[\lambda_0,\lambda]}))$ and $|B|^{M[j(G_{\lambda_0})]} \leq j(\lambda)$. Then the set*

$$\mathcal{I}_B = \{\mathrm{dom}(j(q)) \cap B \mid q \in \mathbb{Q}_{[\lambda_0,\lambda]}\}$$

has size at most λ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$.

Proof. Let B be as in the statement of the lemma and let $B = j(h)(j\text{"}\lambda, \alpha)$ where $h : P_\kappa\lambda^V \times \kappa \to P_{\lambda^+}(\mathrm{dom}(\mathbb{Q}_{[\lambda_0,\lambda]}))^{V[G_{\lambda_0}]}$, $\alpha < F(\lambda)$, and $h \in V[G_{\lambda_0}]$. Then $\bigcup \mathrm{ran}(h)$ is a subset of $\mathrm{dom}(\mathbb{Q}_{[\lambda_0,\lambda]})$ in $V[G_{\lambda_0}]$ with $|\bigcup \mathrm{ran}(h)|^{V[G_{\lambda_0}]} \leq \lambda$. Since $V[G_{\lambda_0}] \models \lambda^{<\lambda} = \lambda$ (in $V[G_{\lambda_0}]$ we have GCH on $[\lambda_0, \lambda]$ and λ is a regular cardinal), it will suffice to show that

$$\mathcal{I}_B \subseteq \{j(d) \cap B \mid d \in P_\lambda(\bigcup \mathrm{ran}(h))^{V[G_{\lambda_0}]}\}.$$

Suppose $\mathrm{dom}(j(q)) \cap B \in \mathcal{I}_B$ where $q \in \mathbb{Q}_{[\lambda_0,\lambda]}$. We will show that $\mathrm{dom}(j(q)) \cap B = j(d) \cap B$ for some $d \in P_\lambda(\bigcup \mathrm{ran}(h))^{V[G_{\lambda_0}]}$. Let $d := \mathrm{dom}(q) \cap \bigcup \mathrm{ran}(h)$, then $\mathrm{dom}(j(q)) \cap B = j(d) \cap B$ since

$$j(d) = \mathrm{dom}(j(q)) \cap \bigcup \mathrm{ran}(j(h)) \supseteq \mathrm{dom}(j(q)) \cap B.$$

□

It now follows from Lemma 3.12 exactly as in [CM13] that K^* is an $M[j_0(G_{\lambda_0})]$-generic filter for $j(\mathbb{Q}_{[\lambda_0,\lambda]})$. Now let us show that $K^* \subseteq j(\mathbb{Q}_{[\lambda_0,\lambda]})$. Suppose $p \in j(\mathbb{Q}_{[\lambda_0,\lambda]})$, then since $|\,\mathrm{dom}(p)|^{M[j(G_{\lambda_0})]} < j(\lambda)$, it follows from Lemma 3.12, that the set $\mathcal{I}_{\mathrm{dom}(p)} := \{\mathrm{dom}(j(q)) \cap \mathrm{dom}(p) \mid q \in \mathbb{Q}_{[\lambda_0,\lambda]}\}$ has size at most λ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$. Let $\langle I_\alpha \mid \alpha < \lambda\rangle \in V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ be an enumeration of $\mathcal{I}_{\mathrm{dom}(p)}$. By the maximality of the filter K, for each $\alpha < \lambda$ we can choose $q_\alpha \in K$ such that $\mathrm{dom}(j(q_\alpha)) \cap p = I_\alpha$. It follows that $\langle j(q_\alpha) \mid \alpha < \lambda\rangle \in M[j(G_{\lambda_0})]$ because $M[j(G_{\lambda_0})]$ is closed under λ-sequences in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$. Since $j(\mathbb{Q}_{[\lambda_0,\lambda]})$ is $<j(\lambda_0)$-directed closed, it follows that the partial master condition $m := \bigcup\{j(q_\alpha) : \alpha < \lambda\}$ is a condition in

$j(\mathbb{Q}_{[\lambda_0,\lambda]})$, and moreover, q^* can be computed in $M[j(G_{\lambda_0})]$ by comparing p and m.

To see that K^* is $M[j(G_{\lambda_0})]$-generic, suppose A is a maximal antichain of $j(\mathbb{Q}_{[\lambda_0,\lambda]})$ in $M[j(G_{\lambda_0})]$. Since $\mathbb{Q}_{[\lambda_0,\lambda]}$ is λ^+-c.c. in $V[G_{\lambda_0}]$, it follows by elementarity that $\mathrm{dom}(A) := \bigcup\{\mathrm{dom}(r) \mid r \in A\}$ has size at most $j(\lambda)$ in $M[j(G_{\lambda_0})]$. Hence by Lemma 3.12, we see that $\mathcal{I}_{\mathrm{dom}(A)} := \{\mathrm{dom}(j(q)) \cap \mathrm{dom}(A) \mid q \in \mathbb{Q}_{[\lambda_0,\lambda]}\}$ has size at most λ in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]}]$ and is therefore in $M[j(G_{\lambda_0})]$. Using this one can show, as in [CM13], that there is a bit-flipping automorphism π_A of $j(\mathbb{Q}_{[\lambda_0,\lambda]})$ in $M[j(G_{\lambda_0})]$ such that if $r \in K$ and $\mathrm{dom}(r) \subseteq \mathrm{dom}(A)$ then $\mathrm{dom}(\pi_A(r)) = \mathrm{dom}(r)$ and $\pi_A(r) = r^*$. Then since $\pi_A^{-1}[A] \in M[j(G_{\lambda_0})]$ is a maximal antichain of $j(\mathbb{Q}_{[\lambda_0,\lambda]})$, and K is generic for $j(\mathbb{Q}_{[\lambda_0,\lambda]})$ over $M[j(G_{[\lambda_0,\lambda]})]$, it follows that there is a condition $s \in K \cap \pi_A^{-1}[A]$. Then $\pi_A(s) = s^* \in K^* \cap A$, and therefore K^* is generic for $j(\mathbb{Q}_{[\lambda_0,\lambda]})$ over $M[j(G_{\lambda_0})]$.

Thus we may lift the embedding to

$$j : V[G_{\lambda_0} * g_{[\lambda_0,\lambda]}] \to M[j(G_{\lambda_0}) * j(g_{[\lambda_0,\lambda]})]$$

where $j(g_{[\lambda_0,\lambda]}) = K^*$ and j is a class of $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]} * J]$. It follows that $M[j(G_{\lambda_0}) * j(g_{[\lambda_0,\lambda]})]$ is closed under λ-sequences in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]} * J]$ and that

$$\begin{aligned} M[j(G_{\lambda_0}) * j(g_{[\lambda_0,\lambda]})] = \{j(h)(j\text{"}\lambda,\alpha) \mid h : P_\kappa\lambda \times \kappa \to V \ \wedge\ \alpha < F(\lambda) \ \wedge \\ h \in V[G_{\lambda_0} * g_{[\lambda_0,\lambda]}]\}. \end{aligned}$$

Since the forcing $g_{[\lambda^+,F(\lambda)]} * J \subseteq \mathbb{Q}_{[\lambda^+,F(\lambda)]} * j_0(\mathbb{Q}_{[\lambda_0,\lambda]})$ is $\leq\lambda$-distributive in $V[G_{\lambda_0} * g_{[\lambda_0,\lambda]}]$, it follows that the pointwise image $j[g_{[\lambda^+,F(\lambda)]} * J]$ generates an $M[j(G_{\lambda_0}) * j(g_{[\lambda_0,\lambda]})]$-generic filter for $j(\mathbb{Q}_{[\lambda^+,F(\lambda)]} * j_0(\mathbb{Q}_{[\lambda_0,\lambda]}))$, denote this filter by $j(g_{[\lambda^+,F(\lambda)]} * J)$. Then the embedding lifts to

$$j : V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]} * J] \to M[j(G_{\lambda_0}) * j(g_{[\lambda_0,F(\lambda)]}) * j(J)]$$

where j is a class of $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]} * J]$, witnessing that κ is λ-supercompact in this model.

3.6. **Controlling the continuum function at $F(\lambda)^+$ and above.** In the model $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]} * J]$ one has $2^\gamma = F(\gamma)$ for every regular cardinal $\gamma \leq F(\lambda)$ and GCH holds at all cardinals greater than or equal to $F(\lambda)^+$. Working in $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]} * J]$, let $\mathbb{E}$ be the Easton-support product of Cohen forcing

$$\mathbb{E} := \prod_{\gamma \in [F(\lambda)^+,\infty) \cap \mathrm{REG}} \mathrm{Add}(\gamma, F(\gamma)).$$

Let E be generic for $\mathbb{E}$ over $V[G_{\lambda_0}*g_{[\lambda_0,F(\lambda)]}*J]$. Standard arguments [Eas70] can be used to see that in $V[G_{\lambda_0}*g_{[\lambda_0,F(\lambda)]}*J*E]$, for every regular cardinal γ we have $2^\gamma = F(\gamma)$. Since $\mathbb{E}$ is $\leq F(\lambda)$-closed in $V[G_{\lambda_0}*g_{[\lambda_0,F(\lambda)]}*J]$, it follows that the pointwise image $j[E]$ generates an $M[j(G_{\lambda_0}) * j(g_{[\lambda_0,F(\lambda)]}) * j(J)]$-generic filter for $j(\mathbb{E})$, which we will denote by $j(E)$. Then j lifts to

$$j : V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]} * J * E] \to M[j(G_{\lambda_0}) * j(g_{[\lambda_0,F(\lambda)]}) * j(J) * j(E)]$$

where j is a class of $V[G_{\lambda_0} * g_{[\lambda_0,F(\lambda)]} * J * E]$ witnessing that κ is λ-supercompact in that model. $\square$

4. Open Questions

First let us discuss the problem of globally controlling the continuum function on the regular cardinals while preserving multiple instances of partial supercompactness. Suppose GCH holds and we have regular cardinals $\kappa_0 < \eta_0 < \kappa_1 < \eta_1$ such that for each $\alpha \in \{0,1\}$, κ_α is η_α-supercompact. Additionally, assume F is a function satisfying the requirements of Easton's theorem (E1) and (E2), and that for each α there is a $j_\alpha : V \to M_\alpha$ with critical point κ_α such that κ_α is closed under F, $M^{\eta_\alpha} \subseteq M$, $H(F(\eta_\alpha)) \subseteq M$, and for each regular cardinal $\gamma \leq \eta_\alpha$, $(|j_\alpha(F)(\gamma)| = F(\gamma))^V$. Then, as a corollary to the proof of Theorem 1.2 above, we obtain the following.

Corollary 4.1. *There is a cardinal preserving forcing extension in which $2^\gamma = F(\gamma)$ for every regular cardinal γ and κ_α remains η_α-supercompact for $\alpha \in \{0,1\}$.*

This corollary can be obtained by essentially applying the above proof of Theorem 1.2 twice. For example, first we carry out the proof of Theorem 1.2 with κ_0 and η_0 in place of κ and λ and where the forcing iteration used terminates before stage κ_1. Lifting the embedding $j_0 : V \to M_0$ witnessing that κ_0 is η_0-supercompact requires the "extra forcing" that depends on j_0. Let $\mathbb{P}_0$ denote the iteration defined so far, including the extra forcing. Since $\mathbb{P}_0$ has size less than the critical point κ_1 of the next embedding $j_1 : V \to M_1$ witnessing the η_1-supercompactness of κ_1, it follows by the Levy-Solovay theorem that j_1 lifts through the iteration performed so far. Next, working in $V^{\mathbb{P}_0}$, we perform an iteration for controlling the continuum function that picks up where the last one left off. Call the iteration $\mathbb{P}_1$, and lift j_1 through the iteration $\mathbb{P}_0 * \mathbb{P}_1$ just as we lifted j_0 through $\mathbb{P}_0$. Furthermore, since $\mathbb{P}_1$ is highly distributive in $V^{\mathbb{P}_0}$ the first embedding j_0 will easily extend to $V^{\mathbb{P}_0 * \mathbb{P}_1}$.

Corollary 4.1 only covers a simple configuration of partially supercompact cardinals. Is a more general result possible? It seems that the need for the "extra forcing" in our proof of Theorem 1.2 prevents the method from providing a clear strategy for obtaining a more general result in which more complicated configurations of partially supercompact cardinals are preserved. It may be the case that the uniformity of the Sacks-forcing method, which is applied in [FH08] to obtain analogous global results for measurable as well as strong cardinals, could lead to an answer to Question 4.2 below. One would desire a two-cardinal version of Sacks forcing for adding subsets to κ that satisfies λ-fusion.

Question 4.2. Assuming GCH, and given a class of partially supercompact cardinals S and a function F from the class of regular cardinals to the class of cardinals satisfying Easton's requirements (E1) and (E2), under what conditions can one force the continuum function to agree with F at all regular cardinals, while preserving cardinals as well as the full degree of partial supercompactness of each cardinal in S?

Another potential way of strengthening Theorem 1.2 is to weaken the hypothesis. This was done for the analagous theorem concerning measurable cardinals in [FH12a]. In this direction, we pose the following question.

Question 4.3. Can the hypothesis of Theorem 1 be weakened by replacing the assumption $H(F(\lambda)) \subseteq M$ by the weaker assumption "V and M have the same cardinals up to and including $F(\lambda)$"? Or, in the special case when $F(\lambda) = \mu^+$ for some regular cardinal μ, by the ostensibly stronger assumption that $H(\mu) \subseteq M$ and $(\mu^+)^M = \mu^+$? (Note however that the latter assumption is actually optimal for the analogous case when one wants to find a model with a measurable cardinal κ with $2^\kappa = \mu^+$, where $\mu = \kappa^{+n}$ for some $n > 0$; see [FH12a] for more details.)

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